THE HOMOMORPHIC MAPPING OF CERTAIN MATRIC ALGEBRAS ONTO RINGS OF DIAGONAL MATRICES

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1. Introduction. The problem of determining the conditions under which a finite set of matrices $A_1, A_2, \ldots, A_k$ has the property that their characteristic roots $\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj}$ $(j = 1, 2, \ldots, n)$ may be so ordered that every polynomial $f(A_1, A_2, \ldots, A_k)$ in these matrices has characteristic roots $f(\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$ $(j = 1, 2, \ldots, n)$ was first considered by Frobenius [4]. He showed that a sufficient condition for the $\langle A_i \rangle$ to have this property is that they be commutative. It may be shown by an example that this condition is not necessary.

J. Williamson [9] considered this problem for two matrices under the restriction that one of them be non-derogatory. He then showed that a necessary and sufficient condition that these two matrices have the above property is that they satisfy a certain finite set of matric equations.

N. H. McCoy [7] showed that a necessary and sufficient condition that $A_1, A_2, \ldots, A_k$ have the above property is that $A_rA_s - A_sA_r$ $(r, s = 1, 2, \ldots, k)$ belong to the radical of the algebra generated by the $\langle A_i \rangle$. It may be noted that while on the one hand McCoy's condition removes the restriction that one of the matrices be non-derogatory, it does not, on the other hand, give a criterion, such as the Williamson condition, which may be easily computed.

In a part of the following investigation it is proved that if $\mathfrak{A}$ is a matric algebra such that the sum of every two matrices of $\mathfrak{A}$ has characteristic roots which are the sum of the characteristic roots of the two matrices, then every finite set of matrices of $\mathfrak{A}$ has the above property. This is a small step forward in an attempt to recover the computability of the Williamson condition.

The following mapping theorem, which is used in the proof of the above theorem, is also proved. Let $\mathfrak{A}$ be an algebra over an algebraically closed field $\mathfrak{F}$. Let $\mathfrak{B}$ be an algebra over $\mathfrak{F}$. Let Φ be a mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ which (1) maps the identity of $\mathfrak{A}$, if any, onto the identity of $\mathfrak{B}$, (2) is linear, and (3) maps zero divisors into zero divisors in a strong sense. Then Φ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ modulo its radical.

Also included in this investigation is a proof of the McCoy condition which is somewhat simpler and more direct than the one originally given by McCoy.

The author wishes to thank the referee for his many helpful suggestions, and in particular for his suggested proofs of Lemma 3.1 and Theorems 4.2 and 5.1.

2. Some known results on the structure of algebras. All the theorems of this section either appear in [1], or are immediate consequences of theorems

Received August 28, 1950.

which appear there. Throughout the discussions of this and subsequent sections $\mathfrak{F}$ shall denote an arbitrary algebraically closed field.

THEOREM 2.1 [**1**, p. 14]. *If $\mathfrak{D}$ is a division algebra over $\mathfrak{F}$, then $\mathfrak{D} = \mathfrak{F}$.*

THEOREM 2.2 [**1**, p. 44]. *If $\mathfrak{A}$ is a semi-simple algebra over $\mathfrak{F}$, then $\mathfrak{A}$ is separable over $\mathfrak{F}$.*

THEOREM 2.3 [**1**, p. 39]. *If $\mathfrak{A}$ is a simple algebra over $\mathfrak{F}$, then $\mathfrak{A}$ is a total matric algebra over $\mathfrak{F}$.*

THEOREM 2.4 [**1**, p. 39]. *If $\mathfrak{A}$ is a semi-simple algebra over $\mathfrak{F}$, then either $\mathfrak{A}$ is a total matric algebra over $\mathfrak{F}$ or $\mathfrak{A}$ is expressible as the direct sum of total matric algebras over $\mathfrak{F}$.*

THEOREM 2.5. *If $\mathfrak{A}$ is an algebra over $\mathfrak{F}$, then*

$$\mathfrak{A} = (\mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \ldots \oplus \mathfrak{M}_k) + \mathfrak{N}$$

where the $\mathfrak{M}_i$ are total matric algebras over $\mathfrak{F}$ and where $\mathfrak{N}$ is the radical of $\mathfrak{A}$. (The symbol $\oplus$ denotes direct sum and the symbol $+$ denotes supplementary sum.)

THEOREM 2.6 [**1**, p. 40]. *A commutative semi-simple algebra is a direct sum of fields.*

THEOREM 2.7 [**1**, p. 44]. *Let $\mathfrak{A}$ be an algebra over $\mathfrak{K}$. Then there exists an algebraic extension $\mathfrak{K}'$ of $\mathfrak{K}$ such that $\mathfrak{A}_{\mathfrak{K}'}$ is a diagonal algebra if and only if $\mathfrak{A}$ is a direct sum of separable fields.*

THEOREM 2.8. *If $\mathfrak{A}$ is a commutative semi-simple algebra over an algebraically closed field, then $\mathfrak{A}$ is isomorphic to a diagonal algebra.*

3. Theorems of Frobenius and McCoy.

THEOREM 3.1 [**4**]. *Let A_i $(i = 1, 2, \ldots, k)$ be a set of commutative matrices. Let $f(x_1, x_2, \ldots, x_k)$ be any polynomial with coefficients in $\mathfrak{F}$. The characteristic roots of A_i, λ_{ij} $(j = 1, 2, \ldots, n)$ may be so ordered that the characteristic roots of $f(A_1, A_2, \ldots, A_k)$ are $f(\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$. This ordering is the same for every f.*

Every finite set of matrices $\langle A_i \rangle$, commutative or otherwise, which enjoys the property of the preceding theorem will be said to have the *Frobenius Property*.

