A PROPERTY OF PLANE SETS OF CONSTANT WIDTH

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1. It is well known that sets of constant width share several properties with spheres. In this note we consider a simple property of the circle and we show that it is shared by every plane set of constant width. As an application we derive a stronger form of the following theorem of D. Gale, [1]: every plane set of diameter 1 is a union of three sets of diameters not exceeding $\sqrt{3}/2$, and this constant is best possible. We shall make free use of the more elementary properties of convex sets and of sets of constant width; for these properties and for the terminology see the standard reference [2], or [3].

2. The class of all plane closed convex sets of constant width will be denoted by $\mathcal{K}$. Greek letters will denote scalars and small Latin letters o, u, v, ... will denote points in the plane. If K is a set then $\mathcal{B}(K)$ and $\mathcal{D}(K)$ are its boundary and its diameter, respectively. The closed circular disk of radius ρ about the centre u will be denoted by $D_\rho(u)$ and its boundary by $C_\rho(u)$. If x and y are two points then xy is the straight segment from x to y and $|xy|$ is its length. Let C be a closed convex curve, and let x and y be two points on C dividing C into two arcs of unequal length; then $C(\widehat{x, y})$ will denote the shorter arc.

3. Let $D = D_{1/2}(o)$, let $x, y, z \in \mathcal{B}(D)$, and suppose that $|xy| = |xz| = \alpha$. Let $C = C_\alpha(x)$; then clearly $C(\widehat{y, z}) \subset D$. We show that this property is shared by every set $K \in \mathcal{K}$.

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THEOREM 1. Let $K \in \mathcal{K}$, $B = \mathcal{B}(K)$, and $x, y, z, \in B$. Suppose that $|xy| = |xz| = \alpha$ and let $C = C_\alpha(x)$; then $C(\widehat{y, z}) \subset K$.

Without loss of generality we may assume that $\alpha < 1$, as otherwise $C(\widehat{y, z}) \subset B$. Let u be the point antipodal to x; if there are several such points let u be any one of them. We show first that $u \in B_1$, where B_1 is that one of the two sub-arcs of B with the end-points y and z which does not contain x. Suppose that $u \notin B_1$ and that the points x, y, z, u are in cyclic order on B. Since xu is a diameter of K, it follows that the angle at x subtended by ux and yx is less than $\pi/2$. Let $C_1 = C_1(w)$ be the circle containing x in its interior and passing through y and u; since $K \in \mathcal{K}$ it is known that $E = C_1(\widehat{y, u}) \subset K$. We observe that as the point t traverses E from y to u the length $|tx|$ increases steadily from α to 1. Hence $|xz| > \alpha$ which is a contradiction.

We have now $u \in B_1$. Let E_1 and E_2 be the arcs defined in the same way as E, with the end-points y and u and u and z, respectively. Then $E_1 \subset K$ and $E_2 \subset K$, so that K contains the closed convex set U bounded by E_1, E_2, xy and xz. It is now a simple matter to verify that $C(\widehat{y, z}) \subset U$. Hence $C(\widehat{y, z}) \subset K$.

4. Let V be a subset of the Euclidean space E^n and let $\mathcal{D}(V) = 1$. Define

$$G_n(V) = \inf \{\alpha : V = \bigcup_{j=1}^{n+1} V_j, \ \mathcal{D}(V_j) \leq \alpha, \ j = 1, \ldots, n+1\} .$$

It has been conjectured by K. Borsuk [4] that $G_n(V) < 1$ for every V. Since every set V, $\mathcal{D}(V) = 1$, is a subset of a set of constant width 1, it suffices to consider the latter sets only. Borsuk's conjecture has been proved so far only for $n = 2$ and $n = 3$, [1], [5]. For $n = 2$ Gale [1] has proved a stronger theorem which may be stated as follows: let $K \in \mathcal{K}$, then $G_2(K) \leq \sqrt{3}/2$; since $D_{1/2}(o)$ cannot be represented as a

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union of three sets of diameters less than $\sqrt{3}/2$, the constant is best possible.

Let $K \in \mathcal{K}$. By a simple continuity argument it is easy to show that there exist equilateral triangles with all vertices on $\mathcal{B}(K)$. Let $X(K)$ denote the side-length of the largest one of all such triangles. Then

THEOREM 2. Let $K \in \mathcal{K}$, then $G_2(K) \leq \min\{ X(K), \sqrt{3} - X(K)\}$.

5. Gale [1] and Gruenbaum [5] use in their proofs of Borsuk's conjecture for $n = 2$ and $n = 3$ the method of universal sets. A set U is called universal in E^n if every set in E^n of constant width 1 is a subset of U. In [1] $n = 2$ and U is an equilateral triangle of side-length $\sqrt{3}$; in [5] $n = 3$ and U is a regular octahedron in which the distance between every pair of opposite walls is 1. In proving Theorem 2 we shall also use the method of universal sets, but instead of considering a single such set for the whole of $\mathcal{K}$ we shall introduce a one-parameter family of such sets. More precisely, every plane set of constant width one will be a subset of at least one set of the family.

Let $K \in \mathcal{K}$, let $X(K) = a$ and let x_1, x_2, x_3 be the vertices of the equilateral triangle $T(a)$ of side-length a, inscribed into K. The set

$$C(a) = \bigcap_{j=1}^{3} D_1(x_j)$$

will be called a caltrop. It follows from the standard properties of sets of constant width that the class of all caltrops $C(a)$, $0 < a \leq 1$, is universal for the class $\mathcal{K}$ in the previously described sense. $\mathcal{B}(C(a))$ consists of three circular arcs; let their mid-points be w_1, w_2, w_3. Let o be the centre of the triangle $T(a)$; the segments ow_1, ow_2, ow_3 divide the caltrop into three congruent sets Q_1, Q_2, Q_3. By an elementary calculation

$$\mathcal{D}(Q_1) = \mathcal{D}(Q_2) = \mathcal{D}(Q_3) = |w_1 w_2| = \sqrt{3} - a.$$

Since $K = \bigcup_{j=1}^{3} (K \cap Q_j)$, we have

LEMMA 1. Let $K \in \mathcal{K}$, then $G_2(K) \leq \sqrt{3} - X(K)$.

