The Conservation Theorems of a Damped Dynamical System.

By E. T. Copson.

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§ 1. The Partial Differential Equations of Physics may be defined as those equations which can be derived from a "least action principle," that is, as those which are obtained by making a certain integral stationary by the methods of the Calculus of Variations. But, generally speaking, such equations belong to conservative physical systems, and not to those which involve dissipation of energy. In this note it is shewn that a certain class of dissipative equation, of which the best known example is the equation of telegraphy, can be derived from such a calculus of variations problem.

When once we know the integral which is stationary in the physical problem considered, it is possible to find certain "divergence relations" and thence "conservation theorems" by means of Lie's theory of continuous groups of transformations.* This method is here applied to find conservation theorems in such dissipative problems as telegraphy, viscous fluid motion and the damped vibrations of a string.

§ 2. Let u denote some function of the two independent variables x_1 and x_2. Further denote $\dfrac{\partial u}{\partial x_1}$ by u_1, $\dfrac{\partial^2 u}{\partial x_1 \partial x_2}$ by u_{12}, and so on. Consider the integral I, where

$$I = \iint (u_1^2 - \alpha u_2^2 + \beta u^2) e^{\gamma x_2} dx_1 \, dx_2$$

where α, β, γ are constants. If we make I stationary by the methods of the calculus of variations, we obtain the partial differential equation

$$u_{11} = \alpha u_{22} + \gamma \alpha u_2 + \beta u.$$

* See *Emmy Noether:* Gött. Nach. (1918), p. 238.

This equation is of the type which occurs in the discussion of the damped vibrations of a string and in the theory of telegraphy. We have here found a "least-action principle" for this type of damped vibration. The integral I has no obvious physical interpretation, but we will now shew how certain "conservation theorems" may be derived from it.

§ 3 We will now find the infinitesimal transformation of that continuous group of transformations which leave unaltered the expression

$$(u_1^2 - \alpha u_2^2 + \beta u^2)\, e^{\gamma x_2} dx_1\, dx_2$$

which occurs in I. The appropriate form for this is*

$$X_1 = x_1 + t\xi$$
$$X_2 = x_2 + t\eta$$
$$U = u + tu\zeta$$

where ξ, η, ζ are functions of x_1 and x_2 to be determined and t is a constant which is so small that its square is neglected.

We find that the following equations have to be satisfied :—

$$\begin{aligned} -\xi_1 + \eta_2 + \gamma\eta + 2\zeta &= 0 \\ -\xi_1 + \eta_2 - \gamma\eta - 2\zeta &= 0 \\ \xi_1 + \eta_2 + \gamma\eta + 2\zeta &= 0 \\ -\eta_1 + \alpha\xi_2 &= 0 \\ \zeta_1 &= 0 \\ \zeta_2 &= 0. \end{aligned}$$

We hence deduce that the required infinitesimal transformation is

$$\Delta x_1 = \epsilon_1\,;\ \Delta x_2 = \epsilon_2\,;\ \Delta u = -\tfrac{1}{2}\gamma\epsilon_2 u$$

where Δx_1 denotes $X_1 - x_1$, and so on. Here ϵ_1 and ϵ_2 are two arbitrary constants which are so small that we neglect their squares.

§ 4. Suppose that an integral

$$I = \iint \dots \int f(x_1,\, x_2,\, \dots\, x_n,\, u,\, u_1,\, u_2,\, \dots\, u_n)\, dx_1 \dots dx_n$$

* See *Lie: Leipz ger Berichte* (1894-95), p. 322.

admits the continuous group of transformations whose infinitesimal transformation is

$$X_r = x_r + \Delta x_r \qquad (r = 1, 2, \dots n)$$
$$U = u + \Delta u$$

u being a function of the n independent variables $x_1, x_2, \dots x_n$, and $\frac{\partial u}{\partial x_r}$ being denoted by u_r.