THEOREM 3.2 [**7**]. *Let $\langle A_i \rangle_{i=1}^{k}$ be an arbitrary set of matrices all of the same order. Let $\mathfrak{R} = \mathfrak{R}[A_1, A_2, \ldots, A_k]$ denote the algebra of all polynomials in the A_i. Let $\mathfrak{N}$ denote the radical of $\mathfrak{R}$. A necessary and sufficient condition that $\langle A_i \rangle$ have the Frobenius Property is that $A_rA_s - A_sA_r \in \mathfrak{N}$ $(r, s = 1, 2, \ldots, k)$.*

Before proceeding to prove these theorems we shall indicate the mode of approach. If the set of matrices $\langle A_i \rangle$ satisfies the Frobenius condition of commutativity or the McCoy condition (i.e., that $A_rA_s - A_sA_r \in \mathfrak{N}$) then by Theorem 2.8 and Wedderburn's Principal Theorem it follows that the algebra

$\mathfrak{R}[A_1, A_2, \ldots, A_k]$ is homomorphic to a diagonal algebra, the kernel of the homomorphism being the radical of $\mathfrak{R}$. Every diagonal algebra clearly has the Frobenius Property. Therefore, if it is shown that the elements of the radical under the operation of addition do not affect the characteristic roots of the elements of the algebra, then both of the above theorems will follow readily.

LEMMA 3.1. *Let $\mathfrak{A}$ be a matric algebra over $\mathfrak{F}$. Let $\mathfrak{N}$ be the radical of $\mathfrak{A}$. Suppose that the identity matrix $I \in \mathfrak{A}$. If $A \in \mathfrak{A}$ and $N \in \mathfrak{N}$, then A and $A + N$ have the same characteristic function.*

Let z be an indeterminate, and $I = A_0$ the unit matrix. Following [3], define matrices A_k and constants c_k recursively as follows:

$$c_0 = 1,\ c_k = (-1/k)\operatorname{tr}(AA_{k-1}),\quad A_k = AA_{k-1} + c_kI.$$

Then we have [3]:

$$P(z, A) = \sum_{k=0}^{n-1} A_k z^{n-1-k},\quad \det(zI - A) = \sum_{k=0}^{n} c_k z^{n-k},$$

where $P(z, A)$ is the adjoint polynomial of $zI - A$.

Now if A is replaced by $A + N$, with N in the radical, the new $(A + N)_k$ differ from the old A_k by elements of the radical, whose trace is zero. Hence the constants c_k are the same for both A and $A + N$, and

$$\det(zI - A) = \det(zI - A - N).$$

LEMMA 3.2. *Let $\mathfrak{A}$ be a semi-simple commutative algebra. Let $\langle A_i\rangle_{i=1}^k$ be a set of matrices with $A_i \in \mathfrak{A}$. Then $\langle A_i\rangle_{i=1}^k$ has the Frobenius Property.*

By Theorem 2.8 $\mathfrak{A}$ is isomorphic to a diagonal algebra. But clearly any finite set of elements of a diagonal algebra has the Frobenius Property. Hence, because of the existing isomorphism, so does $\langle A_i\rangle_{i=1}^k$.

The proof of the sufficiency part of Theorem 3.2, from which Theorem 3.1 follows, may now be given. From Theorem 2.2 and Wedderburn's Principal Theorem it follows that $\mathfrak{R} = \mathfrak{R}' + \mathfrak{N}$ where $\mathfrak{R}' \cong \mathfrak{R} - \mathfrak{N}$. Since $A_rA_s - A_sA_r \in \mathfrak{N}$, it follows that $\mathfrak{R}'$ is a commutative semi-simple algebra. Thus

$$A_i = A_i' + N_i, \qquad A_i' \in \mathfrak{R}',\quad N_i \in \mathfrak{N}.$$

By Lemma 3.1 the characteristic roots of A_i' are the same as those of A_i. By Lemma 3.2 there exists a unique ordering of the roots, λ_{ij}, of the A_i' such that for every polynomial $f(x_1, x_2, \ldots, x_k)$ the characteristic roots of $f(A_1', A_2', \ldots, A_k')$ are $f(\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$ $(j = 1, 2, \ldots, n)$. Note now that

$$\begin{aligned} f(A_1', A_2', \ldots, A_k') &= f(A_1 - N_1, A_2 - N_2, \ldots, A_k - N_k) \\ &= f(A_1, A_2, \ldots, A_k) + N, \qquad N \in \mathfrak{N}. \end{aligned}$$

Again by Lemma 3.1 the characteristic roots of $f(A_1, A_2, \ldots, A_k)$ are $f(\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$. Hence the sufficiency of the stated condition has been shown.

