27

The SVZ expansion

27.1 The anatomy of the SVZ expansion

For definiteness, let us illustrate our discussion from the generic two-point correlator:

$$\Pi_H(q^2) \equiv i \int d^4x \; e^{iqx} \langle 0|\mathcal{T} J_H(x) \, (J_H(0))^\dagger \, |0\rangle \; , \tag{27.1}$$

where $J_H(x)$ is the hadronic current of quark and/or gluon fields. Here, the analysis is in principle much simpler than in the case of deep inelstic scatterings, because one has to sandwich the T-product of currents between the vacuum rather than between two proton states. Following SVZ [1], the breaking of ordinary perturbation theory at low q^2 is due to the manifestation of non-perturbative terms appearing as power corrections in the operator product expansion (OPE) of the Green function à la Wilson [222]. In this way, one can write:

$$\Pi_H(q^2, m^2) \simeq \sum_{D=0,2,4,\ldots} \frac{1}{(m^2 - q^2)^{D/2}} \sum_{dim\mathcal{O}=D} \mathcal{C}(q^2, m^2, \nu^2) \langle \mathcal{O}(\nu) \rangle \; , \tag{27.2}$$

provided that $m^2 - q^2 \gg \Lambda^2$. For simplicity, m is the heaviest quark mass entering into the correlator; ν is an arbitrary scale that separates the long- and short-distance dynamics; $\mathcal{C}$ are the Wilson coefficients calculable in perturbative QCD by means of Feynman diagrams techniques; $\langle \mathcal{O} \rangle$ are the non-perturbative (non-calculable) condensates built from the quarks or/and gluon fields. Though, separately, $\mathcal{C}$ and $\langle \mathcal{O} \rangle$ are (in principle) ν-dependent, this ν-dependence should (in principle) disappear in their product.

- The case $D = 0$ corresponds to the naïve perturbative contribution.
- For $D = 2$ and owing to gauge invariance which forbids the formation of a condensate, one can only have the contributions from the light quark running mass squared. Moreover, one may (or may not) also expect that the summation of perturbation theory via UV renormalon technology (see next section) could also induce such a term, while the possibility from the freezing mechanism of the coupling constant is negligibly small as it is expected to be of the order of $(\alpha_s/\pi)^2$ [338].
- For $D = 4$, the condensates that can be formed are the quark and gluon ones:

$$m\langle \bar{\psi}\psi \rangle \; , \qquad \langle \alpha_s G^2 \rangle \; , \tag{27.3}$$

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where the former can be fixed by pion PCAC (see Part I) in the standard Gell-Man–Oakes–Renner (GMOR) realization of chiral symmetry.

- For $D = 5$, one can only have, in the massless quark limit, the mixed quark-gluon condensate:

$$\left\langle \bar{\psi}\sigma_{\mu\nu}\lambda^a/2G_a^{\mu\nu}\psi\right\rangle . \tag{27.4}$$

- For $D = 6$ one has, in the chiral limit, the triple gluon and the four-quark condensates:

$$g^3 f_{abc}\langle G^a G^b G^c\rangle \ , \qquad \alpha_s\langle\bar{\psi}\Gamma_1\psi\bar{\psi}\Gamma_2\psi\rangle \ , \tag{27.5}$$

where Γ_i are generic notations for any Dirac and/or colour matrices.

The validity of the SVZ expansion has been understood formally, using renormalon technology (see next sections) and the mixing of operators under renormalizations. The SVZ expansion has been also tested in the $\lambda\phi^4$ [367,368] and QCD-like models [369] (Schwinger two-dimensional gauge theories [370], the CP^{N-1} model [371], which both have instantons and θ-vacua; the Gross–Neveu model [372] with dynamical chiral symmetry breaking [373] and 2-d $O(N)$ free non-linear σ model [374,367], where both have the asymptotic freedom property of QCD).

Its phenomenological confirmation can be viewed from the unexpected accurate extraction of α_s from τ decays and from independent measurements of the QCD condensates (see chapter on QCD condensates).