Let $\bar{\delta} u = \Delta u - \sum_r u_r \Delta x_r$, and $\psi = \frac{\partial f}{\partial u} - \sum_r \frac{\partial}{\partial x_r}\left(\frac{\partial f}{\partial u_r}\right)$.

Then $\psi \bar{\delta} u = \sum_r \frac{\partial B_r}{\partial x_r}$, if $B_r = -f\Delta x_r - \frac{\partial f}{\partial u_r}\bar{\delta} u$, where $r = 1, 2, \dots n$.

But $\psi = 0$ is the partial differential equation which arises from annulling the variation of the integral I. Hence the result $\sum_r \frac{\partial B_r}{\partial x_r} = 0$ if u is a solution of $\psi = 0$, will give us certain conservation theorems; whose number is the number of parameters of the group.

This theorem is proved in the paper by Emmy Noether, already quoted. We will now apply the method to the particular equation we are considering.

§ 5. Now $f = (u_1^2 - \alpha u_2^2 + \beta u^2)\, e^{\gamma x_2}$

$$\Delta x_1 = \epsilon_1$$
$$\Delta x_2 = \epsilon_2$$
$$\Delta u = -\tfrac{1}{2}\gamma \epsilon_2 u$$
$$\therefore \quad \bar{\delta} u = -\epsilon_1 u_1 - \epsilon_2 (u_2 + \tfrac{1}{2}\gamma u)$$

and $$\psi = -2e^{\gamma x_2}(u_{11} - \alpha u_{22} - \gamma\alpha u_2 - \beta u).$$

Consequently we have that

$$B_1 = \epsilon_1 (u_1^2 + \alpha u_2^2 - \beta u^2)\, e^{\gamma x_2} + \epsilon_2\, 2u_1 (u_2 + \tfrac{1}{2}\gamma u)\, e^{\gamma x_2}$$
$$B_2 = -\epsilon_1\, 2\alpha u_1 u_2 e^{\gamma x_2} - \epsilon_2 (u_1^2 + \alpha u_2^2 + \beta u^2 + \gamma\alpha u u_2)\, e^{\gamma x_2}.$$

Then if u is a solution of the equation $u_{11} = \alpha u_{22} + \gamma\alpha u_2 + \beta u$, we have the relation $\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} = 0$. As ϵ_1 and ϵ_2 are independent, this is equivalent to the two relations

$$\frac{\partial}{\partial x_1}\left\{e^{\gamma x_2}u_1(2u_2+\gamma u)\right\}-\frac{\partial}{\partial x_2}\left\{e^{\gamma x_2}(u_1^2+\alpha u_2^2+\beta u^2+\gamma\alpha u u_2)\right\}=0$$

$$\frac{\partial}{\partial x_1}\left\{e^{\gamma x_2}(u_1^2+\alpha u_2^2-\beta u^2)\right\}-\frac{\partial}{\partial x_2}\left\{2\alpha e^{\gamma x_2}u_1\,u_2\right\}=0.$$

Hence we have

$$\frac{\partial}{\partial x_1}\{2u_1\,u_2+\gamma u u_1\}-\left(\gamma+\frac{\partial}{\partial x_2}\right)\{u_1^2+\alpha u_2^2+\beta u^2+\gamma\alpha u u_2)\}=0\quad\ldots 5{\cdot}1$$

$$\frac{\partial}{\partial x_2}\{u_1^2+\alpha u_2^2-\beta u^2\}-\left(\gamma+\frac{\partial}{\partial x_2}\right)\{2\alpha u_1\,u_2\}=0.\quad\ldots\ldots\ldots\ldots 5{\cdot}2$$

Integrate the first of these with respect to x_1 between the limits l and l'. We have then

$$\Big[2u_1\,u_2+\gamma u u_1\Big]_l^{l'}=\left(\gamma+\frac{\partial}{\partial x_2}\right)\int_l^{l'}\{u_1^2+\alpha u_2^2+\beta u^2+\gamma\alpha u u_2\}\,dx_1\quad\ldots\ldots 5{\cdot}3$$

Both the identities, of course, reduce to the differential equation. We take then instead of the second identity the equation

$$u_{11}=\alpha u_{22}+\gamma\alpha u_2+\beta u$$

to which it reduces.

