*Bull. Aust. Math. Soc.* **92** (2015), 282–289
doi:10.1017/S0004972715000544

# A NOTE ON CYCLIC AMENABILITY OF THE LAU PRODUCT OF BANACH ALGEBRAS DEFINED BY A BANACH ALGEBRA MORPHISM

**F. ABTAHI✉ and A. GHAFARPANAH**

(Received 12 February 2015; accepted 11 April 2015; first published online 16 June 2015)

**Abstract**

Let $T$ be a Banach algebra homomorphism from a Banach algebra $\mathcal{B}$ to a Banach algebra $\mathcal{A}$ with $\|T\| \leq 1$. Recently, Bhatt and Dabhi ['Arens regularity and amenability of Lau product of Banach algebras defined by a Banach algebra morphism', *Bull. Aust. Math. Soc.* **87** (2013), 195–206] showed that cyclic amenability of $\mathcal{A} \times_T \mathcal{B}$ is stable with respect to $T$, for the case where $\mathcal{A}$ is commutative. In this note, we address a gap in the proof of this stability result and extend it to an arbitrary Banach algebra $\mathcal{A}$.

2010 *Mathematics subject classification*: primary 46H05; secondary 46H99.

*Keywords and phrases*: cyclic amenability, $T$-Lau product.

## 1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and let $T \in \hom(\mathcal{B}, \mathcal{A})$, the set of all Banach algebra homomorphisms from $\mathcal{B}$ into $\mathcal{A}$ with $\|T\| \leq 1$. Following [1, 2], the Cartesian product space $\mathcal{A} \times \mathcal{B}$, equipped with the multiplication

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 + a_1T(b_2) + T(b_1)a_2, b_1b_2) \quad (a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}) \quad (1.1)$$

and the norm

$$\|(a, b)\| = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}},$$

is a Banach algebra, which is denoted by $\mathcal{A} \times_T \mathcal{B}$. Note that our definition of the multiplication $\times_T$ in [1] is slightly different to that given by Bhatt and Dabhi [2], who assumed commutativity of $\mathcal{A}$. However, this assumption is unnecessary and the definition (1.1) applies for an arbitrary Banach algebra $\mathcal{A}$.

Bhatt and Dabhi [2] investigated some algebraic properties of $\mathcal{A} \times_T \mathcal{B}$, such as Arens regularity and some aspects of amenability, for the case where $\mathcal{A}$ is commutative. In the recent work [1], we verified biprojectivity and biflatness of

This research was partially supported by the Centre of Excellence for Mathematics at the University of Isfahan.

$\mathcal{A} \times_T \mathcal{B}$. As an application of these results, we generalised [2, Theorem 4.1, part (1)] for the case where $\mathcal{A}$ is not necessarily commutative.

One of the remarkable results in [2] is that cyclic amenability of $\mathcal{A} \times_T \mathcal{B}$ is stable with respect to $T$. That is, if $\mathcal{A}$ is commutative, then $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ also are. In the present note, we investigate this result and correct a gap in the proof. Moreover, we generalise this result to an arbitrary Banach algebra $\mathcal{A}$.

## 2. Preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $T \in \hom(\mathcal{B}, \mathcal{A})$. Let $\mathcal{A}'$ denote the dual Banach space of $\mathcal{A}$. For $a \in \mathcal{A}$ and $f \in \mathcal{A}'$, $f \cdot a$ and $a \cdot f$ are defined by $f \cdot a(x) = f(ax)$ and $a \cdot f(x) = f(xa)$ for all $x \in \mathcal{A}$. As remarked in [1], the dual space $(\mathcal{A} \times_T \mathcal{B})'$ can be identified with $\mathcal{A}' \times \mathcal{B}'$ via the linear map $\theta : \mathcal{A}' \times \mathcal{B}' \to (\mathcal{A} \times_T \mathcal{B})'$:

