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GENTLE PERTURBATIONS OF $i\frac{d}{dx}$ WITH APPLICATION TO $-\frac{d^2}{dx^2}+q$

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Introduction. The theory of gentle perturbations was introduced by Friedrichs [**3**] as a tool to study the perturbation theory of the absolutely continuous spectrum of a self-adjoint operator H_0 and developed in an abstract form by Rejto [**7**; **8**]. Two examples of gentle structures are well known. In the first of these, the gentle operators have Hölder continuous complex or operator-valued kernels, and in the second, the kernels are Fourier transforms of L_1 functions [**4**].

The gentle structure has traditionally been verified in the case when H_0 is in its spectral representation, that is, when H_0 is the simple differentiation operator. This is not the natural setting for the second example mentioned above where one should consider the simple differentiation operator in a suitable L_2-space and perturbations with L_1 kernels. This point of view also has the advantage of yielding the necessary estimates easily.

We present a development of the theory in this setting in our paper, studying unitary equivalence of the perturbed and unperturbed operators as well as the existence of the scattering operator. We then derive an explicit representation of $-d^2/dx^2$ on $L_2(0,\infty)$ as a simple differentiation operator on a suitable H_2-space and apply the gentle perturbation theory to potentials satisfying suitable smoothness and growth conditions.

Notation. In this paper $\mathscr{R}$ will denote the real numbers, $\mathscr{R}^+$ the non-negative real numbers, and **C** the complex numbers. Let **N** be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. We define the space $L_p(\mathscr{R},\mathbf{N})$ to consist of all measurable functions from the real line into **N** for which

$$\int |f(x)|^p\,dx < \infty .$$

Whenever the space **N** is evident from the context we shall write simply L_p. It can be easily seen that $L_p(\mathscr{R},\mathbf{N})$ is a Banach space whose dual space is $L_q(\mathscr{R},\mathbf{N})$ where $1/p + 1/q = 1$.

In particular, it is well known that $L_2(\mathscr{R},\mathbf{N})$ is a Hilbert space. We shall

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denote this space by $\mathscr{H}$ and its inner product and norm by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively.

In writing integrals we shall use a single integral sign except in the case of iterated integrals and leave the dimension of the integral to be deduced from the domain of integration or the differential element. The range of integration will be omitted if it extends over the whole space under consideration.

If $\mathbf{X}$ is any Hilbert space, we shall let $\mathscr{B}(\mathbf{X})$ denote the space of bounded operators from $\mathbf{X}$ into itself with the operator norm topology. The operator norm in $\mathscr{B}(\mathbf{N})$ will be denoted by $|\cdot|$.

The gentle perturbation theory. We begin by stating the main theorem.

THEOREM 1. *Let A be the unique self-adjoint extension of $i\ d/dx$ in $\mathscr{H} = L_2(\mathscr{R}, \mathbf{N})$ and suppose that $K \in L_1(\mathscr{R}^2, \mathscr{B}(\mathbf{N}))$ has the property that Kf, defined by*

$$Kf(x) = \int K(x, y)f(y)\, dy,$$

maps $L_\infty(\mathscr{R}, \mathbf{N}) \cap L_1(\mathscr{R}, \mathbf{N})$ into $\mathscr{H}$. Then if $|K|_1 < 1$, there exists a projection P which commutes with $A + K$ such that $(A + K)P$ is unitarily equivalent to A. If $|K|_1 < \frac{1}{2}$, then we can conclude that $P = I$.

The proof of this theorem is algebraic after a certain structure has been established and appears at the end. The structure, devised by Friedrichs [**3**] and Rejto [**7**] is known as a *Gentle System*, and is developed in the definitions and lemmas which follow.

Let $\mathbf{B}$ be the set of measurable $\mathscr{B}(\mathbf{N})$-valued functions on $\mathscr{R}^2$ for which

$$|R|_1 = \int |R(x, y)|\, dx\, dy < \infty.$$

It is well known that $(\mathbf{B}, |\cdot|_1)$ is a Banach space.