The proof of the necessity of the condition of Theorem 3.2 is immediate. For if an ordering of the roots exists then clearly all the roots of

$$f(A_1, A_2, \ldots, A_k) \cdot [A_r A_s - A_s A_r]$$

are zero for every $f(A_1, A_2, \ldots, A_k) \in \mathfrak{N}$ so that $A_r A_s - A_s A_r$ is properly nilpotent in $\mathfrak{N}$ and hence is in $\mathfrak{N}$.

It may be interesting to state Theorem 3.2 in the following equivalent form:

THEOREM 3.2a. *A necessary and sufficient condition that a set of matrices $\langle A_i \rangle_{i=1}^k$ have the Frobenius Property is that there exists a homomorphism of the algebra $\mathfrak{N} = \mathfrak{N}[A_1, A_2, \ldots, A_k]$, with kernel the radical of $\mathfrak{N}$, onto a diagonal algebra.*

4. Concerning characteristic vectors. Rademacher [8] proved Frobenius's Theorem, our Theorem 3.1, by first proving:

THEOREM 4.1. *Let $\langle A_i \rangle_{i=1}^k$ be a set of commutative matrices. Then there exists a set of numbers $\langle \mu_i \rangle_{i=1}^k$ and a row vector ψ, such that*

$$\psi A_i = \mu_i \psi \qquad (i = 1, 2, \ldots, k).$$

A row vector $\psi \neq 0$ which has the property that $\psi A_i = \mu_i \psi$ $(i = 1, 2, \ldots, k)$ is called a *characteristic row vector* associated with the set $\langle A_i \rangle_{i=1}^k$. A characteristic column vector associated with $\langle A_i \rangle_{i=1}^k$ may be defined similarly.

A more general form of the above theorem is given in:

THEOREM 4.2. *Suppose that $A_r A_s - A_s A_r$ $(r, s = 1, 2, \ldots, k)$ is in the radical, $\mathfrak{N}$, of $\mathfrak{N} = \mathfrak{N}[A_1, A_2, \ldots, A_k]$. Let n_c (n_r) denote the nullity of the column (row) space of $\mathfrak{N}$. Then there are exactly n_c (n_r) linearly independent characteristic row (column) vectors associated with $\langle A_i \rangle_{i=1}^k$.*

As above, $\mathfrak{N} = \mathfrak{N}' + \mathfrak{N}$ where $\mathfrak{N}' \cong \mathfrak{N} - \mathfrak{N}$ and where $\mathfrak{N}'$ is a commutative semi-simple algebra. By Theorem 2.8 it may be assumed without loss of generality that $\mathfrak{N}'$ is a diagonal algebra.

Since the nullity of the column space of $\mathfrak{N}$ is n_c, there exists a matrix H of rank n_c such that $HN = 0$, for every $N \in \mathfrak{N}$. Clearly the row vectors of H form a basis for the complement of the column space of $\mathfrak{N}$; that is, if ϕ is a row vector such that $\phi N = 0$ for every $N \in \mathfrak{N}$, then ϕ is a linear combination of the row vectors of H; for otherwise the nullity of the column space of $\mathfrak{N}$ would be greater than n_c.

A matrix is in Hermite form if it is "triangular with zeros above the diagonal; with every diagonal element either zero or one; if the diagonal element in any row is zero, the entire row is zero; if the diagonal element in any column is one, every other element of the column is zero" [6, p. 35].

It may be assumed that H is in Hermite form; for otherwise one may multiply H on the left by a non-singular matrix P which brings H into Hermite form [6, p. 35] and then $(PH)N = H'N = 0$ for every $N \in \mathfrak{N}$. It will be shown that each of the n_c non-zero row vectors of H is a characteristic row vector.

Now $A_i = D_i + N_i$ $(i = 1, 2, \ldots, k)$, where D_i is diagonal and $N_i \in \mathfrak{N}$. Note also that since $D_i N \in \mathfrak{N}$ for every $N \in \mathfrak{N}$ it follows that $(HD_i)N = H(D_i N) = 0$ for every $N \in \mathfrak{N}$; and hence it is true that every row vector of HD_i is a linear combination of the row vectors of H. We may therefore write $HD_i = LH$. If z_i is a number such that the diagonal matrix $B_i = z_i I + D_i$ is non-singular, then

$$HB_i = (z_i I + L)H,$$
$$B_i^{-1} HB_i = B_i^{-1}(z_i I + L)H.$$

The matrix $B_i^{-1}HB_i$ on the left is in Hermite form, since this form is still retained after transforming H by a non-singular diagonal matrix. Since the Hermite form is unique, the right member, which is a left multiple of H, can be in Hermite form only if it is equal to H. Hence B_i and, consequently, D_i are commutative with H. From the equation

$$HA_i = HD_i = D_i H$$

it follows, that if ψ_k is a non-vanishing row vector occupying the kth row of H and λ_{ik} is the kth diagonal element of D_i, then

$$\psi_k A_i = \lambda_{ik}\psi_k.$$

Thus the n_c linearly independent vectors ψ_k are characteristic vectors of each of the matrices A_i.

