Glasgow Math. J. **41** (1999) 13–17. © Glasgow Mathematical Journal Trust 1999. Printed in the United Kingdom

PUTNAM'S INEQUALITY FOR p-HYPONORMAL n-TUPLES

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(Received 11 February, 1997; revised 14 June, 1997)

Dedicated to the memory of Professor Katsutoshi Takahashi

Abstract. In this paper, we introduce p-hyponormal tuples in the sense of D. Xia [**6**]. Furthermore we extend Putnam's inequality to these tuples and show an equivalence relation of two spectra.

1. Introduction. Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For an operator $T \in B(\mathcal{H})$ is called *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$. If $p = 1$, T is called *hyponormal*, and if $p = \frac{1}{2}$, T is called *semi-hyponormal*. For an operator $T \in B(\mathcal{H})$, we denote the spectrum of T by $\sigma(T)$. D. Xia, in [**6**], introduced semi-hyponormal tuples and extended the Putnam's inequality to semi-hyponormal tuples. In this paper, we introduce p-hyponormal tuples and extend the Putnam's inequality to p-hyponormal tuples. Throughout this paper, let $\mathbf{U} = (U_1, \ldots, U_n)$ be a fixed commuting n-tuple of unitary operators and, for an operator $T \in B(\mathcal{H})$, we denote the $(n+1)$-tuple $(U_1, \ldots, U_n, T)$ by $(\mathbf{U}, T)$. The operator $\mathbf{Q}_j$ $(j = 1, \cdots, n)$ on $B(\mathcal{H})$ is defined by

$$\mathbf{Q}_j T = T - U_j T U_j^* \quad (T \in B(\mathcal{H})).$$

Let $A \in B(\mathcal{H})$ and $A \geq 0$. $(\mathbf{U}, A)$ is called a *semi-hyponormal tuple* if

$$\mathbf{Q}_{j_1} \cdots \mathbf{Q}_{j_m} A \geq 0$$

for all $1 \leq j_1 < \cdots < j_m \leq n$. If $(\mathbf{U}, A)$ is a semi-hyponormal tuple, then $U_j A$ is semi-hyponormal for every j $(j = 1, \cdots, n)$. For an operator $T \in B(\mathcal{H})$, if

$$\mathcal{S}_j^{\pm}(T) = \mathrm{s} - \lim_{n \to \pm\infty} (U_j^{-n} T U_j^n)$$

exist, then the operators $\mathcal{S}_j^{\pm}(T)$ are called the polar symbols of T. If $U_j A$ is semi-hyponormal, then $\mathcal{S}_j^{\pm}(A)$ exist (cf. [**7**]). For $0 \leq k \leq 1$, we denote

$$(k\mathcal{S}_j^+ + (1-k)\mathcal{S}_j^-)T = k\mathcal{S}_j^+(T) + (1-k)\mathcal{S}_j^-(T).$$

Let $\mathbf{k} = (k_1, \cdots, k_n) \in [0, 1]^n$ and $(\mathbf{U}, A)$ be a semi-hyponormal tuple. Then the generalized polar symbols $A_{\mathbf{k}}$ of A are defined by

$$A_{\mathbf{k}} = \prod_{j=1}^{n} (k_j \mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)A.$$

https://doi.org/10.1017/S0017089599970246 Published online by Cambridge University Press

Then we have a following proposition.

PROPOSITION. *With the above notations*, $(\mathbf{U}, A_{\mathbf{k}})$ *is a commuting* $(n+1)$*-tuple for every* $\mathbf{k} \in [0, 1]^n$.

Proof. Let $\mathbf{k} = (k_1, \ldots, k_n) \in [0, 1]^n$. Since $U_i U_j = U_j U_i$, we have $\mathcal{S}_i^{\pm}(\mathcal{S}_j^{\pm}(\mathrm{A})) = \mathcal{S}_j^{\pm}(\mathcal{S}_i^{\pm}(\mathrm{A}))$ and, from the definition of $\mathcal{S}_j^{\pm}(A)$, we have $U_j\mathcal{S}_j^{\pm}(A) = \mathcal{S}_j^{\pm}(A)U_j$ for every $j = 1, \ldots, n$. Next, let $B = \prod_{i \neq j}(k_i\mathcal{S}_i^{+} + (1-k_i)\mathcal{S}_i^{-})A$. Then we have

$$\begin{aligned} U_jA_{\mathbf{k}} &= U_j(k_j\mathcal{S}_j^{+} + (1-k_j)\mathcal{S}_j^{-})\prod_{i \neq j}(k_i\mathcal{S}_i^{+} + (1-k_i)\mathcal{S}_i^{-})A \\ &= U_j(k_j\mathcal{S}_j^{+}(B) + (1-k_j)\mathcal{S}_j^{-}(B)) = (k_j\mathcal{S}_j^{+}(B) + (1-k_j)\mathcal{S}_j^{-}(B))U_j = A_{\mathbf{k}}U_j. \end{aligned}$$