It is evident that for $R \in \mathbf{B}$, $|R(x, y)|$ will be integrable along lines of the form $x - y = c$ for almost every c. This implies that the integral $\int_0^\infty R(x + u, y + u)\, du$ will exist for almost every pair (x, y) and yield a measurable $\mathscr{B}(\mathbf{N})$-valued function of (x, y). Hence, we can define a mapping Γ from $(\mathbf{B}, |\cdot|_1)$ into the set of measurable $\mathscr{B}(\mathbf{N})$-valued functions on $\mathscr{R}^2$ as follows:

$$\Gamma R(x, y) = -i \int_0^\infty R(x + u, y + u)\, du. \tag{1}$$

Since, for any $f \in L_\infty(\mathscr{R}, \mathbf{N})$, we have

$$\left|\int R(x, y)f(y)\, dy\right| \leqq \int |R(x, y)f(y)|\, dy \leqq \int |R(x, y)|\, |f(y)|\, dy \leqq \|f\|_\infty \int |R(x, y)|\, dy, \tag{2}$$

it is clear that R can be treated as the kernel of an integral operator mapping $L_\infty(\mathscr{R}, \mathbf{N})$ into $L_1(\mathscr{R}, \mathbf{N})$. Now suppose that $f \in L_\infty(\mathscr{R}, \mathbf{N})$ and $g = (\Gamma R)f$.

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Then

$$|g(x)| \leqq \int dy \int_0^\infty du |R(x+u, y+u)|\, |f(y)| \tag{3}$$

$$\leqq \int dy \int_0^\infty du |R(x+u, y+u)|\, ||f||_\infty$$

$$\leqq |R|_1\, ||f||_\infty.$$

It follows that ΓR is the kernel of a bounded operator mapping $L_\infty(\mathscr{R}, \mathbf{N})$ into itself. By a similar inequality we can show that ΓR is also the kernel of a bounded operator mapping $L_1(\mathscr{R}, \mathbf{N})$ into itself.

It follows from a generalization of the Riesz convexity theorem [**1**, Part I, p. 536, problem 39] that ΓR is the kernel of a bounded operator mapping $L_2(\mathscr{R}, \mathbf{N})$ into itself with norm not exceeding $|R|_1$. That is,

$$||\Gamma R||_{2,2} \leqq |R|_1. \tag{4}$$

If we have two $\mathscr{B}(\mathbf{N})$-valued functions $F(x, y)$ and $G(y, z)$ we shall use $(FG)(x, z)$ to denote the product of F and G considered as kernels of operators and defined by:

$$(FG)(x, z) = \int F(x, y)G(y, z)\, dy$$

whenever this integral exists.

In particular, if $C, D \in \mathbf{B}$, we prove below in Lemma 1 that $C(\Gamma D)$ and $(\Gamma C)D$ exist almost everywhere and belong to $\mathbf{B}$. Furthermore, $(\Gamma C)(\Gamma D)$ exists almost everywhere as shown by the following argument.

Suppose $-\infty < \alpha < \beta < \infty$. Since

$$\int_\alpha^\beta dx \int dz \int dy |\Gamma C(x, y)|\, |\Gamma D(y, z)|$$

$$\leqq \int_\alpha^\beta dx \int dz \int dy \int du \int dv |C(x+u, y+u)|\, |D(y+v, z+v)|$$

$$= |\beta - \alpha|\, |C|_1\, |D|_1 < \infty,$$

it follows that $[(\Gamma C)(\Gamma D)](x, z)$ exists for almost every pair (x, z).