Suppose now that ψ is any characteristic row vector associated with $\langle A_i \rangle_{i=1}^k$; $\psi A_i = \lambda_i \psi$. Let N be any element of $\mathfrak{N}$. Since $N \in \mathfrak{N}[A_1, A_2, \ldots, A_k]$ it is true that $N = f(A_1, A_2, \ldots, A_k)$. Thus

$$\psi N = \psi f(A_1, A_2, \ldots, A_k) = f(\lambda_1, \lambda_2, \ldots, \lambda_k)\psi.$$

But since $N \in \mathfrak{N}$, N has only zero as a characteristic root, and hence $f(\lambda_1, \lambda_2, \ldots, \lambda_k) = 0$. Therefore ψ annihilates every element of $\mathfrak{N}$. But this means that ψ is a linear combination of the n_c vectors ψ_k considered above. This completes the proof of the theorem.

In the example given below $n_c \neq n_r$. This indicates that one cannot in general expect to get an expression for n_c or n_r in terms of the Weyr or Segre characteristics of the matrices involved; for the latter invariants do not differentiate between the structure of the row spaces and the column spaces of the matrices.

Example: $$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\mathfrak{R} = \mathfrak{R}[A_1, A_2], \quad \mathfrak{R}' = \mathfrak{R}[I], \quad \mathfrak{N} = \mathfrak{N}[N_1, N_2],$$

$$A_1 = I + N_1, \quad A_2 = I + N_2.$$

Thus $n_c = 1$, $n_r = 2$. Also note that

$$[1,\ 0,\ 0] \cdot A_i = 1 \cdot [1,\ 0,\ 0] \qquad (i = 1, 2),$$

$$A_i \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_i \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (i = 1, 2).$$

5. A mapping theorem. $\mathfrak{X}$ is said to be a module over $\mathfrak{F}$ if $\mathfrak{X}$ is a linear subset of an algebra over $\mathfrak{F}$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be modules over $\mathfrak{F}$. Let Φ be a mapping of $\mathfrak{X}$ onto $\mathfrak{Y}$ which satisfies the following conditions:

(1) If $\mathfrak{X}$ has a unit ϵ, then $\mathfrak{Y}$ has a unit ϵ', and $\Phi(\epsilon) = \epsilon'$.

C: (2) Φ is linear, i.e., if $X_i \in \mathfrak{X}$ and $a_i \in \mathfrak{F}$, then $\Phi(\sum_{i=1}^{k} a_i X_i) = \sum_{i=1}^{k} a_i \Phi(X_i)$.

(3) If $X_i \in \mathfrak{X}$ and if $\prod_{i=1}^{k} X_i = 0$, then $\prod_{i=1}^{k} \Phi(X_i) = 0$.

THEOREM 5.1. *Let $\mathfrak{M}$ be a total matric algebra over $\mathfrak{F}$, $\mathfrak{Y}$ a module over $\mathfrak{F}$, and Φ a mapping of $\mathfrak{M}$ onto $\mathfrak{Y}$ which satisfies conditions* C. *If $A, A' \in \mathfrak{M}$ then $\Phi(A \cdot A') = \Phi(A)\Phi(A')$. Thus $\mathfrak{Y}$ is an algebra and Φ maps $\mathfrak{M}$ homomorphically onto $\mathfrak{Y}$.*

$\mathfrak{M}$ has a basis E_{ij} $(i, j = 1, 2, \ldots, n)$ where $E_{ij}E_{km} = \delta_{jk}E_{im}$ and where δ_{jk} is Kronecker's delta.

Since Φ is linear it will be sufficient to show that

$$\Phi(E_{ij}E_{km}) = \Phi(E_{ij})\Phi(E_{km}) = \delta_{jk}\Phi(E_{im}).$$

If I is the unit matrix, each of the following products vanishes:

$$E_{ij}E_{km} = 0 \quad \text{for } j \neq k,$$
$$(E_{ii} - I)E_{ik} = E_{ii}E_{ik} - IE_{ik} = 0,$$
$$(E_{ii} - E_{ij})(E_{ik} + E_{jk}) = E_{ii}E_{ik} + E_{ii}E_{jk} - E_{ij}E_{ik} - E_{ij}E_{jk} = 0.$$

Hence the image under the mapping Φ of each of these products vanishes. We obtain successively:

$$\Phi(E_{ij})\Phi(E_{km}) = 0 \quad \text{for } j \neq k,$$
$$\Phi(E_{ii})\Phi(E_{ik}) = \Phi(I)\Phi(E_{ik}) = \Phi(E_{ik}),$$
$$\Phi(E_{ij})\Phi(E_{jk}) = \Phi(E_{ii})\Phi(E_{ik}) + \Phi(E_{ii})\Phi(E_{jk}) - \Phi(E_{ij})\Phi(E_{ik}) = \Phi(E_{ik}).$$

THEOREM 5.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras over $\mathfrak{F}$ with radicals $\mathfrak{N}$ and $\mathfrak{N}'$ respectively. Let Φ be a mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ which satisfies conditions* C. *If $A, A' \in \mathfrak{A}$ then $\Phi(A \cdot A') = \Phi(A)\Phi(A')$* mod $\mathfrak{N}'$.

By Theorem 2.5, $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$ where $\mathfrak{S} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \ldots \oplus \mathfrak{M}_k$. Thus if $A \in \mathfrak{A}$ then A is uniquely expressible as $A = S + N$, where $S \in \mathfrak{S}$ and $N \in \mathfrak{N}$.