27.2 SVZ expansion in the $\lambda\phi^4$ model

For a simple pedagogical introduction, let us illustrate the SVZ expansion for scalar field theory.[1] The bare Lagrangian of the theory reads:

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi_B)^2 - \frac{1}{2}m_B^2\phi_B^2 - \frac{\lambda_B}{4!}\phi_B^4 \ , \tag{27.6}$$

where ϕ is the scalar field, m and λ are its mass and coupling. B refers to bare quantity. It is known that for $m_B^2 < 0$, one has a spontaneous breaking mechanism where the field acquires a non-vanishing expectation value, which is non-analytical in the coupling, such that the model mimics non-perturbative effects. In order to further simplify our discussions, let us, however, work in the case $m_B^2 > 0$, where no condensate breaks spontaneously the symmetry and let us ignore (for the moment) renormalization effects. We shall be concerned with the propagator:

$$\mathcal{D}(q^2) = i\int d^4x e^{iqx}\langle 0|\mathbf{T}\phi(x)\phi(0)|0\rangle \ , \tag{27.7}$$

which we shall evaluate in two different ways. In the first one, we evaluate it using the standard perturbative expansion in λ (Fig. 27.1).

[1] We shall ignore in this illustrative example the radiative corrections discussed in [367,368].

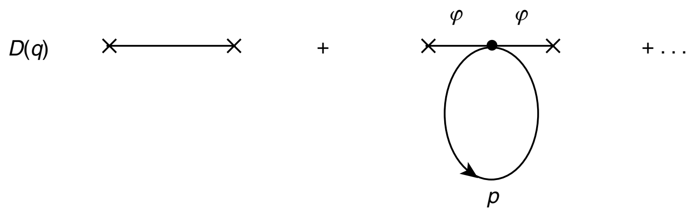

Fig. 27.1. Lowest order perturbative contribution to the scalar correlator.

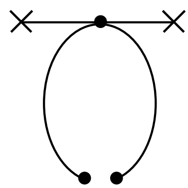

Fig. 27.2. Lowest order scalar condensate contribution to the scalar correlator.

Using, for instance, a Pauli–Villars regularization (the following conclusion is regularization invariant), one obtains for $-q^2 \gg m_B^2 \equiv m^2$:

$$\mathcal{D}(q^2) \simeq \frac{1}{q^2} + \frac{m^2}{q^4}\left\{1 - \frac{\lambda}{32\pi^2}\left(\log\frac{\Lambda^2}{m^2} - \frac{\Lambda^2}{m^2}\right)\right\}, \tag{27.8}$$

where Λ is an arbitrary UV cut-off.

In the second method, one evaluates the propagator using the SVZ expansion for $-q^2 \gg m^2$. Therefore, it reads:

$$\mathcal{D}(q^2) \simeq C_1 \mathbf{1} + C_\phi \langle\phi^2\rangle + \cdots \tag{27.9}$$

By introducing the scale ν, which separates the long- and short-wavelength fluctuations, one can extract C_1 from the perturbative graph for $p > \nu$ (short fluctuations):

$$C_1 \simeq \frac{1}{q^2} + \frac{m^2}{q^4}\left\{1 - \frac{\lambda}{32\pi^2}\left(\log\frac{\Lambda^2}{\nu^2} - \frac{\Lambda^2 - \nu^2}{m^2}\right)\right\}. \tag{27.10}$$

The Wilson coefficient C_ϕ is associated to the ϕ^2 'condensate':

$$C_\phi \simeq \frac{\lambda}{2q^4}, \tag{27.11}$$

and comes from Fig. 27.2.

The condensate $\langle\phi^2\rangle$ corresponds to the evaluation of the tadpole-like graph for $p < \nu$ (large fluctuations), from which, one obtains:

$$\langle\phi^2\rangle \simeq \frac{1}{16\pi^2}\left(\nu^2 - m^2\log\frac{\nu^2}{m^2}\right). \tag{27.12}$$

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These results show that in this simple example, the SVZ expansion recovers (to this approximation) the usual calculation. However, the coincidence of the series is not trivial if one goes to higher order. This will be the subject of the next discussion in QCD.

27.3 Renormalization group invariant (RGI) condensates

Now, the next step is to see if the condensates can be well-defined in perturbation theory, namely if one can form quantities which are invariant under the RGE. This discussion has already been anticipated when we discussed the renormalization of composite operators, where it has been shown that, in general, these operators can mix under renormalizations [126–131].

27.3.1 Scale invariant $D = 4$ condensates

- **Generalities and definitions**

In the previous section, it was demonstrated that the condensates:

$$m_j\langle\bar{\psi}_j\psi_j\rangle\ ,$$
$$\langle\theta_\mu^\mu\rangle = \frac{1}{4}\langle\beta(\alpha_s)GG\rangle + \sum_j\gamma_m(\alpha_s)m_j\langle\bar{\psi}_j\psi_j\rangle\ , \tag{27.13}$$

are renormalization group invariant [126,128]. However, perturbative evaluations of the quark-mass corrections to the correlation functions give rise to IR logarithms of the form $m^4\alpha_s^n(\nu)\log^k(m/\nu)$ $(k \leq n+1)$, where ν is the $\overline{MS}$ renormalization scale [167,399]. The mass singularities arise from the region of small loop-momenta in the relevant Feynman diagrams, and therefore should be absorbed (like the IR renormalons) into the non-perturbative condensates $\langle\mathcal{O}(\nu)\rangle$. The IR logarithms are nothing more than the perturbative contributions to the $D = 4$ vacuum condensates.