$$\langle \theta(f, g), (a, b)\rangle = \langle f, a\rangle + \langle g, b\rangle,$$

where $a \in \mathcal{A}, f \in \mathcal{A}', b \in \mathcal{B}$ and $g \in \mathcal{B}'$. Moreover, $(\mathcal{A} \times_T \mathcal{B})'$ is a $(\mathcal{A} \times_T \mathcal{B})$-bimodule with natural module actions of $A \times_T B$ on its dual. In fact, it is easily verified that

$$(f, g) \cdot (a, b) = (f \cdot (a + T(b)), T^*(f \cdot a) + g \cdot b) \tag{2.1}$$

and

$$(a, b) \cdot (f, g) = ((a + T(b)) \cdot f, T^*(a \cdot f) + b \cdot g), \tag{2.2}$$

where $a \in \mathcal{A}, b \in \mathcal{B}, f \in \mathcal{A}'$ and $g \in \mathcal{B}'$. Furthermore, $\mathcal{A} \times_T \mathcal{B}$ is a Banach $\mathcal{A}$-bimodule under the module actions

$$c \cdot (a, b) := (c, 0) \cdot (a, b) \quad \text{and} \quad (a, b) \cdot c := (a, b) \cdot (c, 0)$$

for all $a, c \in \mathcal{A}$ and $b \in \mathcal{B}$. Similarly, $\mathcal{A} \times_T \mathcal{B}$ can be made into a Banach $\mathcal{B}$-bimodule.

We also introduce some further maps similar to those defined in [5]. Let $p_{\mathcal{A}} : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}$ and $p_{\mathcal{B}} : \mathcal{A} \times_T \mathcal{B} \to \mathcal{B}$ be the usual projections, which are defined by $p_{\mathcal{A}}((a, b)) = a$ and $p_{\mathcal{B}}((a, b)) = b$, respectively, for $a \in \mathcal{A}, b \in \mathcal{B}$. Let $q_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \times_T \mathcal{B}$ and $q_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \times_T \mathcal{B}$ be the usual injections, defined by $q_{\mathcal{A}}(a) = (a, 0)$ and $q_{\mathcal{B}}(b) = (0, b)$, respectively. Finally, define the mapping $r_{\mathcal{A}} : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}$ by $r_{\mathcal{A}}((a, b)) := a + T(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. One can simply check that $q_{\mathcal{A}}$ and $r_{\mathcal{A}}$ are Banach $\mathcal{A}$-bimodule maps and $p_{\mathcal{B}}$ and $q_{\mathcal{B}}$ are Banach $\mathcal{B}$-bimodule maps.

## 3. Main results

Let $\mathcal{A}$ be a Banach algebra and let $X$ be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to X$ is called a derivation if $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathcal{A}$. Given $x \in X$, let $ad_x : \mathcal{A} \to X$ be given by $ad_x(a) = a \cdot x - x \cdot a$ for $a \in \mathcal{A}$. Then $ad_x$ is a derivation, which is called an inner derivation at $x$. Recall from [3] that a derivation $D : \mathcal{A} \to \mathcal{A}^*$ is called cyclic if

$$\langle D(a), b\rangle + \langle D(b), a\rangle = 0$$

https://doi.org/10.1017/S0004972715000544 Published online by Cambridge University Press

for all $a, b \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is called cyclic amenable if every cyclic derivation is inner.

In [2, Theorem 4.1, part (4)], it has been proved that if $\mathcal{A}$ is commutative, then $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ also are. There appear to be some gaps in the proof presented in [2]. In the first part of the proof, it has been assumed that if $D : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}^* \times \mathcal{B}^*$ is a cyclic derivation, then $D_{|\mathcal{A}} : \mathcal{A} \to \mathcal{A}^*$ and $D_{|\mathcal{B}} : \mathcal{B} \to \mathcal{B}^*$ are also cyclic derivations. However, $D_{|\mathcal{A}}$ and $D_{|\mathcal{B}}$ do not necessarily map into $\mathcal{A}^*$ and $\mathcal{B}^*$. Dabhi kindly provided us with a new proof of his result for the case where $\mathcal{A}$ is commutative, but with an extra assumption, which seems to be necessary. Here, we adapt his proof to the general case where $\mathcal{A}$ is an arbitrary Banach algebra. First we introduce the concept of a faithful dual space.