Therefore, U_j commutes with $A_{\mathbf{k}}$ for every j and every $\mathbf{k} \in [0, 1]^n$.

We denote the *Taylor spectrum* of a commuting m-tuple $\mathbf{S} = (S_1, \ldots, S_m)$ of operators by $\sigma_T(\mathbf{S})$. For an m-tuple $\mathbf{S} = (S_1, \ldots, S_m)$, let $\sigma_{ja}(\mathbf{S})$ be the *joint approximate point spectrum* of $\mathbf{S}$, i.e., the set of all points $(z_1, \cdots, z_m)$ for which there exits a sequence $\{x_k\}$ of unit vectors such that

$$\lim_{k\to\infty} \| (\mathcal{S}_j - z_j)x_k \| = 0 \quad (j = 1, \ldots, m).$$

It is well known that $\sigma_T(\mathbf{S}) = \sigma_{ja}(\mathbf{S})$ if $\mathbf{S}$ is a commuting m-tuple of normal operators. Hence, from the Proposition, it holds that $\sigma_T(\mathbf{U}, A_{\mathbf{k}}) = \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}})$ for every $\mathbf{k} \in [0, 1]^n$ (cf. [**2**],[**5**]). Next, D. Xia in [**7**] defined the joint spectrum for a non-commuting commuting $(n+1)$-tuple $(\mathbf{U}, A) = (U_1, \ldots, U_n, A)$ as follows: Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ and let $E(\cdot)$ be the spectral measure of $\mathbf{U}$. For $\mathbf{z} = (z_1, \cdots, z_n) \in \sigma_T(\{\mathbf{U})$, the set of all products $\Delta = \gamma_1 \times \cdots \times \gamma_n$ of open arcs $\gamma_j \subset \mathbf{T}$, containing z_j $(j = 1, \cdots, n)$ is denoted by $\Gamma(\mathbf{z})$. The set

$$\{(\mathbf{z}, r) \ : \ \mathbf{z} \in \sigma_T(\mathbf{U}), r \in \bigcap_{\Delta \in \Gamma(\mathbf{z})} \sigma(E(\Delta)AE(\Delta))\}$$

is called the *Xia spectrum* of $(\mathbf{U}, A)$ and we denote it by $\sigma_X(\mathbf{U}, A)$. Also D. Xia proved the following result.

THEOREM A (Th. 2 of [**6**]). *Let* $(\mathbf{U}, A)$ *be a semi-hyponormal tuple. Then*

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k} \in [0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}}).$$

Let m_j $(j = 1, \cdots, n)$ be the normalized Haar measure in the unit circle $\mathbf{T}$, i.e.,

$$dm_j = \frac{1}{2\pi} d\theta_j \ \ (e^{i\theta_j} \in \mathbf{T})$$

and $m = m_1 \times \cdots \times m_n \times dr$, where dr is the Lebesgue measure on $\mathbf{R}$. Then D. Xia also proved the following

THEOREM B (Th. 5 of [**6**]). *Let* $(\mathbf{U}, A)$ *be a semi-hyponormal tuple. Then*

$$\| \mathbf{Q}_1 \ldots \mathbf{Q}_n A \| \leq m(\sigma_X(\mathbf{U}, A)).$$

We now introduce p-hyponormal tuples. Let $A \in B(\mathcal{H})$ and $A \geq 0$. $(\mathbf{U}, A)$ is called a *p-hyponormal tuple* if

$$\mathbf{Q}_{j1} \ldots \mathbf{Q}_{jm} A^{2p} \geq 0$$

for all $1 \leq j_1 < \cdots < j_m \leq n$. Evidently, if $(\mathbf{U}, A)$ is a p-hyponormal tuple, then $(\mathbf{U}, A^{2p})$ is a semi-hyponormal tuple. Let $(\mathbf{U}, A)$ be a p-hyponormal tuple and $0 \leq k \leq 1$. We denote

$$(k\mathcal{S}_j^+ + (1-k)\mathcal{S}_j^-)_p A = \{k\mathcal{S}_j^+(A^{2p}) + (1-k)\mathcal{S}_j^-(A^{2p})\}^{\frac{1}{2p}}.$$