However, one should notice that the calculation of the $D = 2$ quark-mass corrections does not produce any logarithms $\log^k(m/\nu)$, which is a consequence of the absence of the $D = 2$ operators in QCD.

In order to be explicit, let us consider the pseudoscalar two-point correlator defined in Eq. (8.29). At $q = 0$, one obtains from a perturbative calculation of the correlator [167]:

$$\Psi_5^R(0)|_{\text{pert}} = \frac{3}{4\pi^2}(m_i + m_j)\left(m_i^3 Z_i + m_j^3 Z_j\right)\ , \tag{27.14}$$

with:

$$Z_i = 1 - \log\frac{m_i^2}{\nu^2} + \frac{2}{3}\left(\frac{\alpha_s}{\pi}\right)\left(5 - 5\log\frac{m_i^2}{\nu^2} + 3\log^2\frac{m_i^2}{\nu^2}\right)\ , \tag{27.15}$$

which improves the non-perturbative Ward identity in Eq. (2.17), and which indicates that, in order to absorb the mass singularities, one should add a perturbative piece to the quark condensate. In a similar way, the perturbative piece to the gluon condensate reads [325]:

$$\langle GG(\nu)\rangle_{\text{pert}}^{\overline{MS}} = -\frac{1}{2\pi^2}\left(\frac{\alpha_s}{\pi}\right)\sum_i m_i^4(\nu)\left[9 - 8\log\frac{m_i^2}{\nu^2} + 3\log^2\frac{m_i^2}{\nu^2}\right]\ . \tag{27.16}$$

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The summation of the log-terms using the RGE becomes more convenient by working with the scale invariant *non-normal ordered* condensates [325]:[2]

$$\langle\overline{\alpha_s GG}\rangle \equiv \left(1+\frac{16}{9}\alpha_s(\nu)+O\left(\alpha_s^2\right)\right)\frac{\alpha_s(\nu)}{\pi}\langle\ :GG(\nu):\ \rangle^{\overline{MS}}$$
$$-\frac{16}{9}\frac{\alpha_s(\nu)}{\pi}\left(1+\frac{19}{24}\frac{\alpha_s(\nu)}{\pi}+O\left(\alpha_s^2\right)\right)\sum_i m_i\langle\bar\psi_i\psi_i\rangle^{\overline{MS}}(\nu)$$
$$-\frac{1}{3\pi}\left(1+\frac{4}{3}\frac{\alpha_s(\nu)}{\pi}+O\left(\alpha_s^2\right)\right)\sum_i m_i^4(\nu)\,, \tag{27.17}$$

and:

$$\langle\overline{m_i\bar\psi_j\psi_j}\rangle \equiv m_i\langle\ :\bar\psi_j\psi_j:\ \rangle^{\overline{MS}}(\nu)$$
$$+\frac{3}{7\pi\alpha_s(\nu)}\left(1-\frac{53}{24}\frac{\alpha_s(\nu)}{\pi}+O\left(\alpha_s^2\right)\right)m_i(\nu)m_j^3(\nu)\,, \tag{27.18}$$

where one can notice the inverse power of α_s in the expression for the quark condensate. The use of these scale-invariant condensates in the OPE implies that the coefficient functions obey an homogeneous RGE, which then facilitates the summation of the $\log(Q/\nu)$ terms in the analysis.

Analogously to the invariant mass $\hat m_i$ in Eq. (11.77), a spontaneous RGI mass $\hat\mu_i$ associated to the quark vacuum condensate can also be introduced by taking into account the fact that the product $m_i\langle\bar\psi\psi\rangle$ is RGI (at least to leading order in m_i) [28,110,2]. Then, one obtains:

$$\langle\overline{\ :\bar\psi_i\psi_i:\ }\rangle(\nu) = -\hat\mu_i^3\left(-\beta_1 a_s(\nu)\right)^{\gamma_1/\beta_1}\left\{1+\frac{\beta_2}{\beta_1}\left(\frac{\gamma_1}{\beta_1}-\frac{\gamma_2}{\beta_2}\right)a_s(\nu)\,,\right.$$
$$+\frac{1}{2}\left[\frac{\beta_2^2}{\beta_1^2}\left(\frac{\gamma_1}{\beta_1}-\frac{\gamma_2}{\beta_2}\right)^2-\frac{\beta_2^2}{\beta_1^2}\left(\frac{\gamma_1}{\beta_1}-\frac{\gamma_2}{\beta_2}\right)+\frac{\beta_3}{\beta_1}\left(\frac{\gamma_1}{\beta_1}-\frac{\gamma_3}{\beta_3}\right)\right]a_s^2(\nu)$$
$$\left.+\,\mathcal{O}\left(a_s^3,\ m_i^3\right)\right\}^{-1}\,, \tag{27.19}$$

where the values of the β functions and mass anomalous dimensions can be found in Table 11.1.

- **Values of the light quark condensates**
  Assuming a GMOR realization of chiral symmetry as commonly accepted, the light quark condensate can be estimated, to leading order of the light quark mass, from the PCAC relation given in Eq. (2.22):

$$(m_u+m_d)\langle\ :\bar\psi_u\psi_u+\bar\psi_d\psi_d:\ \rangle \simeq -2m_\pi^2 f_\pi^2\,. \tag{27.20}$$

Anticipating the values of the running masses in the next chapter, one can deduce:

$$\frac{1}{2}\langle\bar uu+\bar dd\rangle(2\ \text{GeV}) = -[(254\pm15)\ \text{MeV}]^3\,. \tag{27.21}$$

We shall also see, in the next chapter, that one has a large breaking from the $SU(3)_F$ flavour symmetric value of the condensates as first noticed in [400] from the pseudoscalar sum rule, where

[2] One should notice that the use of the Wick's theorem in the evaluation of the Feynman diagrams generates automatically *normal-ordered* condensates which we shall denote as $\langle\ :\mathcal{O}:\ \rangle$

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a recent update estimate [354,419–421] (see also [423]) leads to the ratio of the *normal ordered* condensates:

$$\langle\ :\bar{s}s:\ \rangle/\langle\ :\bar{u}u:\ \rangle = 0.66 \pm 0.10\ , \tag{27.22}$$

in agreement within the errors with different baryon sum rules results [426–430]. Combining Eq. (27.17) with this result, one can also deduce the ratio of the *non-normal ordered* condensates:

$$\langle\ \bar{s}s\ \rangle/\langle\ \bar{u}u\ \rangle = 0.75 \pm 0.12\ . \tag{27.23}$$

- **Value of the gluon condensate**
  The gluon condensate has been orginally estimated by SVZ from charmonium sum rules [1]. We shall see in the next chapter that a re-extraction of this quantity from the $e^+e^- \to I = 1$ hadrons data and from the heavy quarkonia mass-splittings, lead to [329,313] (Sections 51.3 and 52.10):

$$\langle\alpha_s G^2\rangle = (7.1 \pm .9)10^{-2}\ \text{GeV}^4\ , \tag{27.24}$$

  as expected from various post-SVZ estimates [3,356–365], but about a factor of two higher than the original SVZ estimate.

27.3.2 $D = 5$ mixed quark-gluon condensate

The renormalization of the mixed quark-gluon condensate has been studied in [130], where it has been shown that the scale invariant, which one can form, is the combination:

$$\langle\bar{O}_5\rangle = \alpha_s^{\gamma_M/-\beta_1}\langle O_4 + x\ O_1 + y\ O_2\rangle \tag{27.25}$$

where the dimension-five gauge invariant operators are:

$$\begin{aligned}
O_1 &\equiv i m_j^2 \langle\bar{\psi}_j\psi_j\rangle\ ,\\
O_2 &\equiv -\frac{i}{4} m_j \langle G^2\rangle\ ,\\
O_3 &\equiv -m_j\bar{\psi}_j(\hat{D} + i m_j)\psi_j\ ,\\
O_4 &\equiv g\left\langle\bar{\psi}_j\sigma^{\mu\nu}\frac{\lambda^a}{2}\psi_j G^a_{\mu\nu}\right\rangle\ .
\end{aligned} \tag{27.26}$$

For $SU(3)_C$:

$$x = -\frac{1944}{315} \qquad y = -\frac{72}{63} \qquad \gamma_M = -\frac{1}{3}\ , \tag{27.27}$$

which indicates that working with $\langle O_4\rangle$, is only valid to leading order in the quark mass-expansion.