On integrating with respect to x_1 from l to l', we have

$$\Big[\;u_1\;\Big]_l^{l'}=\frac{d}{dx_2}\int_l^{l'}\alpha\,u_2dx_1+\int_l^{l'}\gamma\alpha\,u_2dx_1+\int^{l'}\beta u dx_1.\quad\ldots\ldots 5{\cdot}4$$

We shall see that this method of using the second identity possesses a real physical interpretation.

§ 6. Let us consider the damped transverse vibrations of a stretched string of density ρ at a tension of ρc^2. Let the damping force be $\rho\gamma x$ (velocity) per unit length. Further suppose that the string is attracted to its equilibrium position by a force $\rho k^2 c^2 x$ (displacement) per unit length of string. The transverse displacement satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}+\frac{\gamma}{c^2}\frac{\partial u}{\partial t}+k^2 u\quad\ldots\ldots\ldots\ldots 6\,1$$

We can identify this with the equation $\psi=0$ of § 5, by taking $\alpha=\frac{1}{c^2}$, $x_1=x$, $x_2=t$, $\beta=k^2$. The conservation theorems now

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become

$$\left[\gamma u\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right]_l^{l'}=\left(\gamma+\frac{d}{dt}\right)\int_l^{l'}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2\right.$$

$$\left.+k^2u^2+\frac{\gamma}{c^2}u\frac{\partial u}{\partial t}\right\}dx \quad \ldots\ldots\ldots 6{\cdot}2$$

and

$$\left[\frac{\partial u}{\partial x}\right]_l^{l'}=\frac{d}{dt}\int_l^{l'}\frac{1}{c^2}\frac{\partial u}{\partial t}\,dx+\int_l^{l'}\frac{\gamma}{c^2}\frac{\partial u}{\partial t}\,dx+\int_l^{l'}k^2u\,dx \quad \ldots\ldots\ldots 6{\cdot}3$$

On multiplying through equation 6·3 by ρc^2, we see that it is merely the expression of the fact that the rate of change of momentum of the string is equal to the impressed forces acting on it. Equation 6·2 is simply the energy equation of this damped motion. To study the equation more easily, we will suppose that the ends of the string are fixed at $x=l$ and $x=l'$. We have then, that $\dfrac{dE}{dt}+\gamma E=0$ if we denote by E the expression

$$\int_l^{l'}\tfrac{1}{2}\rho\left\{c^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2+k^2c^2u^2+\gamma u\frac{\partial u}{\delta t}\right\}dx,$$

and $\therefore\ E=E_0e^{-\gamma t}$.

This quantity E is not the total energy of the system. For a force $\rho\gamma\dfrac{\partial u}{\partial t}$ per unit length cannot be derived from a potential. The other terms occurring are however the energies, kinetic and potential, of the system excluding the damping force. We can find an interpretation of the equation by introducing Rayleigh's Dissipation Function.*

Denote by H the true energy of the string, *i.e.*

$$\int_l^{l'}\tfrac{1}{2}\rho c^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2+k^2u^2\right\}dx. \quad \text{Then}$$

$$-\frac{dH}{dt}=\gamma E+\frac{d}{dt}\int_l^{l'}\tfrac{1}{2}\gamma\rho u\frac{\partial u}{\partial t}\,dx$$

$$=\gamma E+\int_l^{l'}\tfrac{1}{2}\rho\gamma\left(\frac{\partial u}{\partial t}\right)^2dx+\int_l^{l'}\tfrac{1}{2}\rho\gamma c^2\left\{\frac{\partial^2u}{\partial x^2}-\frac{\gamma}{c^2}\frac{\partial u}{\partial t}-k^2u\right\}u\,dx$$

$$=\int_l^{l'}\rho\gamma\left(\frac{\partial u}{\partial t}\right)^2dx, \text{ on integrating by parts.}$$

* *Proc. Lond. Math. Soc.* (1) 4 (1873).