For $\mathbf{k} = (k_1, \cdots, k_n) \in [0, 1]^n$, the generalized polar symbols $A_{(\mathbf{k})}$ of A are defined by

$$A_{(\mathbf{k})} = \prod_{j=1}^{n} (k_j\mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)_p A.$$

We remark that if $p = \frac{1}{2}$,

$$(k_j\mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)_p A = (k_j\mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)A \quad \text{and} \quad A_{(\mathbf{k})} = A_{\mathbf{k}}.$$

Since, for every i, j, it holds that

$$\begin{aligned}(k_i\mathcal{S}_i^+ + (1-k_i)\mathcal{S}_i^-)_p(k_j\mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-)_p A = \{k_i\mathcal{S}_i^+[k_j\mathcal{S}_j^+(A^{2p}) + \\ (1-k_j)\mathcal{S}_j^-(A^{2p})] + (1-k_i)\mathcal{S}_i^-[k_j\mathcal{S}_j^+(A^{2p}) + (1-k_j)\mathcal{S}_j^-(A^{2p})]\}^{\frac{1}{2p}},\end{aligned}$$

we also have that $(\mathbf{U}, A_{(\mathbf{k})}) = (U_1, \cdots, U_n, A_{(\mathbf{k})})$ is a commuting $(n+1)$-tuple for every $\mathbf{k} \in (0, 1)^n$. And the *Xia spectrum* $\sigma_X(\mathbf{U}, A)$ is defined by

$$\sigma_X(\mathbf{U}, A) = \{(\mathbf{z}, r) \ : \ \mathbf{z} \in \sigma_T(\mathbf{U}), r \in \bigcap_{\Delta \in \Gamma(\mathbf{z})} \sigma((E(\Delta)A^{2p}E(\Delta))^{\frac{1}{2p}})\}.$$

In this paper, we show the following theorems.

THEOREM 1. *Let* $(\mathbf{U}, A)$ *be a p-hyponormal tuple. Then*

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k} \in [0,1]^n} \sigma_{ja}(\mathbf{U}, A_{(\mathbf{k})}).$$

THEOREM 2. *Let* $(\mathbf{U}, A)$ *be a p-hyponormal tuple. Then*

$$\| \mathbf{Q}_1 \cdots \mathbf{Q}_n A^{2p} \| \leq \frac{2p}{(2\pi)^n} \int \cdots \int_{\sigma_X(\mathbf{U}, A)} r^{2p-1} d\theta_1 \cdots d\theta_n dr.$$

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2. Proof. The following lemma is well known, but we include a proof for completeness.

LEMMA. *Let A be a self-adjoint operator. Suppose that $r \in \mathbf{R}$ and $\{x_n\}$ is a sequence of unit vectors such that $\lim_{n\to\infty} \| (A - r)x_n \| = 0$. Then for any $f \in C(\sigma(A))$, $\lim_{n\to\infty} \| (f(A) - f(r))x_n \| = 0$, where $C(\sigma(A))$ denotes the set of all continuous functions on $\sigma(A)$.*

Proof. Since for any polynomial p, there is a polynomial q such that $p(z) - p(r) = q(z)(z - r)$, then $\lim_{n\to\infty} \| (p(A) - p(r))x_n \| = 0$. We choose a sequence $\{p_m\}$ of polynomials such that $\lim_{m\to\infty} \| p_m - f \| = 0$ in $C(\sigma(A))$. By a standard argument using $\{p_m\}$, we have $\lim_{n\to\infty} \| (f(A) - f(r))x_n \| = 0$.

Proof of Theorem 1. By the definition of $\sigma_X(\mathbf{U}, A)$, we have

$$\sigma_X(\mathbf{U}, A) = \{(\mathbf{z}, r^{\frac{1}{2p}}) \ : \ (\mathbf{z}, r) \in \sigma_X(\mathbf{U}, A^p)\}.$$

Since $(\mathbf{U}, A^{2p})$ is semi-hyponormal, Theorem A implies that

$$\sigma_X(\mathbf{U}, A^{2p}) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, (A^{2p})_{\mathbf{k}}).$$

Since, from the Lemma, we have

$$\sigma_{ja}(\mathbf{U}, A_{(\mathbf{k})}) = \{(\mathbf{z}, r^{\frac{1}{2p}}) \ : \ (\mathbf{z}, r) \in \sigma_{ja}(\mathbf{U}, (A^{2p})_{\mathbf{k}})\}$$

for every $\mathbf{k} \in [0, 1]^n$, it follows that

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{(\mathbf{k})}).$$

Hence the theorem is proved.