Definition 3.1. Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ has a left (respectively right) faithful dual space if for each nonzero $f \in \mathcal{A}^*$, there exists $a \in \mathcal{A}$ such that $a \cdot f \neq 0$ (respectively $f \cdot a \neq 0$). We say that $\mathcal{A}$ has a faithful dual space if $\mathcal{A}$ has both a left and a right faithful dual space.

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with faithful dual spaces and $T \in$ $\mathrm{hom}(\mathcal{B}, \mathcal{A})$. If $\mathcal{A}$ and $\mathcal{B}$ are cyclic amenable, then $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable.*

Proof. Suppose that $D : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}^* \times \mathcal{B}^*$ is a cyclic derivation. Then

$$D = (D_1, D_2) = (q_{\mathcal{A}}^* \circ D, q_{\mathcal{B}}^* \circ D).$$

Using (2.1) and (2.2), for all $(a, b), (c, d) \in \mathcal{A} \times_T \mathcal{B}$,

$$\begin{aligned} D((a,b)(c,d)) &= (a,b) \cdot (D_1(c,d), D_2(c,d)) + (D_1(a,b), D_2(a,b)) \cdot (c,d) \\ &= ((a + T(b)) \cdot D_1(c,d), T^*(a \cdot D_1(c,d)) + b \cdot D_2(c,d)) \\ &\quad + (D_1(a,b) \cdot (c + T(d)), T^*(D_1(a,b) \cdot c) + D_2(a,b) \cdot d). \end{aligned}$$

It follows that

$$D_1((a,b)(c,d)) = (a + T(b)) \cdot D_1(c,d) + D_1(a,b) \cdot (c + T(d)) \tag{3.1}$$

and

$$D_2((a,b)(c,d)) = T^*(a \cdot D_1(c,d)) + b \cdot D_2(c,d) + T^*(D_1(a,b) \cdot c) + D_2(a,b) \cdot d. \tag{3.2}$$

Let

$$d_1 = q_{\mathcal{A}}^* \circ D \circ q_{\mathcal{A}} = D_1 \circ q_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^*$$

and

$$d_2 = q_{\mathcal{B}}^* \circ D \circ q_{\mathcal{B}} = D_2 \circ q_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^*.$$

Choosing $b = d = 0$ in (3.1) and $a = c = 0$ in (3.2),

$$d_1(ac) = a \cdot d_1(c) + d_1(a) \cdot c \quad \text{and} \quad d_2(bd) = b \cdot d_2(d) + d_2(b) \cdot d.$$

Thus, $d_1$ and $d_2$ are derivations. Also, by the fact that $D$ is cyclic, for all $a, c \in \mathcal{A}$ and $b, d \in \mathcal{B}$,

$$\langle a, d_1(c)\rangle + \langle c, d_1(a)\rangle = \langle (a,0), D(c,0)\rangle + \langle (c,0), D(a,0)\rangle = 0$$

and

$$\langle b, d_2(d)\rangle + \langle d, d_2(b)\rangle = \langle (0,b), D(0,d)\rangle + \langle (0,d), D(0,b)\rangle = 0.$$

