Note on Partial Fractions and Determinants.

By Professor H. W. Turnbull.

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Introduction.

In looking for a compact way of writing down the partial fraction formula in general, with repeated factors, I noticed how the expansion of a determinant by its top or bottom row suggested a method. The following gives a formula perfectly easy to write down in any given case where the factors of the denominator of the fraction are known. Incidentally it gives, as a determinant, the integral of a rational fraction $f(x)/Q(x)$ where $f(x)$ and $Q(x)$ are polynomials, $Q(x)$ having higher order.

Most probably the results are not new, but they cannot readily be traced. I find hints of allied things in Sir Thomas Muir's History of Determinants, Vol. I p. 339, Jacobi (1841); II 175, 181, 183; III 133, 144, 152, 154.

The method seems fruitful, and the matter might be pursued further, for instance in dealing with complex roots of $Q(x) = 0$, and in deriving algebraic identities as in § 2 for the case of repeated factors.

§ 1. If we expand by the method of partial fractions when a, b, c are unequal we find

$$\frac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)} = \frac{c-b}{x-a} + \frac{a-c}{x-b} + \frac{b-a}{x-c},$$

whence

$$\frac{1}{(x-a)(x-b)(x-c)} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ \dfrac{1}{x-a} & \dfrac{1}{x-b} & \dfrac{1}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Further if $b = a + h$ and $h \to 0$, $a \neq c$, we obtain

$$\frac{1}{(x-a)^2(x-c)} = \lim \begin{vmatrix} 1 & 1 & 1 \\ a & a+h & c \\ \dfrac{1}{x-a} & \dfrac{1}{x-a-h} & \dfrac{1}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ a & a+h & c \\ a^2 & (a+h)^2 & c^2 \end{vmatrix}.$$

Subtract the first column from the second in each determinant, and divide each by h; then the limit is

$$\begin{vmatrix} 1 & 0 & 1 \\ a & 1 & c \\ \dfrac{1}{x-a} & \dfrac{1}{(x-a)^2} & \dfrac{1}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & 0 & 1 \\ a & 1 & c \\ a^2 & 2a & c^2 \end{vmatrix}.$$

On expanding the first determinant by its last row, we have the partial fractions of the left hand expression. This simple device is perfectly general, leading at once to a compact formula for partial fractions of

$$\frac{f(x)}{Q(x)} \equiv \frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_n)}$$

where $f(x)$ is a polynomial in x of degree less than n, having no common factor with $Q(x)$.

The general formula is

$$\frac{f(x)}{Q(x)} = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ \dfrac{f(a_1)}{x-a_1} & \dfrac{f(a_2)}{x-a_2} & \ldots & \dfrac{f(a_n)}{x-a} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^{n-1} \end{vmatrix} \qquad (1)$$

So the theory of partial fractions is closely related to that of alternants, the name given to determinants like this second one Δ, which is readily seen to have $\frac{1}{2}n(n-1)$ linear factors. In fact

$$\begin{aligned} \Delta = (a_2 - a_1)(a_3 - a_1) \ldots (a_{n-1} - a_1)(a_n - a_1) \\ \times (a_3 - a_2) \ldots (a_{n-1} - a_2)(a_n - a_2) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \times (a_{n-1} - a_{n-2})(a_n - a_{n-2}) \\ \times (a_n - a_{n-1}). \end{aligned} \qquad (2)$$

For Δ vanishes if $a_i = a_j$, $(i \neq j)$ because two columns are then equal; the dimensions agree; and the coefficient of the chief term

$$a_2 a_3^2 \ldots a_n^{n-1}$$

is unity in both the determinant and the product.

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The proof of (1) is now immediate by the method of partial fractions. Thus if the left side of (1) is written

$$\sum_{i=1}^{n} \frac{f(a_i)}{A_i(x-a_i)}$$

then

$$A_i = (a_i - a_1)(a_i - a_2) \ldots (a_i - a_{i-1})(a_i - a_{i+1}) \ldots (a_i - a_n).$$

In particular $A_n = (a_n - a_1) \ldots (a_n - a_{n-1})$ which is precisely the quotient of Δ divided by the cofactor of $\dfrac{f(a_n)}{x-a_n}$ in the other determinant, since this cofactor is the alternant of the first $n-1$ quantities $a_1, a_2, a_3 \ldots$. Symmetry shews that all the coefficients will agree in the same way.