(1) If $N \in \mathfrak{N}$, then $\Phi(N) \in \mathfrak{N}'$. For suppose that $N \in \mathfrak{N}$. It is sufficient to show that if $B \in \mathfrak{B}$, then $[B\Phi(N)]^l = 0$ for some positive integer l. Since Φ maps $\mathfrak{A}$ *onto* $\mathfrak{B}$ it follows that if $B \in \mathfrak{B}$ then there exists an $A \in \mathfrak{A}$ such that $B = \Phi(A)$. Since $N \in \mathfrak{N}$ it is true that there exists an l such that

$$[AN]^l = ANAN \ldots AN = 0.$$

By C_3, $\Phi(A)\Phi(N) \ldots \Phi(A)\Phi(N) = 0$, or $[\Phi(A)\Phi(N)]^l = [B\Phi(N)]^l = 0$. Therefore $\Phi(N) \in \mathfrak{N}'$.

(2) With the use of Theorem 5.1 it may be shown quite easily that if $S, S' \in \mathfrak{S}$, then $\Phi(S)\Phi(S') = \Phi(S{\cdot}S')$.

(3) Suppose now that $A, A' \in \mathfrak{A}$. Then

$$A = S + N, \quad A' = S' + N', \quad S,S' \in \mathfrak{S}, \quad N,N' \in \mathfrak{N},$$
$$AA' = S'' + N'' = (A-N)(A'-N') + N'', \quad S'', (A-N), (A'-N') \in \mathfrak{S}, \quad N'' \in \mathfrak{N}.$$

Using (1) and (2) we may write

$$\Phi(AA') = \Phi(A-N)\Phi(A'-N') + \Phi(N''),$$
$$\Phi(A)\Phi(A') = [\Phi(A-N)+\Phi(N)][\Phi(A'-N')+\Phi(N')],$$
$$\Phi(AA') - \Phi(A)\Phi(A') = \Phi(N'') - \Phi(N)\Phi(A'-N') - \Phi(A-N)\Phi(N') - \Phi(N)\Phi(N') \equiv 0 \mod \mathfrak{N}'.$$

In the preceeding theorem it has been assumed that the field $\mathfrak{F}$ was algebraically closed. The following example shows that Theorem 5.2 is not necessarily true if the field is not algebraically closed. Thus some condition on $\mathfrak{F}$ is necessary. It may be proved that the theorem still holds if the condition of algebraic closure is replaced by the somewhat weaker condition that the characteristic roots of every element of $\mathfrak{A}$ all lie in $\mathfrak{F}$.

Let Ra denote the rational field and let $\mathfrak{A}$ be an algebra over Ra with basis elements I and A, where I is the identity and $A^2 = -I$. Define a mapping Φ of $\mathfrak{A}$ onto $\mathfrak{A}$ as follows:

$$\Phi(aI + bA) = (a + b)I + bA, \qquad a,b \in Ra.$$

Clearly $\Phi(I) = I$; Φ is linear; and since A has no proper zero divisors, Φ satisfies C_3 vacuously. The radical of $\mathfrak{A}$ is zero. Note however that

$$\Phi(A^2) = \Phi(-I) = -I \neq \Phi(A)\Phi(A) = 2A.$$

Note also that if the complex field were used instead of the rational field, and Φ defined similarly, then condition C_3 would not be satisfied. For

$$(iI + A)(iI - A) = 0 \qquad (i^2 = -1),$$

whereas

$$\Phi(iI + A)\Phi(iI - A) = [(i + 1)I + A][(i - 1)I - A] = -I.$$

https://doi.org/10.4153/CJM-1952-003-0 Published online by Cambridge University Press

6. The assignment of a common order to the characteristic roots of certain sets of matrices. Let $\mathfrak{A}$ be a subalgebra of a total matric algebra $\mathfrak{M}$ of order n^2 over an algebraically closed field $\mathfrak{F}$. Suppose that the identity matrix I is in $\mathfrak{A}$. Let λ_{ij} $(j = 1, 2, \ldots, n)$ denote the characteristic roots of $A_i \in \mathfrak{A}$.

$\mathfrak{A}$ is said to have property P_0 if the characteristic roots of every pair of matrices $A_1, A_2 \in \mathfrak{A}$ may be so ordered that the characteristic roots of $A_1 + A_2$ are $\lambda_{1j} + \lambda_{2j}$ $(j = 1, 2, \ldots, n)$.

$\mathfrak{A}$ is said to have property P_1 if the characteristic roots of every finite set of matrices $\langle A_i \rangle_{i=1}^{k} \in \mathfrak{A}$ may be so ordered that the characteristic roots of $\sum_{i=1}^{k} a_i A_i$ are $\sum_{i=1}^{k} a_i \lambda_{ij}$ $(j = 1, 2, \ldots, n)$ for all $a_i \in \mathfrak{F}$.