The value of the mixed quark-gluon condensate has been estimated from baryon sum rules [424–430] to be about 0.8 GeV^2, and alternatively from the heavy-light quark systems B and B^* [401]. Further discussions will be given in the next chapter. The QSSR value of the mixed quark-gluon condensate is:

$$g\left\langle\bar{\psi}_j\sigma^{\mu\nu}\frac{\lambda^a}{2}\psi_j G^a_{\mu\nu}\right\rangle \equiv M_0^2\ \alpha_s^{\gamma_M/-\beta_1}\langle\bar{\psi}_j\psi_j\rangle\ , \tag{27.28}$$

where:

$$M_0^2 = (0.8 \pm 0.01)\ \text{GeV}^2\ . \tag{27.29}$$

For a conservative result, we shall adopt:

$$M_0^2 = (0.8 \pm 0.1)\ \text{GeV}^2\ , \tag{27.30}$$

assuming a 10% uncertainty typical for the sum rule approach.

27.3.3 $D = 6$ gluon condensates

- **Triple gluon condensate:**
  It reads:

$$\langle \mathcal{O}_G \rangle \equiv g^3 f_{abc} \langle G^a G^b G^c \rangle\ , \tag{27.31}$$

  and does not mix under renormalization with the other $D = 6$ operators. Its renormalization improved expression is [130]:

$$\langle \bar{\mathcal{O}}_G \rangle = \alpha_s^{-\gamma_G/\beta_1} \langle \mathcal{O}_G \rangle\ , \tag{27.32}$$

  where for $SU(N)$:

$$\gamma_G = \frac{1}{6}(2 + 7N)\ . \tag{27.33}$$

  A crude estimate of this condensate can be obtained using a dilute gas instanton approximation model (DIGA) (see next chapter). For an instanton size $\rho_c \approx 200$ MeV, one obtains [382]:[3]

$$\langle \mathcal{O}_G \rangle \approx (1.5 \pm 0.5)\ \text{GeV}^2 \langle \alpha_s G^2 \rangle\ , \tag{27.34}$$

  where we have assumed a 30% error. This estimate is in good agreement with the $SU(2)$ lattice estimate [402]:

$$\langle \mathcal{O}_G \rangle \approx (1.2\ \text{GeV}^2) \langle \alpha_s G^2 \rangle\ , \tag{27.35}$$

  although one should be careful in using this result (in particular the sign) as the latter has been obtained in the Euclidean region.
- **Gluon derivative condensate:**

$$g^2 \langle DG\ DG \rangle \equiv g^2 \big\langle D_\mu G_a^{\alpha\mu}\ D^\nu G^a_{\alpha\nu} \big\rangle\ , \tag{27.36}$$

  It reduces to the four-quark condensates:

$$g^4 \left\langle \sum \bar{\psi} \gamma_\mu \frac{\lambda_a}{2} \psi \right\rangle^2 , \tag{27.37}$$

  after the use of the equation of motion.

[3] The effect of the anomalous dimension has not been included in the analysis but it is negligible.

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27.3.4 $D = 6$ *four-quark condensates*

On the contrary, the four-quark condensates mix with each other [130,131]. In the chiral limit and after the use of the equation of motion, one remains only with five operators:

$$O_\Gamma \equiv \langle \bar\psi^\alpha \Gamma_1 \psi_\alpha \bar\psi^\beta \Gamma_2 \psi_\beta \rangle \ , \tag{27.38}$$

where the colour indices (α, β) run from 1 to N. Γ_i is any Dirac and colour indices. One can choose the basis:

$$\begin{aligned} &\Gamma_S = 1 \ , \quad \Gamma_V = \gamma_\mu \ , \\ &\Gamma_P = \gamma_5 \ , \quad \Gamma_A = \gamma_\mu \gamma_5 \ , \\ &\Gamma_T = \sigma_{\mu\nu} \equiv \tfrac{i}{2}[\gamma_\mu, \gamma_\nu] \ . \end{aligned} \tag{27.39}$$

In the large N-limit, where the vacuum saturation is expected, one has:

$$\langle \bar\psi \Gamma_1 \psi \bar\psi \Gamma_2 \psi \rangle = \frac{1}{16N^2}[\mathbf{Tr}\ \Gamma_1\ \mathbf{Tr}\ \Gamma_2 - \mathbf{Tr}(\Gamma_1 \Gamma_2)]\langle \bar\psi \psi \rangle^2 \ . \tag{27.40}$$