COROLLARY. We immediately integrate $\dfrac{f(x)}{Q(x)}$ and obtain

$$\int \frac{f(x)\,dx}{(x-a_1)(x-a_2)\ldots(x-a_n)} = \begin{vmatrix} 1 & \ldots\ldots, & 1 \\ a_1 & \ldots\ldots, & a_n \\ a_1^2 & \ldots\ldots, & a_n^2 \\ \ldots & \ldots & \ldots \\ a_1^{n-2} & \ldots\ldots, & a_n^{n-2} \\ f(a_1)\log(x-a_1), & \ldots, & f(a_n)\log(x-a_n) \end{vmatrix} \div \Delta$$

where Δ is the alternant $|\,1\ a_2\ a_3^2 \ldots a_n^{n-1}\,|$.

HOMOGENEOUS PRODUCTS.

§ 2. If in (1) we take $x = \dfrac{1}{y}$ and $f(x) = 1$, and then expand both sides in ascending powers of y, we obtain on the left

$$\frac{y^n}{(1-a_1y)(1-a_2y)\ldots(1-a_ny)} = \sum_{p=0}^{\infty} {}_nH_p\, y^{n+p}$$

where ${}_nH_p$ is the sum of homogeneous products of $a_1, a_2, \ldots a_n$ of degree p. But on the right of (1) the coefficient of y^{n+p} in the last row gives

$$a_1^{n+p-1},\ a_2^{n+p-1},\ \ldots\ a_n^{n+p-1}.$$

This leads to the well known result[1]

$${}_nH_p = \begin{vmatrix} 1 & \ldots & 1 \\ a_1 & \ldots & a_n \\ a_1^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots \\ a_1^{n-2} & \ldots & a_n^{n-2} \\ a_1^{n+p-1} & \ldots & a_n^{n+p-1} \end{vmatrix} \div \begin{vmatrix} 1 & \ldots & 1 \\ a_1 & \ldots & a_n \\ a_1^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots \\ a_1^{n-1} & \ldots & a_n^{n-1} \end{vmatrix}.$$

[1] *History*, I, p. 339. Jacobi (1841).

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For example

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^5 & b^5 & c^5 & d^5 \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd.$$

REPEATED FACTORS IN $Q(x)$.

§ 3. The method of limits yields a result immediately. Thus, as in the case first explained, we should have, if $f(x)$ is a cubic in x,

$$\frac{f(x)}{(x-a)^2(x-b)(x-c)} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ a & 1 & b & c \\ a^2 & 2a & b^2 & c^2 \\ \frac{f(a)}{x-a}, & \left(\frac{f(a)}{x-a}\right)', & \frac{f(b)}{x-b}, & \frac{f(c)}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & 0 & 1 & 1 \\ a & 1 & b & c \\ a^2 & 2a & b^2 & c^2 \\ a^3 & 3a^2 & b^3 & c^3 \end{vmatrix}.$$

Here $\left(\frac{f(a)}{x-a}\right)'$ denotes $\frac{\partial}{\partial a}\frac{f(a)}{x-a}$ *i.e.* $\frac{f'(a)}{x-a} + \frac{f(a)}{(x-a)^2}$.

If now $b = a + h$ and $h \rightarrow 0$ we obtain, in each third column, linear multiples of the previous columns. So h^2 is the first power of h in the resulting development of both determinants in ascending powers of h. If $(\phi)''$ denotes $\frac{\partial^2\phi}{\partial a^2}$ we therefore obtain

$$\frac{f(x)}{(x-a)^3(x-c)} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ a & 1 & 0 & c \\ a^2 & 2a & 1 & c^2 \\ \frac{f(a)}{x-a}, & \left(\frac{f(a)}{x-a}\right)', & \frac{1}{2!}\left(\frac{f(a)}{x-a}\right)'', & \frac{f(c)}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & 0 & 0 & 1 \\ a & 1 & 0 & c \\ a^2 & 2a & 1 & c^2 \\ a^3 & 3a^2 & 3a & c^3 \end{vmatrix}.$$

Manifestly the same argument, assumed for $k-1$ repetitions of $(x-a)$, will apply for k, by making the next factor tend to equality with $x-a$. Thus the k^{th} entry in the last row will be

$$\frac{f(b)}{x-b} = \frac{f(a+h)}{x-a-h} = \frac{f(a)}{x-a} + h\left(\frac{f(a)}{x-a}\right)' + \dots + \frac{h^k}{k!}\left(\frac{f(a)}{x-a}\right)^{(k)} + \dots$$

In both determinants the k^{th} column will then involve linear multiples of all preceding columns. So these may be discarded. After cancelling h^k and then making $h = 0$ we obtain a definite result. So the theorem is true by induction. It can then be extended to cover cases when several sets of repeated factors occur.