The ordering of the roots in property P_0 is not assumed to be unique. It is conceivable, *a priori*, that the jth characteristic root of $A_1 + A_2$ is $\lambda_{1j} + \lambda_{2j}$ but that for some $a \in \mathfrak{F}$ the jth characteristic root of $A_1 + aA_2$ is $\lambda_{1j} + a\lambda_{2k}$; thus it seems possible that the jth root of A_1 may associate with the jth root of A_2 but that for some $a \in \mathfrak{F}$ the jth root of A_1 will associate with the kth root of aA_2. That this is not so is proved in

LEMMA 6.1. *Suppose that $\mathfrak{A}$ has property P_0. Suppose that the jth characteristic root of $A_1 + A_2$ is $\lambda_{1j} + \lambda_{2j} (j = 1, 2, \ldots, n)$. Then the jth characteristic root of $aA_1 + bA_2$ is $a\lambda_{1j} + b\lambda_{2j}$ for all $a,b \in F$ $(j = 1, 2, \ldots, n)$.*

Denote the jth characteristic root of

$$cA_1 + [(a - c)A_1 + bA_2], \quad (a - c)A_1 + bA_2, \text{ and } aA_1 + bA_2$$

by

$$c\lambda_{1j} + (a - c)\lambda_{1l} + b\lambda_{2m}, \quad (a - c)\lambda_{1l} + b\lambda_{2m}, \text{ and } a\lambda_{1p} + b\lambda_{2q}$$

respectively. It would seem that the subscripts l, m, p, and q depend on the values of j, a, b, and c; that is, $l = l(j, a, b, c)$, $m = m(j, a, b, c)$, $p = p(j, a, b, c)$, $q = q(j, a, b, c)$, where the functions involved are integral valued and assume values only between 1 and n inclusive. Since

$$cA_1 + [(a-c)A_1 + bA_2] = aA_1 + bA_2,$$

it is true that

$$c\lambda_{1j} + (a - c)\lambda_{1l(j,a,b,c)} + b\lambda_{2m(j,a,b,c)} = a\lambda_{1p(j,a,b,c)} + b\lambda_{2q(j,a,b,c)}. \tag{6.1}$$

Now let j, b, and c be arbitrary but fixed. Consider the quadruplet of integers $[l(a), m(a), p(a), q(a)]$. Since $l(a)$, $m(a)$, $p(a)$, and $q(a)$ are integers between 1 and n it follows that at most n^4 distinct quadruplets can be obtained by letting a run over $\mathfrak{F}$. Since $\mathfrak{F}$ is algebraically closed it is an infinite field and hence there exist an infinite number of distinct $a_i \in \mathfrak{F}$ such that $[l(a_i), m(a_i), p(a_i), q(a_i)] = [l_0, m_0, p_0, q_0]$ for some fixed l_0, m_0, p_0, and q_0. Thus for an infinite number of distinct $a_i \in \mathfrak{F}$

$$c\lambda_{1j} + (a_i - c)\lambda_{1l_0} + b\lambda_{2m_0} = a\lambda_{1p_0} + b\lambda_{2q_0}. \tag{6.2}$$

From (6.2) it follows immediately that

$$\lambda_{1l_0} = \lambda_{1p_0}. \tag{6.3}$$

Furthermore, $\lambda_{1l(a)}$ *may be taken equal* to λ_{1l_0}, and $\lambda_{1p(a)}$ *may be taken equal* to λ_{1p_0} for all $a \in \mathfrak{F}$. For

$$\det(cA_1 + (x-c)A_1 + bA_2 - c\lambda_{1j}I - (x-c)\lambda_{1l_0}I - b\lambda_{2m_0}I)$$
$$= \det(xA_1 + bA_2 - x\lambda_{1p_0}I - b\lambda_{2q_0}I) = 0$$

for an infinite number of distinct $x \in \mathfrak{F}$. Hence the above determinants are identically zero, and thus $cA_1 + (a_1 - c)A_1 + bA_2$ and $a_iA_1 + bA_2$ have respectively the characteristic roots $c\lambda_{1j} + (a_i - c)\lambda_{1l_0} + b\lambda_{2m_0}$ and $a_i\lambda_{1p_0} + b\lambda_{2q_0}$ for all $a \in \mathfrak{F}$. Consequently one may, without loss of generality, redefine the functions l, m, p, and q so that $[l(a_i), m(a_i), p(a_i), q(a_i)] = [l_0, m_0, p_0, q_0]$ for all $a_i \in \mathfrak{F}$. From this and the fact that the choice of j, b, and c was arbitrary it follows from (6.3) that

$$\lambda_{1l(j,a,b,c)} = \lambda_{1p(ja,b,c)} \text{ for all } a, b, c \in \mathfrak{F} \qquad (j = 1, 2, \ldots, n).$$

Similarly if j, a, and c are kept fixed it can be shown that

$$\lambda_{2m(j,a,b,c)} = \lambda_{2q(j,a,b,c)} \text{ for all } a, b, c \in \mathfrak{F} \qquad (j = 1, 2, \ldots, n).$$

It has been proved that if the jth root of $(a-c)A_1 + bA_2$ is $(a-c)\lambda_{1l} + b\lambda_{2m}$ then the jth root of $aA_1 + bA_2$ is a $\lambda_{1l} + b\lambda_{2m}$. But the jth root of

$$[a-(a-1)]\, A_1 + A_2$$

is $\lambda_{1j} + \lambda_{2j}$, and hence the jth root of $aA_1 + A_2$ is $a\lambda_{1j} + \lambda_{2j}$. Applying the same process to $[b-(b-1)]A_1 + aA_2$ one obtains the desired result that the jth root of $aA_1 + bA_2$ is $a\lambda_{1j} + b\lambda_{2j}$ for all $a, b \in \mathfrak{F}$ $(j = 1, 2, \ldots, n)$.

