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ON A PROBLEM OF RANDOM WALK IN SPACE(1)

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Introduction. The following theorem is well known [1, p. 66]: Suppose that, in a ballot, candidate P scores p votes and candidate Q scores q votes, where $p>q$. The probability that throughout the counting there are always more votes for P than for Q, equals $(p-q)/(p+q)$.

The process of counting the votes is called a path. If the above condition is fulfilled during the processes we say that candidate P "leads". The proof of this so-called "Ballot theorem" is based on a Reflection principle, a method credited in the probability literature to Andre (1887).

A result of the above theorem is the following: Among the $\binom{2n}{n}$ different random walks describing the vote-counting process in a ballot in which there are $2n$ votes for two candidates, with each candidate scoring n votes at the end of the counting—the number of the walks ("paths") in which A leads throughout the process is ([1, p. 71]):

$$L_{2n} = \frac{1}{n+1}\binom{2n}{n}.$$

We will discuss a natural generalization of these theorems from two dimensions to three and more, i.e. the case that throughout the counting process candidate A leads the rest, candidate B leads relative to candidate C, and so on.

We did not succeed in applying the Reflection principle to this case, and eventually used a result of Young [2](2).

The idea of generalizing the classical Ballot theorem, an idea important in itself, arose in the context of a traffic-light problem. Yadin used the two-dimensional Ballot theorem for solving such a problem [4], and our own version was constructed in the course of generalizing his results to space.

1. A Ballot theorem in space.

THEOREM. *Suppose that in a ballot, candidate A_1 scores α_1 votes, A_2 scores α_2 votes and, in general, candidate A_k scores α_k votes $(k=1, 2, \ldots, h)$, where* $\alpha_1 \geq$

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(1) The idea of this study was suggested by Prof. H. Hanani, at the Faculty of Mathematics, Technion, Israel Institute of Technology, Haifa, Israel.

(2) A direct result of [2] helped Schwarz in counting the number of matrices composed of n^2 pairwise different elements arranged in decreasing order in the rows and columns [3]. This was needed for finding extremal eigenvalues of the matrices composed of those elements.

$\alpha_2 \geq \cdots \geq \alpha_h$. *The total of* $(\sum_{i=1}^{h} \alpha_i)!/\prod_{i=1}^{h} \alpha_i!$ *different random walks describing the vote-counting process in a ballot comprising* $\sum_{i=1}^{h} \alpha_i$ *votes, contains*

$$(1) \qquad \left(\sum_{i=1}^{h} \alpha_i\right)! \frac{\prod_{r,s} (\alpha_r - \alpha_s - r + s)}{\prod_r (\alpha_r + h - r)!}$$

paths in which A_1 *leads the rest,* A_2 *leads relative to* A_3, *and so on.*

Proof. To prove this theorem, we start with the following: In [2, p. 260] it was proved that (1) is the number of standard tables belonging to $T\alpha_1, \alpha_2, \ldots, \alpha_h$. In other words, if we construct from $\sum_{i=1}^{h} \alpha_i$ given pairwise different numbers, tables of the form:

$$(2) \qquad \begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,\alpha_1} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,\alpha_2} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,\alpha_3} \\ \vdots & & & & \\ a_{h,1} & a_{h,2} & a_{h,3} & \cdots & a_{h,\alpha_h} \end{matrix}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_h$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_h$, are given natural numbers, then by standard tables we mean those in which the elements are arranged in decreasing order in the rows and columns. Noting this remarkable fact, our proof is within reach. Instead of counting the vote processes in analogy to the Reflection principle (which seems too complicated in this case), we assert a one-to-one correlation between them and the standard tables. Let us arrange all $a_{i,j}$, $i \in \{1, 2, \ldots, h\}, j \in \{1, 2, \ldots, \alpha_i\}$ in a row in decreasing order:

$$\{\omega_1, \omega_2, \ldots, \omega_l\}, \qquad l = \sum_{i=1}^{h} \alpha_i.$$

Now $\omega_1 = a_{1,1}$ denotes a vote for A_1, and in every table of type (2) $a_{1,1}$ has to be the same, namely ω_1. The place of ω_2 signifies the second vote in the counting process. By the condition of leading, it may be a vote for candidate A_2 or again for A_1; accordingly, we put ω_2 as $a_{1,2}$ or $a_{2,1}$ respectively and proceed to the next element. Clearly, every counting process obeying our rule of leading yields a table of type (2) and vice versa, and our proof is completed.

2. **A geometrical interpretation.** The classical counting process is usually described along a straight line, such that a vote for candidate A is represented by a unit displacement to the right from the origin, and a vote for candidate B by the same unit displacement to the left. This Random Walk is described in literature [1] in the plane as well, a vote for A being a unit displacement at 45° to the X-axis, and a vote for B by one at $-45°$ to it. In this way, a position right or left of the origin in the first description, and above or below the X-axis in the second, determines the leading candidate.