Hence the energy of the system is being dissipated at the rate $\int_{l}^{l'} \rho\gamma\left(\frac{\partial u}{\partial t}\right)^2 dx$. The quantity $\int_{l}^{l'} \frac{1}{2}\rho\gamma\left(\frac{\partial u}{\partial t}\right)^2 dx$ was called by Rayleigh the Dissipation Function, denoted usually by F. Then $\frac{dH}{dt} = -2F$, a well known property of the Dissipation Function.

§ 7. The propagation of signals in a telegraph cable will next be considered. Suppose that the cable has capacity C, self-inductance L, resistance R, and leakage A, per unit length, where C, L, R and A are constants. Let V be the potential and j the current at time t at a distance x from some fixed point in the cable. Then it can be shewn* that V and j are connected by the relations

$$V = \frac{\partial\theta}{\partial x} \qquad \text{.......................................} 7{\cdot}1$$

$$j = -A\theta - C\frac{\partial\theta}{\partial t} \qquad \text{.......................................} 7{\cdot}2$$

where θ is a solution of the partial differential equation

$$\frac{\partial^2\theta}{\partial x^2} = LC\frac{\partial^2\theta}{\partial t^2} + (LA+RC)\frac{\partial\theta}{\partial t} + RA\theta \qquad \text{.......................................} 7{\cdot}3$$

The equation 5·4 gives us that

$$\left[\frac{\partial\theta}{\partial x}\right]_{l}^{l'} = \frac{d}{dt}\int_{l}^{l'} LC\frac{\partial\theta}{\partial t}\,dx + \int_{l}^{l'} (LA+RC)\frac{\partial\theta}{\partial t}\,dx + \int_{l}^{l'} RA\theta\,dx$$

or, using equations 7·1 and 7·2, that

$$\left[V\right]_{l}^{l'} = -\frac{d}{dt}\int_{l}^{l'} Lj\,dx - \int_{l}^{l'} Rj\,dx \qquad \text{.......................................} 7{\cdot}4$$

This is merely the appropriate form of Neumann's Law of Electromagnetic Induction.

From the equation 5·3, we have that

$$\left[\frac{\partial\theta}{\partial x}\left\{(LA+RC)\theta + 2LC\frac{\partial\theta}{\partial t}\right\}\right]_{l}^{l'} = \left\{(LA+RC) + LC\frac{d}{dt}\right\} \int_{l}^{l'}\left\{\left(\frac{\partial\theta}{\partial x}\right)^2 + LC\left(\frac{\partial\theta}{\partial t}\right)^2 + RA\theta^2 + (LA+RC)\,\theta\frac{\partial\theta}{\partial t}\right\}dx.$$

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* See *Rayleigh's Sound* I., p. 467.

After some transformations by means of the equations 7·1 and 7·2, this becomes

$$\frac{d}{dt}\int_l^{l'} \tfrac{1}{2}(Lj^2 + CV^2)\,dx + \int_l^{l'} AV^2\,dx + \int_l^{l'} Rj^2\,dx = \Big[\,Vj\,\Big]_{l'}^{l} \quad \ldots\ldots 7{\cdot}5$$

The equation 7·5 completely accounts for the energy of the system. The term on the right hand side is the rate at which energy is put into the cable by the external source of current. On the left hand side, the term $\frac{d}{dt}\int_l^{l'} \frac{1}{2}(Lj^2 + CV^2)\,dx$ is the rate at which the total energy resident in the cable increases. Then $\int_l^{l'} Rj^2\,dx$ is the rate at which electrical energy is turned into heat, because of the resistance of the wire. Lastly the term $\int_l^{l'} AV^2\,dx$ is the rate at which energy is being lost because of the leakage due to faulty insulation. The equation shows how the energy put into the cable is used up; it is the energy equation.