LEMMA 6.2. *Suppose that* $\mathfrak{A}$ *has property* P_0. *Suppose that the jth characteristic root of* $aA_1 + bA_2$ *is* $a\lambda_{1j} + b\lambda_{2j}$ *and that the jth characteristic root of* $aA_2 + bA_3$ *is* $a\lambda_{2j} + b\lambda_{3j}$ *for all* $a, b \in \mathfrak{F}$ $(j = 1, 2, \ldots, n)$. *Then the jth characteristic root of* $aA_1 + bA_2 + cA_3$ *is* $a\lambda_{1j} + b\lambda_{2j} + c\lambda_{3j}$, *for all* $a, b, c \in \mathfrak{F}$.

Note that $aA_1 + bA_2 + cA_2 = [aA_1 + bA_2] + cA_3 = aA_1 + [bA_2 + cA_3]$. Then as in Lemma 6.1, $a\lambda_{1j} + b\lambda_{2j} + c\lambda_{3l(j,a,b,c)} = a\lambda_{1p(m,a,b,c)} + b\lambda_{2m(j,a,b,c)} + c\lambda_{3m(j,a,b,c)}$. Now keep j, a, and b fixed and consider the triplet $[l(c), m(c), p(m, c)]$. Proceeding as in Lemma 6.1, one obtains that $\lambda_{3l(j,a,b,c)} = \lambda_{3m(j,a,b,c)}$ for all $a, b, c \in \mathfrak{F}$ $(j = 1, 2, \ldots, n)$. Similarly keeping j, a, and c fixed gives the result that $\lambda_{2j} = \lambda_{2m(j,a,b,c)}$. From these facts the desired result follows readily.

THEOREM 6.1. *Properties* P_0 *and* P_1 *are equivalent.*

Clearly P_1 implies P_0. The fact that P_0 implies P_1 follows from a simple induction on the number of matrices in Lemma 6.2.

THEOREM 6.2. *Property* P_1 *and the Frobenius Property are equivalent.*

Obviously the Frobenius Property implies property P_1. Suppose that $\mathfrak{A}$ has property P_1. If it is shown that there exists a mapping Φ which

(a) maps $\mathfrak{A}$ onto an algebra $\mathfrak{B}$ which is semi-simple and has the Frobenius Property,

(b) preserves characteristic roots, i.e., A and $\Phi(A)$ have the same characteristic roots, and

(c) satisfies conditions C,

then it will follow from Theorem 5.2 that $\mathfrak{A}$ has the Frobenius Property.

A mapping Φ satisfying these conditions will now be shown to exist.

Let E_i $(i = 1, 2, \ldots, k)$ be a basis for $\mathfrak{A}$. Let ρ_{ij} $(j = 1, 2, \ldots, n)$ denote the characteristic roots of E_i. Define

$$\Phi(E_i) = \begin{bmatrix} \rho_{i1} & 0 & 0 & \ldots & 0 \\ 0 & \rho_{i2} & 0 & \ldots & 0 \\ 0 & 0 & \rho_{i3} & \ldots & 0 \\ & \ldots & & \ldots & \\ 0 & 0 & 0 & \ldots & \rho_{in} \end{bmatrix}$$

where the ρ_{ij} are so ordered that $\Phi(\sum_{i=1}^{k} a_i E_i) = \sum_{i=1}^{k} a_i \Phi(E_i)$ for all $a_i \in \mathfrak{F}$. Since A satisfies P_1 this is possible. Let $\mathfrak{B}$ be the set of all matrices $\langle \Phi(A) \rangle$ with $A \in \mathfrak{A}$.

(a) $\mathfrak{B}$ is a semi-simple algebra with the Frobenius Property.

To prove that $\mathfrak{B}$ is an algebra it will be sufficient to show that if $A_1, A_2 \in \mathfrak{A}$, then there exists an $A_3 \in \mathfrak{A}$ such that $\Phi(A_3) = \Phi(A_1)\Phi(A_2)$, i.e., that $\mathfrak{B}$ is closed under multiplication. Now if $A_1, A_2 \in \mathfrak{A}$, then since $\mathfrak{A}$ has property P_1 it is true that the jth characteristic root of

$$a(A_1 + A_2)^2 + b(A_1 + A_2) + c(A_1^2 + A_2^2)$$

is $a(\lambda_{1j} + \lambda_{2j})^2 + b(\lambda_{1j} + \lambda_{2j}) + c(\lambda_{1j}^2 + \lambda_{2j}^2)$. Letting $a = \frac{1}{2}$, $b = 0$, and $c = -\frac{1}{2}$ one obtains the result that $\Phi(\frac{1}{2}[A_1A_2 + A_2A_1]) = \Phi(A_1)\ \Phi(A_2)$. Thus $\mathfrak{B}$ is an algebra. Furthermore, since $\mathfrak{B}$ consists of diagonal matrices only, it is semi-simple and has the Frobenius Property.

(b) Φ, by construction, preserves characteristic roots.

(c) Φ satisfies conditions C.