Thus, $d_1$ and $d_2$ are cyclic derivations. By the hypothesis, there are $\varphi \in \mathcal{A}^*$ and $\psi \in \mathcal{B}^*$ such that $d_1 = ad_\varphi$ and $d_2 = ad_\psi$. It follows that

$$D_1(a,0) = a \cdot \varphi - \varphi \cdot a \quad \text{and} \quad D_2(0,b) = b \cdot \psi - \psi \cdot b \tag{3.3}$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. By using (3.1),

$$\begin{aligned} aT(b) \cdot \varphi - \varphi \cdot aT(b) &= D_1(aT(b),0) = D_1((a,0)(0,b)) \\ &= a \cdot D_1(0,b) + D_1(a,0) \cdot T(b) \\ &= a \cdot D_1(0,b) + a \cdot \varphi \cdot T(b) - \varphi \cdot aT(b). \end{aligned}$$

Thus,

$$a \cdot (D_1(0,b) - ad_\varphi(T(b))) = 0 \quad (a \in \mathcal{A}).$$

Since $\mathcal{A}$ has a faithful dual space,

$$D_1(0,b) = ad_\varphi(T(b)) = T(b) \cdot \varphi - \varphi \cdot T(b) \tag{3.4}$$

and (3.3) and (3.4) imply that

$$D_1(a,b) = D_1(a,0) + D_1(0,b) = ad_\varphi(a + T(b)). \tag{3.5}$$

From (3.1) to (3.3), for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

$$\begin{aligned} D_2(aT(b),0) &= D_2((a,0)(T(b),0)) = T^*(a \cdot D_1(T(b),0)) + T^*(D_1(a,0) \cdot T(b)) \\ &= T^*(a \cdot D_1(T(b),0) + D_1(a,0) \cdot T(b)) = T^*(D_1(aT(b),0)) \\ &= T^*(aT(b) \cdot \varphi - \varphi \cdot aT(b)). \end{aligned}$$

Thus, again using (3.2) and (3.4),

$$\begin{aligned} T^*(aT(b) \cdot \varphi - \varphi \cdot aT(b)) &= D_2(aT(b),0) \\ &= D_2((a,0)(0,b)) \\ &= T^*(a \cdot D_1(0,b)) + D_2(a,0) \cdot b \\ &= T^*(a \cdot (T(b) \cdot \varphi - \varphi \cdot T(b))) + D_2(a,0) \cdot b. \end{aligned}$$

Consequently,

$$D_2(a,0) \cdot b = T^*(a \cdot \varphi \cdot T(b) - \varphi \cdot aT(b)). \tag{3.6}$$

One can easily see that

$$T^*(a \cdot \varphi \cdot T(b) - \varphi \cdot aT(b)) = T^*(a \cdot \varphi - \varphi \cdot a) \cdot b. \tag{3.7}$$

https://doi.org/10.1017/S0004972715000544 Published online by Cambridge University Press

Now, (3.6) and (3.7) together with the fact that $\mathcal{B}$ has a faithful dual space yield

$$D_2(a,0) = T^*(a\cdot\varphi - \varphi\cdot a). \tag{3.8}$$

From (3.3) and (3.8),

$$D_2(a,b) = D_2(a,0) + D_2(0,b) = T^*(ad_\varphi(a)) + ad_\psi(b) \tag{3.9}$$

for all $(a,b)\in\mathcal{A}\times_T\mathcal{B}$. Now, we have the tools to prove that $D$ is inner. Suppose that $(a,b)\in\mathcal{A}\times_T\mathcal{B}$. From (2.1), (2.2), (3.5) and (3.9), for each $(x,y)\in\mathcal{A}\times_T\mathcal{B}$,