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If in the above example we multiply each third column by $2!$ we obtain a pair of determinants easier to form but not quite so easy to compute. It gives the rule: *if* $(x-a)$ *is a factor occurring* k *times in* $Q(x)$, *but not* $k+1$ *times, form* k *columns of each determinant by* $k-1$ *successive differentiations of the normal column as found when* $k=1$.

Since

$$\left(\frac{f(a)}{x-a}\right)' = \frac{f'(a)}{x-a} + \frac{f(a)}{(x-a)^2},$$

$$\left(\frac{f(a)}{x-a}\right)'' = \frac{f''(a)}{x-a} + \frac{2f'(a)}{(x-a)^2} + \frac{2f(a)}{(x-a)^3},$$

$$\left(\frac{f(a)}{x-a}\right)''' = \frac{f'''(a)}{x-a} + \frac{3f''(a)}{(x-a)^2} + \frac{6f'(a)}{(x-a)^3} + \frac{3!\,f(a)}{(x-a)^4}, \text{ etc.}$$

We can easily derive the usual partial fractions by picking out the necessary terms from the determinants. One further example is appended:—

$$\frac{f(x)}{(x-a)^3(x-b)^2(x-c)} \text{ gives}$$

$$\frac{\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ a & 1 & 0 & b & 1 & c \\ a^2 & 2a & 1 & b^2 & 2b & c^2 \\ a^3 & 3a^2 & 3a & b^3 & 3b^2 & c^3 \\ a^4 & 4a^3 & 6a^2 & b^4 & 4b^3 & c^4 \\ \dfrac{f(a)}{x-a}, & \left(\dfrac{f(a)}{x-a}\right)', & \dfrac{1}{2!}\left(\dfrac{f(a)}{x-a}\right)'', & \dfrac{f(b)}{x-b}, & \left(\dfrac{f(b)}{x-b}\right)', & \dfrac{f(c)}{x-c} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ a & 1 & 0 & b & 1 & c \\ a^2 & 2a & 1 & b^2 & 2b & c^2 \\ a^3 & 3a^2 & 3a & b^3 & 3b^2 & c^3 \\ a^4 & 4a^3 & 6a^2 & b^4 & 4b^3 & c^4 \\ a^5 & 5a^4 & 10a^3 & b^5 & 5b^4 & c^5 \end{vmatrix}}.$$

These determinants, and all such, simplify by replacing row_q by

$$\text{row}_q - (q-1)a\,\text{row}_{q-1} + \binom{q-1}{2}a^2\,\text{row}_{q-2} - \ldots - (-)^q a^{q-1}\,\text{row}_1$$

where in the upper determinant $q = 4, 3, 2$ in succession, and in the lower, $q = 5, 4, 3, 2$. Then the denominator becomes[1]

$$\Delta' = (b-a)^6(c-a)^3(c-b)^2.$$

[1] *Cf.* Muir. *Historg* IV, p. 178. Schendel appears first to have discussed this type of determinant.

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The first five rows of the numerator become

$$\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & (b-a) & 1 & (c-a) \\
0 & 0 & 1 & (b-a)^2 & 2(b-a) & (c-a)^2 \\
0 & 0 & 0 & (b-a)^3 & 3(b-a)^2 & (c-a)^3 \\
0 & 0 & 0 & (b-a)^4 & 4(b-a)^3 & (c-a)^4.
\end{array}$$

If for example the partial fraction involving $(x-a)^{-2}$ is required we have at once $A_2/(x-a)^2$, where

$$A_2 = \begin{vmatrix}
1 & . & . & 1 & . & 1 \\
. & 1 & . & (b-a) & 1 & (c-a) \\
. & . & 1 & (b-a)^2 & 2(b-a) & (c-a)^2 \\
. & . & . & (b-a)^3 & 3(b-a)^2 & (c-a)^3 \\
. & . & . & (b-a)^4 & 4(b-a)^3 & (c-a)^4 \\
. & f(a) & f'(a) & . & . & .
\end{vmatrix} \div \Delta'$$

$$= \frac{f(a)(3a-b-2c)}{(b-a)^3(c-a)^2} + \frac{f'(a)}{(b-a)^2(c-a)}.$$

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