(1) $I \in \mathfrak{A}$ and $\Phi(I) = I$, so that Φ satisfies C_1.

(2) Φ, by construction is linear. Hence Φ satisfies C_2.

(3) It is required to show that if $\prod_{i=1}^{h} A_i = 0$, then $\prod_{i=1}^{h} \Phi(A_i) = 0$; by the

https://doi.org/10.4153/CJM-1952-003-0 Published online by Cambridge University Press

construction of Φ it is thus required to prove that if $\prod_{i=1}^{h} A_i = 0$, then $\prod_{i=1}^{h} \lambda_{ij} = 0$ $(j = 1, 2, \ldots, n)$, where the λ_{ij} are so ordered that $\sum_{i=1}^{h} a_i A_i$ has characteristic roots $\sum_{i=1}^{h} a_i \lambda_{ij}$. This shall be proved by an induction on h.

Let $h = 2$ and suppose that $A_1 A_2 = 0$. Consider $A_2 A_1 = N$. N is nilpotent; for $N^2 = 0$. Furthermore, if $f(A_1, A_2) \in \mathfrak{R}[A_1, A_2]$ then since $A_1 A_2 = 0$, $f(A_1, A_2) \cdot [A_2 A_1] = \sum_i a_i A_2^{r_i} A_1$, where $r_i > 0$ for all i. But $(\sum_i a_i A_2^{r_i} A_1)^2 = 0$. Therefore $N = A_2 A_1$ is the radical of $\mathfrak{R}[A_1, A_2]$. Then $A_2 A_1 - A_1 A_2 = N$ is in the radical of $\mathfrak{R}[A_1, A_2]$ and by Theorem 3.2, $\lambda_{1j}\lambda_{2j} = 0$ $(j = 1, 2, \ldots, n)$, where the λ_{ij} are so ordered that the characteristic roots of $a_1 A_1 + a_2 A_2$ are $a_1 \lambda_{1j} + a_2 \lambda_{2j}$ for all $a_1, a_2 \in \mathfrak{F}$.

Assume now that if $\prod_{i=1}^{h} A_i = 0$, then $\prod_{i=1}^{h} \lambda_{ij} = 0$ $(j = 1, 2, \ldots, n)$. Suppose that $\prod_{i=1}^{h+1} A_i = 0$. Then $(A_1 A_2) \prod_{i=3}^{h+1} A_i = 0$. By the induction assumption $\mu_j \prod_{i=3}^{h+1} \lambda_{ij} = 0$ where μ_j is the characteristic root of $A_1 A_2$ associated with λ_{ij} $(i = 1, 2, \ldots, h + 1)$. Suppose that for some j, $\prod_{i=3}^{h+1} \lambda_{ij} \neq 0$. Then $\mu_j = 0$. It must be shown that either λ_{1j} or λ_{2j} (or both) equals zero.

Consider the matrix

$$\begin{aligned} B_a &= A_1 A_2 - a/(a-1) \cdot \lambda_{2j} A_1 - a\lambda_{1j} A_2 + a^2/(a-1) \cdot \lambda_{1j}\lambda_{2j} I \\ &= [A_1 - a\lambda_{1j} I][A_2 - a/(a-1) \cdot \lambda_{2j} I], \qquad a \neq 1. \end{aligned}$$

Since $\mu_j = 0$ and since $\mathfrak{A}$ has property P_1, B_a has for each $a \in \mathfrak{F}$, $a \neq 1$, a characteristic root equal to zero. Thus for every $a \neq 1$ there exists a vector $\phi_a \neq 0$ such that $B_a \phi_a = 0$. Thus

$$[A_1 - a\lambda_{1j} I][A_2 - a/(a-1) \cdot \lambda_{2j} I]\phi_a = 0.$$

Let $[A_2 - a/(a-1) \cdot \lambda_{2j} I]\phi_a = \psi_a$. Now clearly if $\psi_a \neq 0$, then

$$[A_1 - a\lambda_{1j} I]\, \psi_a = 0.$$

Thus either $[A_2 - a/(a-1) \cdot \lambda_{2j} I]\phi_a = 0$, $\phi_a \neq 0$ for an infinite number of distinct $a \in \mathfrak{F}$, or $[A_1 - a\lambda_{1j} I]\psi_a = 0$, $\psi_a \neq 0$ for an infinite number of distinct $a \in \mathfrak{F}$ (or both). Suppose, say, that $[A_2 - a/(a-1) \cdot \lambda_{2j} I]\phi_a = 0$, $\phi_a \neq 0$ for an infinite number of distinct $a \in \mathfrak{F}$. Then A_2 has characteristic roots $a/(a-1) \cdot \lambda_{2j}$ for an infinite number of distinct $a \in \mathfrak{F}$. But A_2 has only a finite number of distinct characteristic roots. Therefore, for some $a_1, a_2, a_1 \neq a_2$ it is true that $a_1/(a_1 - 1) \cdot \lambda_{2j} = a_2/(a_2 - 1) \cdot \lambda_{2j}$. From this it follows that $\lambda_{2j} = 0$.

By induction it follows that Φ satisfies C_3. Hence the theorem.

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As a corollary to the two preceeding theorems we have:

THEOREM 6.3. P_0, P_1, *and the Frobenius Property are equivalent.*

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