$$\begin{aligned}
&\langle D(a,b),(x,y)\rangle\\
&\quad = \langle (D_1(a,b), D_2(a,b)),(x,y)\rangle\\
&\quad = \langle D_1(a,b),x\rangle + \langle D_2(a,b),y\rangle\\
&\quad = \langle (a+T(b))\cdot\varphi - \varphi\cdot(a+T(b)),x\rangle + \langle T^*(a\cdot\varphi-\varphi\cdot a)+b\cdot\psi-\psi\cdot b,y\rangle\\
&\quad = \langle a\cdot\varphi + T(b)\cdot\varphi, x\rangle + \langle T^*(a\cdot\varphi)+b\cdot\psi,y\rangle\\
&\qquad - \langle \varphi\cdot a+\varphi\cdot T(b),x\rangle - \langle T^*(\varphi\cdot a)+\psi\cdot b,y\rangle\\
&\quad = \langle (a\cdot\varphi+T(b)\cdot\varphi, T^*(a\cdot\varphi)+b\cdot\psi),(x,y)\rangle\\
&\qquad - \langle (\varphi\cdot a+\varphi\cdot T(b), T^*(\varphi\cdot a)+\psi\cdot b),(x,y)\rangle\\
&\quad = \langle (a,b)\cdot(\varphi,\psi) - (\varphi,\psi)\cdot(a,b),(x,y)\rangle\\
&\quad = \langle ad_{(\varphi,\psi)}(a,b),(x,y)\rangle.
\end{aligned}$$

Thus, $D = ad_{(\varphi,\psi)}$ and so $D$ is inner. Therefore, $\mathcal{A}\times_T\mathcal{B}$ is cyclic amenable, as claimed. □

In the next result, we prove the converse of Theorem 3.2, without any extra assumption. This generalises the converse [2, Theorem 4.1, part (4)] for an arbitrary Banach algebra $\mathcal{A}$.

Theorem 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $T\in\mathrm{hom}(\mathcal{B},\mathcal{A})$. If $\mathcal{A}\times_T\mathcal{B}$ is cyclic amenable, then both $\mathcal{A}$ and $\mathcal{B}$ are cyclic amenable.*

Proof. Suppose that $d_1:\mathcal{A}\to\mathcal{A}^*$ is a cyclic derivation and let $D_1 = r_\mathcal{A}^*\circ d_1\circ r_\mathcal{A}$. We show that $D_1:\mathcal{A}\times_T\mathcal{B}\to\mathcal{A}^*\times\mathcal{B}^*$ is a cyclic derivation. It is easily verified that for all $f\in\mathcal{A}^*$ and $(a,b)\in\mathcal{A}\times_T\mathcal{B}$,

$$(a,b)\cdot r_\mathcal{A}^*(f) = r_\mathcal{A}^*((a+T(b))\cdot f) \tag{3.10}$$

and

$$r_\mathcal{A}^*(f)\cdot(a,b) = r_\mathcal{A}^*(f\cdot(a+T(b))). \tag{3.11}$$

https://doi.org/10.1017/S0004972715000544 Published online by Cambridge University Press

Using (3.10) and (3.11), for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A} \times_T \mathcal{B}$,

$$\begin{aligned}
&(a_1, b_1) \cdot D_1(a_2, b_2) + D_1(a_1, b_1) \cdot (a_2, b_2) \\
&\quad = (a_1, b_1) \cdot [r^*_{\mathcal{A}}(d_1(a_2 + T(b_2)))] + [r^*_{\mathcal{A}}(d_1(a_1 + T(b_1)))] \cdot (a_2, b_2) \\
&\quad = r^*_{\mathcal{A}}[(a_1 + T(b_1)) \cdot (d_1(a_2) + d_1(T(b_2)))] \\
&\qquad + r^*_{\mathcal{A}}[(d_1(a_1) + d_1(T(b_1))) \cdot (a_2 + T(b_2))] \\
&\quad = r^*_{\mathcal{A}}[d_1(a_1 a_2) + d_1(a_1 T(b_2)) + d_1(T(b_1) a_2) + d_1(T(b_1) T(b_2))] \\
&\quad = r^*_{\mathcal{A}} \circ d_1(a_1 a_2 + a_1 T(b_2) + T(b_1) a_2 + T(b_1 b_2)) \\
&\quad = r^*_{\mathcal{A}} \circ d_1 \circ r_{\mathcal{A}}(a_1 a_2 + a_1 T(b_2) + T(b_1) a_2, b_1 b_2) \\
&\quad = D_1((a_1, b_1)(a_2, b_2)).
\end{aligned}$$