Proof of Theorem 2. Since $(\mathbf{U}, A^{2p})$ is semi-hyponormal, by Theorem B, we have that

$$\| \mathbf{Q}_1 \cdots \mathbf{Q}_n A^{2p} \| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma_X(\mathbf{U}, A^{2p})} d\theta_1 \cdots d\theta_n dr.$$

Also it follows that

$$\sigma_X(\mathbf{U}, A^{2p}) = \{(\mathbf{z}, r^{2p}) \ : \ (\mathbf{z}, r) \in \sigma_X(\mathbf{U}, A)\}.$$

Hence Theorem 2 is proved by the transformation of the variables.

3. Application. In this section, we study a relation of the usual spectrum $\sigma(T)$ and the Xia spectrum $\sigma_X(\mathbf{U}, |T|)$ of a p-hyponormal operator $T = U|T|$ with unitary U, where $T = U|T|$ is the polar decomposition of T.

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THEOREM 3. *Let $T = U|T|$ be p-hyponormal with unitary U. Then*

$$re^{i\theta} \in \sigma(T) \text{ if and only if } (e^{i\theta}, r) \in \sigma_X(U, |T|).$$

We need the following result.

THEOREM C (See [**1**]). *Let $T = U|T|$ be p-hyponormal with unitary U. Then*

$$\sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(T_{(k)}).$$

Proof of Theorem 3. For $q > 0$, let

$$\mathcal{S}_U^{\pm}(|T|^q) = \text{s} - \lim_{n \to \pm\infty} (U^{-n}\,|T|^q\, U^n).$$

For $0 \leq k \leq 1$, we define an operator $|T|_{(k)}$ as follows:

$$|T|_{(k)} = \{k\mathcal{S}_U^{+}(|T|^{2p}) + (1-k)\mathcal{S}_U^{-}(|T|^{2p})\}^{\frac{1}{2p}}.$$

Also let $T_{(k)} = U|T|_{(k)}$ for $0 \leq k \leq 1$. By Theorem 1 we have

$$\sigma_X(U, |T|) = \bigcup_{k \in [0,1]} \sigma_{ja}(U, |T|_{(k)}). \qquad (*)$$

From the above, $(U, |T|_{(k)})$ is a commuting pair. Since U is unitary and $|T|_{(k)} \geq 0$, it holds that $\sigma_T(U, |T|_{(k)}) = \sigma_{ja}(U, |T|_{(k)})$, where $\sigma_{\mathrm{T}}(U, |T|_{(k)})$ denotes the Taylor spectrum of $(U, |T|_{(k)})$ (cf. [**5**]). Since $U|T|_{(k)} = T_{(k)}$, it follows from the spectral mapping theorem of the Taylor spectrum that

$$(e^{i\theta}, r) \in \sigma_{ja}(U, |T|_{(k)}) \text{ if and only if } re^{i\theta} \in \sigma(T_{(k)}).$$

Hence by (*) and Theorem C, the proof is complete.

REFERENCES

1. M. Chō and M. Itoh, On spectra of p-hyponormal operators, *Integral Equations and Operator Theory* **23** (1995), 287–293.

2. R. Curto, On the connectedness of invertible n-tuples, *Indiana Univ. Math. J.* **29** (1980), 393–406.

3. R. Curto, P. Muhly and D. Xia, A trace estimate for p-hyponormal operators, *Integral Equations and Operator Theory* **6** (1983), 507–514.

4. C. R. Putnam, *Commutation properties of Hilbert space operators* (Springer-Verlag, 1967).

5. J. L. Taylor, A joint spectrum for several commuting operators, *J. Funct. Anal.* **6** (1970), 172–191.

6. D. Xia, On the semi-hyponormal n-tuple of operators, *Integral Equations and Operator Theory* **6** (1983), 879–898.

7. D. Xia, *Spectral theory of hyponormal operators* (Birkhäuser, 1983).

https://doi.org/10.1017/S0017089599970246 Published online by Cambridge University Press