With the inclusion of the next $1/N$-correction, one has [130]:

$$\begin{aligned} &O_S = \langle \bar\psi \psi \rangle^2 \left(1 - \tfrac{1}{4N}\right) \ , \quad O_P = \langle \bar\psi \psi \rangle^2 \left(-\tfrac{1}{4N}\right) \ , \\ &O_V = \langle \bar\psi \psi \rangle^2 \left(-\tfrac{1}{N}\right) \ , \qquad O_A = \langle \bar\psi \psi \rangle^2 \left(\tfrac{1}{N}\right) \ , \\ &O_T = \langle \bar\psi \psi \rangle^2 \left(-\tfrac{3}{4N}\right) \ , \end{aligned} \tag{27.41}$$

indicating that in the large N-limit, only O_S is relevant. However for finite N, the operators mix with each other and signal the fact that the vacuum saturation is inconsistent with the RGE, as one cannot form a RGI condensate. In the solely case of $N \to \infty$, the situation improves, as one can construct the RGI condensate:

$$\langle \bar O_2 \rangle = \alpha_s^{\gamma_6/-\beta_1} \langle g^2 \bar\psi \psi \bar\psi \psi \rangle \ , \tag{27.42}$$

where:

$$\gamma_6(N \to \infty) = \frac{143}{33} N \ . \tag{27.43}$$

The size of the four-quark condensates has been estimated from the $e^+e^- \to$ hadrons data [403,404], [405–409] and from the $\tau \to$ hadrons decay width [328,33]. It has been noticed for *the first time* in [404] that the vacuum saturation assumption, used previously, underestimates the real value of the four-quark condensate by a factor:

$$\rho \simeq 2 - 3 \ , \tag{27.44}$$

while in the e^+e^- analysis, there is a strong correlation between the four-quark and the $\langle \alpha_s G^2 \rangle$ condensates, as they appear in opposite signs in the OPE. This result has been also found from baryon sum rules [424,426]. A recent analysis of the e^+e^- data gives [329] (reprinted paper):

$$\rho \alpha_s \langle \bar\psi \psi \rangle^2 = (5.8 \pm 0.9) 10^{-4}\ \text{GeV}^6 \ , \tag{27.45}$$

a result confirmed by the ALEPH and OPAL measurement from τ-decay data [328,33].

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27.3.5 Higher dimensions gluonic condensates

The dimension-eight gluonic condensates have been discussed in [410–412]. In general, one can form eight operators:

$$
\begin{aligned}
O_1 &= \langle \mathbf{Tr}\, G^2\ \mathbf{Tr}\, G^2 \rangle\ , \\
O_2 &= \langle \mathbf{Tr}\, G_{\nu\mu} G^{\rho\mu}\ \mathbf{Tr}\, G_{\nu\tau} G^{\rho\tau} \rangle\ , \\
O_3 &= \langle \mathbf{Tr}\, G_{\nu\mu} G^{\tau\rho}\ \mathbf{Tr}\, G_{\nu\mu} G^{\tau\rho} \rangle\ , \\
O_4 &= \left\langle \mathbf{Tr}\, G_{\nu\mu} G^{\tau\rho}\ \mathbf{Tr}\, G^{\nu}_{\tau} G^{\mu}_{\rho} \right\rangle , \\
O_5 &= \langle \mathbf{Tr}\, G_{\nu\mu} G^{\mu\rho}\ G_{\rho\tau} G^{\tau\nu} \rangle\ , \\
O_6 &= \langle \mathbf{Tr}\, G_{\nu\mu} G^{\nu\mu} G^{\tau\rho} G_{\tau\rho}\ \rangle\ , \\
O_7 &= \langle \mathbf{Tr}\, G_{\nu\mu} G^{\nu\rho}\ G_{\mu\tau} G^{\rho\tau} \rangle\ , \\
O_8 &= \langle \mathbf{Tr}\, G_{\nu\mu} G^{\rho\tau}\ G^{\nu\mu} G^{\rho\tau} \rangle\ .
\end{aligned} \tag{27.46}
$$

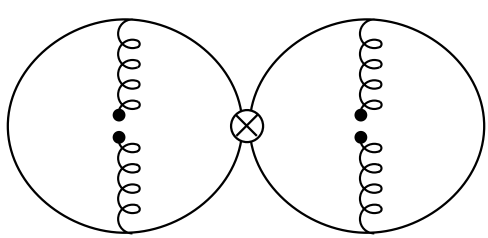
Using the symmetry properties of the colour indices and an explicit evaluation of the trace, one can show that one has only six independent operators and the relation for $N = 3$ [411]:

$$
\begin{aligned}
O_5 + 2O_7 &= O_2 + \frac{1}{2} O_4\ , \\
O_8 + 2O_6 &= O_3 + \frac{1}{2} O_1\ .
\end{aligned} \tag{27.47}
$$