LEMMA 1. *Whenever C and D belong to* $\mathbf{B}$ *we have*

$$|C(\Gamma D)|_1 \leqq |C|_1 |D|_1 \quad \textit{and} \quad |(\Gamma C)D|_1 \leqq |C|_1 |D|_1. \tag{5}$$

Proof. Certainly

$$|C(\Gamma D)(x, z)| \leqq \int dy \int du |C(x, y)|\, |D(y+u, z+u)|.$$

If we now integrate this inequality over x and z, and make the change of variables

$$z' = z + u, \qquad u' = y + u,$$

a suitable change in the order of integration yields

$$|C(\Gamma D)|_1 \leqq \int dx \int dy |C(x, y)| \, |D|_1 = |C|_1 |D|_1$$

which is the first inequality. The second is proved similarly.

The identity proved in the following lemma is basic for the theory.

LEMMA 2. *Whenever $C, D \in \mathbf{B}$ we have*

$$[(\Gamma C)(\Gamma D)](x, z) = [\Gamma\{C(\Gamma D) + (\Gamma C)D\}](x, z)$$

for almost every pair (x, z).

Proof. From the definitions we have

$$(6) \quad [\Gamma\{C(\Gamma D)\}](x, z) = -\int_0^\infty du \int dy \int_0^\infty dv C(x + u, y) D(y + v, z + u + v).$$

We make the following variable changes:

$$\begin{aligned} x + u_1 &= x + u, \\ y_1 + u_1 &= y, \\ y_1 + v_1 &= y + v, \\ z + v_1 &= z + u + v, \end{aligned}$$

that is

$$u_1 = u, \qquad v_1 = u + v, \qquad y_1 = y - u.$$

The old region $\{u \geqq 0, v \geqq 0\}$ maps into $\{u_1 \geqq 0, v_1 - u_1 \geqq 0\}$, which we denote by T_1. The right side of (6) then becomes

$$(6') \qquad -\int_{T_1} du_1 \, dv_1 \, dy_1 \, C(x + u_1, y_1 + u_1) D(y_1 + v_1, z + v_1).$$

In a similar fashion we can show that

$$(7) \quad [\Gamma\{(\Gamma C)D\}](x, z) = -\int_{T_2} du_1 dv_1 dy_1 \, C(x + u_1, y_1 + u_1) D(y_1 + v_1, z + v_1),$$

where $T_2 = \{(u_1, v_1): u_1 \geqq 0, u_1 - v_1 \geqq 0\}$. It is clear that T_1 and T_2 intersect in the line $u_1 = v_1$ and have the first quadrant as their union.

Combining (6′) and (7) we obtain

$$\begin{aligned} [\Gamma\{(\Gamma C)D + C(\Gamma D)\}](x, z) &= \int_0^\infty du \int_0^\infty dv \int dy C(x + u, y + u) D(y + v, z + v) \\ &= [(\Gamma C)(\Gamma D)(x, z)] \end{aligned}$$

which completes the proof of the lemma.

We complete our discussion of $(\mathbf{B}, |\cdot|_1)$ and Γ with the following straightforward result.

LEMMA 3. *Given* $K \in \mathbf{B}$ *with* $|K|_1 < 1$, *there exists a unique* $R \in \mathbf{B}$ *such that* $R = K + K(\Gamma R)$.

Proof. We solve the equation by iteration, defining

$$R_0 = K, \qquad R_{n+1} = K + K(\Gamma R_n), \qquad n \geqq 0.$$

Clearly $R_n \in \mathbf{B}$ for all $n \geqq 0$ and $R_{n+1} - R_n = K\Gamma(R_n - R_{n-1})$.