Thus, $D_1$ is a derivation. We next show that $D_1$ is cyclic. Since $d_1$ is a cyclic derivation, for all $(a_1, b_1), (a_2, b_2) \in \mathcal{A} \times_T \mathcal{B}$,

$$\begin{aligned}
&\langle D_1(a_1, b_1), (a_2, b_2)\rangle + \langle D_1(a_2, b_2), (a_1, b_1)\rangle \\
&\quad = \langle r^*_{\mathcal{A}}(d_1(a_1 + T(b_1))), (a_2, b_2)\rangle + \langle r^*_{\mathcal{A}}(d_1(a_2 + T(b_2))), (a_1, b_1)\rangle \\
&\quad = \langle d_1(a_1 + T(b_1)), (a_2 + T(b_2))\rangle + \langle d_1(a_2 + T(b_2)), (a_1 + T(b_1))\rangle \\
&\quad = 0,
\end{aligned}$$

which implies that $D_1$ is cyclic. Since $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable, it follows that $D_1$ is inner. Thus, there are $\varphi_1 \in \mathcal{A}^*$ and $\psi_1 \in \mathcal{B}^*$ such that $D_1 = ad_{(\varphi_1, \psi_1)}$. Consequently, for each $a \in \mathcal{A}$,

$$D_1(a, 0) = (a, 0) \cdot (\varphi_1, \psi_1) - (\varphi_1, \psi_1) \cdot (a, 0).$$

Using this equality together with (2.1) and (2.2),

$$D_1(a, 0) = r^*_{\mathcal{A}}(d_1(a)) = (a \cdot \varphi_1 - \varphi_1 \cdot a, T^*(a \cdot \varphi_1 - \varphi_1 \cdot a)). \tag{3.12}$$

Moreover, for all $(c, d) \in \mathcal{A} \times_T \mathcal{B}$,

$$\begin{aligned}
\langle r^*_{\mathcal{A}}(d_1(a)), (c, d)\rangle &= \langle d_1(a), c + T(d)\rangle \\
&= \langle d_1(a), c\rangle + \langle T^*(d_1(a)), d\rangle \\
&= \langle (d_1(a), T^*(d_1(a))), (c, d)\rangle.
\end{aligned}$$

Thus,

$$r^*_{\mathcal{A}} \circ d_1(a) = (d_1(a), T^*(d_1(a))). \tag{3.13}$$

Now, (3.12) and (3.13) imply that $d_1 = ad_{\varphi_1}$ and so $d_1$ is inner. Therefore, $\mathcal{A}$ is cyclic amenable. Similarly, we show that $\mathcal{B}$ is cyclic amenable. Suppose that $d_2 : \mathcal{B} \to \mathcal{B}^*$ is a cyclic derivation and let $D_2 = p^*_{\mathcal{B}} \circ d_2 \circ p_{\mathcal{B}}$. It is not hard to see that for all $(a, b) \in \mathcal{A} \times_T \mathcal{B}$ and $g \in \mathcal{B}^*$,

$$(a, b) \cdot p^*_{\mathcal{B}}(g) = p^*_{\mathcal{B}}(b \cdot g) \quad \text{and} \quad p^*_{\mathcal{B}}(g) \cdot (a, b) = p^*_{\mathcal{B}}(g \cdot b).$$

By an argument similar to the proof of the first part, $D_2 : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}^* \times \mathcal{B}^*$ is a cyclic derivation. It follows that there are $\varphi_2 \in \mathcal{A}^*$ and $\psi_2 \in \mathcal{B}^*$ such that $D_2 = ad_{(\varphi_2,\psi_2)}$. Using (2.1) and (2.2), for all $b \in \mathcal{B}$,

$$D_2(0, b) = (T(b) \cdot \varphi_2 - \varphi_2 \cdot T(b), b \cdot \psi_2 - \psi_2 \cdot b).$$