We next prove

LEMMA 2. If $K \in \mathcal{K}$ and $T(b) \subset K$ with at least one vertex of $T(b)$ in the interior of K, then $X(K) > b$.

This is proved by a simple continuity argument. We firs move $T(b)$ so that, remaining in K, it has two vertices, say x_1 and x_2, on $\mathcal{B}(K)$. Then x_2 is moved in $\mathcal{B}(K)$ away fro x_1, while x_1 itself is fixed; eventually the third vertex x_3 will cross $\mathcal{B}(K)$.

LEMMA 3. Let $K \in \mathcal{K}$ and $X(K) = a$, let $T(a)$ be an equilateral triangle of side-length a inscribed into K, and let o be its centre. Then $\max_{x \in \mathcal{B}(K)} |ox| < a$.

For the radius $r(K)$ of the inscribed circle C of K w have the estimates

(1) $$1 - 3^{-1/2} \leq r(K) \leq 1/2 .$$

Let the vertices of $T(a)$ be x_1, x_2, x_3, and let $y_i = C \cap ox_i$ $i = 1, 2, 3$. Then

$$|ox_1| \geq \max_{i=1,2,3} |oy_i| \geq r(K) ,$$

and since $a = \sqrt{3}\, |ox_i|$, we get from (1)

(2) $$a \geq \sqrt{3} - 1 .$$

Let $C_1 = C_1(w)$ pass through x_2 and x_3, and let x_1 be inside C_1. Put $E = C_1(\widehat{x_2, x_3})$, so that $E \subset K$. A simple calculation yields

$$\min_{v \in E} |ov| = 1 - [(4 - a^2)^{1/2} - 3^{-1/2} a]/2 .$$

Hence

$$(3) \qquad \min_{x \in \mathcal{B}(K)} |ox| \geq 1 - [(4 - a^2)^{1/2} - 3^{-1/2} a]/2 = f(a)$$

say. Since $1 - f(a)$ is monotone decreasing, it follows from (2) and (3) that

$$\min_{x \in \mathcal{B}(K)} |ox| \geq f(\sqrt{3} - 1)$$

and so

$$\max_{x \in \mathcal{B}(K)} |ox| \leq 1 - f(\sqrt{3} - 1) < \sqrt{3} - 1 .$$

This together with (2) proves the lemma.

Let $B = \mathcal{B}(K)$, $X(K) = a$, and let x_1, x_2, x_3 be, as before, the vertices of the triangle $T(a)$ inscribed into K.

LEMMA 4. $\displaystyle \max_{x, y \in B(\widehat{x_1, x_2})} |xy| = a$.

Let this maximum occur for $x = u$ and $y = v$. Suppose first that $u \neq x_1$ and $v \neq x_2$. Then through u and v there pass two parallel supporting lines to $B(\widehat{x_1, x_2})$, orthogonal to uv, and containing the arc $B(\widehat{x_1, x_2})$ between them. Since neither x_1 nor x_2 lie on these supporting lines, it is clear that a suitable translation will carry the triangle $T(a)$ into the interior of K. By Lemma 2 this contradicts the maximality of $T(a)$.

Suppose next that $u = x_1$ but $v \neq x_2$. Let $C = C_\alpha(x_1)$; then by Theorem 1 $C(\widehat{x_2, x_3}) \subset K$. Since, by the hypothesis, $|x_1 v| > a$, it follows that by rotating $T(a)$ about x_1 until

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x_2 lies on $x_1 v$ we get an equilateral triangle of side-length a, with a vertex inside K and the other two vertices in K. This again contradicts the maximality assumption $X(K) = a$.

Hence $u = x_1$ and $v = x_2$ and the lemma follows.

LEMMA 5. $G_2(K) \leq X(K)$.

Let $X(K) = a$, let o be the centre of $T(a)$ and x_1, x_2, x_3 its vertices. The segments ox_1, ox_2, ox_3 divide K into three closed convex sets R_1, R_2, R_3, with R_i being disjoint from x_i. Let also $B = \mathcal{B}(K)$ and $B_1 = B(\widehat{x_2, x_3})$, $B_2 = B(\widehat{x_1, x_3})$, $B_3 = B(\widehat{x_1, x_2})$. Then $\mathcal{D}(R_1) = \max\{\max_{x \in B_1} |ox|, \max_{x,y \in B_1} |xy|\}$.

Therefore by Lemmas 3 and 4 we have

$$\mathcal{D}(R_1) = \mathcal{D}(R_2) = \mathcal{D}(R_3) = a$$

and the lemma is proved.

Theorem 2 is now an immediate consequence of Lemmas 1 and 5.

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REFERENCES

1. D. Gale, Proc. A.M.S., 4(1953), 222-225.

2. T. Bonnesen and W. Fenchel, Theorie der konvexen Koerper, Leipzig, (1934).

3. H. Eggleston, Convexity, No. 47, Cambridge Tracts, (1958).

4. K. Borsuk, Fund. Math., 20(1933), 177-190.

5. B. Gruenbaum, Proc. Cambr. Phil. Soc., 53(1957), 776-778.

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https://doi.org/10.4153/CMB-1963-036-6 Published online by Cambridge University Press