The three-dimensional case obeying the above conditions may also be described in the plane (Fig. 1).

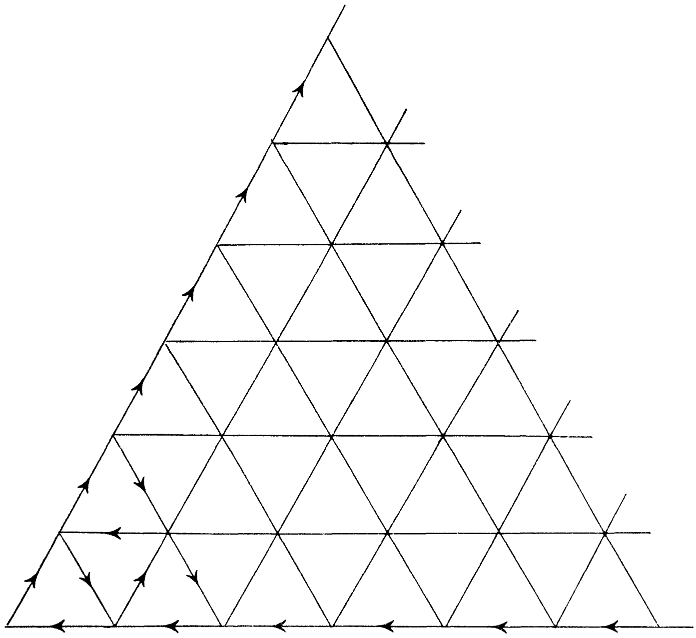

Figure 1.

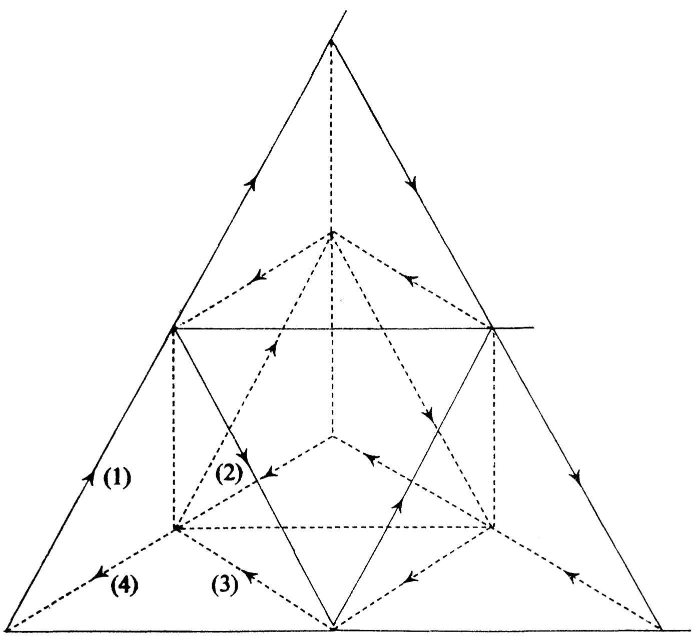

Figure 2.

There are three alternatives of movement over the polygon, the first representing a vote for candidate A, the second–one for B and the third–one for C. The condition of A leading the rest and of B leading relative to C, is automatically fulfilled by the geometry of the scheme. A similar description is possible for a four-dimensional case, using a tetrahedron with four main directions only, as shown in Fig. 2. (Movement along unmarked edges and along their counterparts on the extension of the tetrahedron is ruled out.)

3. **Some direct combinatorial results.** If we set, in (1), $\alpha_1=\alpha_2=\cdots=\alpha_h=m$, $h=n$, we get the following number of counting processes:

$$(3)\qquad (mn)!\frac{1^{n-1}\cdot 2^{n-2}\dots(n-1)}{(m+n-1)!(m+n-2)!\dots m!}=\frac{(mn)!\,1!\,2!\dots(n-1)!}{m!(m+1)!\dots(m+n-1)!}$$

and if we set $m=2$, we get:

$$(4)\qquad (2n)!\frac{1}{n!(n+1)}=\frac{1}{n+1}\binom{2n}{n}$$

which is the case of the classical Ballot theorem [1, p. 71].

In the case of square matrices, i.e. (in ballot terminology) the case where not only does every candidate score the same number of votes in the end, but this number equals the number of candidates, equation (3) becomes:

$$(5)\qquad \frac{(n^2)!\,1!\,2!\dots(n-1)!}{n!(n+1)!\dots(2n-1)!}$$

so that the number of matrices composed of n^2 elements, in which the extremal eigenvalues can appear, is clearly less than $(n^2)!$, the initial number of matrices, as mentioned in [3].

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