Thus, for all $b, d \in \mathcal{B}$,

$$\begin{aligned}\langle D_2(0, b), (0, d)\rangle &= \langle (T(b) \cdot \varphi_2 - \varphi_2 \cdot T(b), b \cdot \psi_2 - \psi_2 \cdot b), (0, d)\rangle \\ &= \langle T(b) \cdot \varphi_2 - \varphi_2 \cdot T(b), 0\rangle + \langle b \cdot \psi_2 - \psi_2 \cdot b, d\rangle \\ &= \langle b \cdot \psi_2 - \psi_2 \cdot b, d\rangle.\end{aligned}$$

On the other hand, by the definition of $D_2$,

$$\langle D_2(0, b), (0, d)\rangle = (d_2(b), d)$$

and consequently

$$d_2(b) = b \cdot \psi_2 - \psi_2 \cdot b$$

for all $b \in \mathcal{B}$. It follows that

$$d_2 = ad_{\psi_2},$$

which implies that $d_2$ is inner. Therefore, $\mathcal{B}$ is cyclic amenable, as claimed. □

Remark 3.4. In [4, Theorem 2.2], part (iii), it is mentioned that part (4) of [2, Theorem 4.1] is valid for an arbitrary Banach algebra $\mathcal{A}$ with the same proof as given in [2]. However, in the light of the earlier discussion and Theorem 3.2, the result given in part (iii) of [4, Theorem 2.2] may suffer from the same gap as the proof in [2]. We have not yet been able to prove or provide a counterexample for these results in [4].

Theorem 3.2 leads us to the following natural question.

Question 3.5. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $T \in \hom(\mathcal{B}, \mathcal{A})$ be such that $\mathcal{A}$ and $\mathcal{B}$ are cyclic amenable. Is $\mathcal{A} \times_T \mathcal{B}$ always cyclic amenable?

## Acknowledgement

The authors thank Dr P. A. Dabhi for his suggestion of adding an extra condition in the assumptions of Theorem 3.2 and for providing the proof for the commutative case. His proof led us to present the result for the general case.

## References

[1] F. Abtahi, A. Ghafarpanah and A. Rejali, ‘Biprojectivity and biflatness of Lau product of Banach algebras defined by a Banach algebra morphism’, *Bull. Aust. Math. Soc.* **91** (2015), 134–144.
[2] S. J. Bhatt and P. A. Dabhi, ‘Arens regularity and amenability of Lau product of Banach algebras defined by a Banach algebra morphism’, *Bull. Aust. Math. Soc.* **87** (2013), 195–206.
[3] N. Gronbaek, ‘Weak and cyclic amenability for non-commutative Banach algebras’, *Proc. Edinb. Math. Soc.* (2) **35** (1992), 315–328.

[4] H. Javanshiri and M. Nemati, 'On a certain product of Banach algebras and some of its properties', *Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci.* **15**(3) (2014), 219–227.

[5] A. R. Khoddami and H. R. Ebrahimi Vishki, 'Biflatness and biprojectivity of Lau product of Banach algebras', *Bull. Iranian Math. Soc.* **39**(3) (2013), 559–568.

F. ABTAHI, Department of Mathematics,
University of Isfahan, PO Box 81746-73441,
Isfahan, Iran
e-mail: abtahi.fatemeh@yahoo.com, f.abtahi@sci.ui.ac.ir

A. GHAFARPANAH, Department of Mathematics,
University of Isfahan, PO Box 81746-73441,
Isfahan, Iran
e-mail: ghafarpanah@sci.ui.ac.ir, ghafarpanah2002@gmail.com

https://doi.org/10.1017/S0004972715000544 Published online by Cambridge University Press