The use of the vacuum saturation in the large N-limit gives:

$$
\begin{aligned}
O_1 &= \langle G^2\rangle^2 \tfrac{1}{4}\left(1 + \tfrac{1}{3}\tfrac{1}{N^2-1}\right)\ , &\quad O_2 &= \langle G^2\rangle^2 \tfrac{1}{4}\left(\tfrac{1}{4} + \tfrac{1}{3}\tfrac{1}{N^2-1}\right)\ , \\
O_3 &= \langle G^2\rangle^2 \tfrac{1}{4}\left(\tfrac{1}{6} + \tfrac{7}{6}\tfrac{1}{N^2-1}\right)\ , &\quad O_4 &= \langle G^2\rangle^2 \tfrac{1}{4}\left(\tfrac{1}{12} + \tfrac{1}{2}\tfrac{1}{N^2-1}\right)\ , \\
O_5 &= \langle G^2\rangle^2 \tfrac{1}{4N}\left(\tfrac{1}{2} - \tfrac{1}{12}\tfrac{1}{N^2-1}\right)\ , &\quad O_6 &= \langle G^2\rangle^2 \tfrac{1}{4N}\left(\tfrac{7}{6} - \tfrac{1}{6}\tfrac{1}{N^2-1}\right)\ , \\
O_7 &= \langle G^2\rangle^2 \tfrac{1}{4N}\left(\tfrac{1}{3} - \tfrac{1}{4}\tfrac{2}{N^2-1}\right)\ , &\quad O_8 &= \langle G^2\rangle^2 \tfrac{1}{4N}\left(\tfrac{1}{3} - \tfrac{1}{N^2-1}\right)\ ,
\end{aligned} \tag{27.48}
$$

which indicates that only the first four operators are leading in $1/N$, and they do not satisfy the previous constraints. Moreover, the $1/N^2$ corrections to these leading-term are also large for $N = 3$ in the case of O_3 and O_4, and put some doubts on the validity of the $1/N$-approximation. A modified factorization has been therefore proposed [411] based on the evaluation of the typical G^4 one-point function:



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within a heavy-quark mass expansion and on the approximate validity of the factorization of the four-quark operator. In this way, one obtains the constraints:

$$
\begin{aligned}
O_1 &= \frac{\langle G^2 \rangle^2}{4} \,, \\
O_3 &= 2 O_4 \,, \\
O_6 &= \frac{1}{4}\frac{1}{N}\langle G^2 \rangle^2 \,, \\
O_5 &= O_7 \,, \\
O_8 &= 4 O_7 - O_6 \,, \qquad (27.50)
\end{aligned}
$$

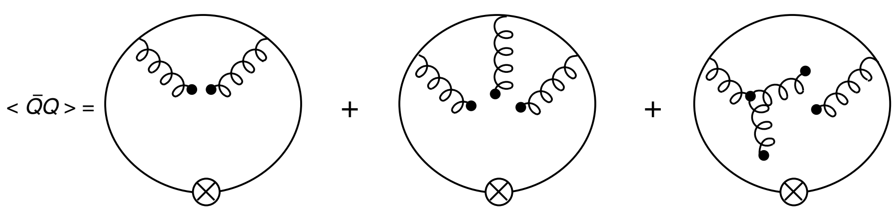
which leave one operator unconstrained.

Although theoretically interesting, these results are not very rewarding in practice. However, phenomenological fits from the τ decay [407,328,346] and $e^+e^- \to$ hadrons data [329] (Section 52.10) indicate that the estimate based on the factorization assumption gives about a factor 5 underestimate of the real value of the dimension-eight operators.

27.3.6 Relations among the different condensates

The heavy quark expansion has been used to derive the relations among the different condensates by studying the OPE in the inverse of the quark mass of the corresponding one-point function. More rigorously, the following results apply in the condensates of a heavy quark Q, although it has been extended to the light quark one in the literature, in an attempt to derive the light quark constituent mass from an effective QCD action [413].

A study of the $\langle \bar{Q} Q \rangle$ one-point function as shown in the following figure:



(27.51)

gives:

$$
M_Q \langle \bar{Q} Q \rangle = -\frac{1}{12\pi}\langle \alpha_s G^2 \rangle - \frac{1}{1440\pi^2}\frac{g^3\langle G^3 \rangle}{M_Q^2} - \frac{1}{120\pi^2}\frac{\langle DGDG \rangle}{M_Q^2} \,, \qquad (27.52)
$$

where the first term has been obtained orginally by SVZ [1] and the higher dimensions corrections have been evaluated in [410,411].