By Lemma 1,

$$\begin{aligned} |R_{n+1} - R_n|_1 &\leqq |K|_1 |R_n - R_{n-1}|_1 \\ &\leqq |K|_1^n |R_1 - R_0|_1. \end{aligned}$$

It follows then that $R_0 + (R_1 - R_0) + (R_2 - R_1) + \ldots$ is dominated in norm by the convergent series

$$|R_0|_1 + \sum_n |K|_1^n |R_1 - R_0|_1$$

and therefore $\lim_{n\to\infty} R_n = R$ exists. By taking the limit as $n \to \infty$ in $R_{n+1} = K + K(\Gamma R_n)$ we deduce $R = K + K(\Gamma R)$. R is unique since if there were two solutions R_1 and R_2 we would have $R_1 - R_2 = K[\Gamma(R_1 - R_2)]$, from which $|R_1 - R_2|_1 < |R_1 - R_2|_1$, implying $R_1 - R_2 = 0$.

We now proceed to a study of the behaviour of elements of $\mathbf{B}$ and $\Gamma\mathbf{B}$ as kernels of operators in $\mathscr{H}$. Let $L_\infty^0(\mathscr{R}, \mathbf{N})$ denote the linear manifold of bounded $\mathbf{N}$-valued functions with compact support on the real line. Given $R \in \mathbf{B}$, and $f \in L_\infty^0$, we shall consider an operator U in $\mathscr{H}$ given by

$$Uf(x) = f(x) + \int \Gamma R(x, t) f(t)\, dt. \tag{8}$$

Although U as defined by (8) has domain $L_\infty^0(\mathscr{R}, \mathbf{N})$, it can be extended as a bounded operator to all of $\mathscr{H}$ because of (4). When we refer to U we shall mean this extension.

Let $R^*(x, y) = \overline{R(y, x)}$ denote the adjoint kernel of R.

LEMMA 4. *If we assume that* $R \in \mathbf{B}$ *satisfies*

$$\Gamma R(x, y) - \Gamma R^*(x, y) = [(\Gamma R^*)(\Gamma R)](x, y) \quad \textit{for almost every pair } (x, y),$$

then U *is isometric as an operator in* L_2.

Proof. Let f and g be bounded and of compact support. Then $Uf, Ug \in L_2$ and, using (8),

$$\begin{aligned} (Uf, Ug) = \int dx \{ f(x)\overline{g(x)} &+ \left(\int \Gamma R(x, y) f(y)\, dy\right) \overline{g(x)} \\ &+ f(x) \int \overline{\Gamma R(x, z)}\, \overline{g(z)}\, dz + \left(\int \Gamma R(x, y) f(y)\, dy\right)\left(\int \overline{\Gamma R(x, z)}\, \overline{g(z)}\, dz\right)\}. \end{aligned} \tag{9}$$

It is a simple matter to check that all the terms in (9) are absolutely integrable so that orders of integration can be interchanged at will. Making use of this fact, we find that after a suitable change of variables the last three terms in (9) add up to zero by virtue of our hypothesis. That is

$$(Uf, Ug) = (f, g). \tag{10}$$

Since U is bounded and L_∞^0 is dense in $\mathscr{H}$, this relation holds on all of $\mathscr{H}$. The proof is complete.

Remark. It is evident that if $|R|_1 < 1$, then ΓR is a contraction and therefore $U = I + \Gamma R$ must have full range. This implies, of course, that U is unitary.

In case $R = K + K(\Gamma R)$ and $|K|_1 < \frac{1}{2}$, we have

$$\begin{aligned}|R|_1 &\leqq |K|_1 + |K|_1|R|_1 \\ &< \tfrac{1}{2} + \tfrac{1}{2}|R|_1\end{aligned}$$

so that $|R|_1 < 1$ and U is unitary.

We complete our study of U with the following lemma.

Lemma 5. *The hypotheses of Lemma 4 are satisfied if R is a solution of $R = K + K(\Gamma R)$ and K is formally self-adjoint, i.e. $K = K^*$.*

Proof. By Lemma 2,

$$\begin{aligned}\Gamma R - \Gamma R^* - (\Gamma R^*)(\Gamma R) &= \Gamma R - \Gamma R^* - \Gamma\{R^*(\Gamma R) + (\Gamma R^*)R\} \\ &= \Gamma\{(I - \Gamma R^*)R - R^*(I + \Gamma R)\}.\end{aligned}$$

Now $R = K + K(\Gamma R)$ implies

$$R^* = K^* - (\Gamma R^*)K^* = (I - \Gamma R^*)K,$$

from which $(I - \Gamma R^*)R = R^*(I + \Gamma R)$, completing the proof.