§ 8. We have already shown that, if θ is a solution of the equation $\frac{\partial^2\theta}{dx^2} = \alpha\frac{\partial^2\theta}{\partial t^2} + f\frac{\partial\theta}{\partial t} + k^2\theta$, there are two conservation theorems, viz.,

$$\left[f\theta\frac{\partial\theta}{\partial x} + 2\alpha\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}\right]_l^{l'} = \left(f + \alpha\frac{d}{dt}\right)\int_l^{l'}\left\{\left(\frac{\partial\theta}{\partial x}\right)^2 + \alpha\left(\frac{\partial\theta}{\partial t}\right)^2 + k^2\theta^2 + f\theta\frac{\partial\theta}{\partial t}\right\}dx$$

and
$$\left[\frac{\partial\theta}{\partial x}\right]_l^{l'} = \alpha\frac{d}{dt}\int_l^{l'}\frac{\partial\theta}{\partial t}dx + \int_l^{l'} f\frac{\partial\theta}{\partial t}dx + \int_l^{l'} k^2\theta\,dx.$$

Now put $\alpha = 0$. Then if θ is a solution of the equation

$$\frac{\partial^2\theta}{\partial x^2} = f\frac{\partial\theta}{\partial t} + k^2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 8{\cdot}1$$

then
$$\left[\theta\frac{\partial\theta}{\partial x}\right]_l^{l'} = \int_l^{l'}\left\{\left(\frac{\partial\theta}{\partial x}\right)^2 + k^2\theta^2 + f\theta\frac{\partial\theta}{\partial t}\right\}dx \quad \ldots\ldots\ldots\ldots\ldots 8{\cdot}2$$

and
$$\left[\frac{\partial\theta}{\partial x}\right]_l^{l'} = \int_l^{l'} f\frac{\partial\theta}{\partial t}\,dx + \int_l^{l'} k^2\theta\,dx. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 8{\cdot}3$$

Although the equation 8·1 cannot be derived from a calculus of variations problem, we have found two conservation theorems by this somewhat indirect method.

§ 9. The laminar motion of a viscous incompressible fluid under no external forces satisfies an equation of the type considered in § 8.* Suppose that the fluid moves parallel to the axis of y with a velocity v which is a function of x and t only. All the stress components vanish except $p_{xy} = \rho\nu\dfrac{\partial v}{\partial x}$, where ρ is the density of the fluid, and ν is Maxwell's kinematic coefficient of viscosity. Then v satisfies the partial differential equation $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{1}{\nu}\dfrac{\partial v}{\partial t}$.

The conservation theorem 8·2 gives us then that

$$\left[v\frac{\partial v}{\partial x}\right]_l^{l'} = \int_l^{l'} \left\{\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{\nu} v \frac{\partial v}{\partial t}\right\} dx.$$

Consequently we have

$$\left[v p_{xy}\right]_l^{l'} = \int_l^{l'} \nu\rho \left(\frac{\partial v}{\partial x}\right)^2 dx + \frac{d}{dt}\int_l^{l'} \tfrac{1}{2}\rho v^2\, dx.$$

The left hand side is the rate at which energy is added to the fluid between the planes $x = l$ and $x = l'$. The second term on the right hand side is the rate at which the kinetic energy of the fluid increases, whilst the first term on that side is the rate at which the energy is dissipated in the form of heat. The first term on the right hand side is merely the form of Rayleigh's Dissipation Function appropriate to this problem.† The conservation theorem is the energy equation.

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* See Lamb's *Hydrodynamics* (4th Edn.), p. 609.

† See Lamb, *loc. cit.*, p. 575.

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