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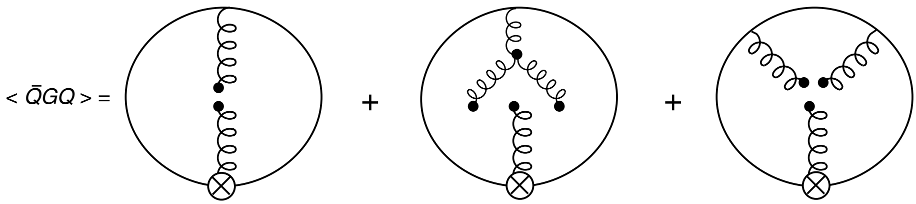
A similar relation has been also derived from the one-point function of the mixed quark-gluon condensate [410,411,414]:



(27.53)

It reads:

$$M_Q \left\langle \bar{Q}\sigma^{\mu\nu}\frac{\lambda_a}{2}QG^a_{\mu\nu}\right\rangle = -\frac{5}{24}\left(\frac{\alpha_s}{\pi}\right)g\langle G^3\rangle\ , \tag{27.54}$$

where we have dropped the non-local *perturbative* term, which has no physical relevance, though its rôle is useful for absorbing the $m^k \log^n(m/\nu)$ mass singularities (subtleties related to these terms have been discussed in detail in [411]).

The previous relations show that the quark and mixed quark-gluon condensates vanish in the world with an infinitely heavy quark mass. Due to the positivity of the G^3-condensate, the previous relation also shows that the mixed quark-gluon condensate is positive, which is a less trivial result. Finally, a relation among the condensates has also been derived using a Cauchy–Scharwz-like inequality [415]:

$$\langle g\bar{\psi}G\psi\rangle^2 \leq 16\pi\,\langle\alpha_s G^2\rangle|\langle\bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi\rangle|\ , \tag{27.55}$$

which we should regard with a great caution as we can not control the error in deriving this formula. However, it indicates that the previous values of the condensates obtained from QSSR are self-consistent and disfavours the SVZ *standard* value of the gluon and four-quark condensates.

27.3.7 Non-normal ordered condensates and cancellation of mass singularities

We have discussed in previous sections the mixing among different condensates and the neccessity to form scale-invariant quantities which facilitates the log-resummation. This definition is also intimately connected with the absence of quark-mass singularities in the OPE. In order to show explicitly how these IR singularities are absorbed, it is informative to write the renormalized *non-normal-ordered* condensates in terms of the *normal-ordered* ones denoted by $\langle\ :\mathcal{O}:\ \rangle$, where the latter appear naturally in the use of the Wick's theorem

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for the calculation of the Feynman diagrams. To order α_s, one has:

$$
\begin{aligned}
\langle\, \bar\psi\psi\,\rangle(\nu) &= \langle\ :\bar\psi\psi\ :\ \rangle - \frac{3}{4\pi^2} m^3 \left[\log\frac{m^2}{\nu^2} - 1\right] - \frac{1}{12\pi}\frac{1}{m}\langle\ :\alpha_s\, GG:\ \rangle\ , \\
\langle\, \alpha_s\, GG\,\rangle(\nu) &= \langle\ :\alpha_s\, GG:\ \rangle + \mathcal{O}\big(\alpha_s^2\big)\ , \\
g\left\langle\, \bar\psi\sigma^{\mu\nu}\frac{\lambda_a}{2}G^a_{\mu\nu}\psi\,\right\rangle(\nu) &= g\left\langle\ :\bar\psi\sigma^{\mu\nu}\frac{\lambda_a}{2}G^a_{\mu\nu}\psi:\ \right\rangle + \frac{m}{2\pi}\log\frac{m^2}{\nu^2}\langle\ :\alpha_s\, GG:\ \rangle\ ,
\end{aligned}
\tag{27.56}
$$

where the non-local logarithms and additionnal terms are just those necessary to render the results of the OPE free from IR singularities. Careful handling of these quantities are required for correct treatments of the Green's functions in which one or more internal fermion masses are much smaller than the QCD scale. We also expect that most of the perturbative results available in the literature (e.g. supersymmetric calculations or QCD high-energy processes, . . .), which are strongly affected by the change of the light quark masses (or the IR scale of the theory), should be treated in an analogous way in order to absorb such divergences.