The next lemma establishes the link between the gentle structure and the operator A. As is customary, $C_0^1(\mathscr{R}, \mathbf{N})$ denotes the set of continuously differentiable functions of compact support on the real line.

Lemma 6. *If $f \in C_0^1(\mathscr{R}, \mathbf{N})$ and $Rf \in \mathscr{H}$, then $(\Gamma R)f \in \mathscr{D}(A)$, and $(\Gamma R)Af - A(\Gamma R)f = Rf$.*

Proof. Since f' is both bounded and integrable, it follows that $(\Gamma R)Af \in L_1 \cap L_\infty$ also and therefore $(\Gamma R)Af \in L_2$.

We now consider the second term. Again we have

$$f \in L_1 \cap L_\infty \Rightarrow (\Gamma R)f \in L_1 \cap L_\infty \subset L_2.$$

In order to justify integration by parts in the formula

$$(\Gamma Rf)(x) = \int dy\left(\int_0^\infty (-iR(x + u, y + u))\, du\right)f(y), \tag{11}$$

we fix a value of x and let

$$G(y) = \int_{-\infty}^{y} d\eta \int_0^\infty (-iR(x + u, \eta + u))\, du.$$

Clearly, $|G(y)| \leqq |R|_1$ and if $[c, d]$ is any interval, then

$$|G(d) - G(c)| \leqq \int_c^d d\eta \int_0^\infty |R(x + u, \eta + u)| \, du.$$

Consequently we have the estimate

$$\begin{aligned}|G(d)f(d) - G(c)f(c)| &\leqq |G(d) - G(c)| \, |f(d)| + |G(c)| \, |f(d) - f(c)| \\ &\leqq |f(d)| \left| \int_c^d dy \int_0^\infty |R(x+u, y+u)| \, du + |R|_1 \, |f(d) - f(c)| \right|.\end{aligned}$$

Since f is assumed strongly continuously differentiable, it follows easily from this estimate that $G(y)f(y)$ is of strongly bounded variation and weakly absolutely continuous [**5**, Definitions 3.2.4 and 3.6.2].

From [**5**, p. 88, Corollary 2] we conclude that

$$G'(y) = \int_0^\infty (-iR(x + u, y + u)) \, du \quad \text{for almost every } x$$

with convergence in the $|\cdot|$-norm ($\mathscr{B}(\mathbf{N})$-norm). Consequently, $G(y)f(y)$ is strongly differentiable in the $\mathbf{N}$-norm with derivative

$$G'(y)f(y) + G(y)f'(y).$$

It is easy to see that each term above is Bochner integrable. Consequently, using [**5**, Theorem 3.86], and choosing (c, d) to contain the support of f, we conclude that

$$0 = \int_c^d \frac{d}{dy} [G(y)f(y)] \, dy = \int_c^d G'(y)f(y) \, dy + \int_c^d G(y)f'(y) \, dy.$$

This is the required integration by parts formula. Using it, we deduce from (11) that

$$(\Gamma Rf)(x) = \int dy \left(\int_{-\infty}^y d\eta \int_0^\infty iR(x + u, \eta + u) \, du \right) f'(y).$$

This has been preparatory to showing that $(\Gamma Rf)(x)$ is differentiable (absolutely continuous) and that its derivative is square integrable.

It follows from [**5**, Theorem 3.7.12] that if the $\mathscr{B}(\mathbf{N})$-valued function $F(x, y)$ is strongly absolutely continuous and differentiable for almost every pair (x, y) and $(\partial F/\partial x)(x, y)$ is locally Bochner integrable with respect to $dx\,dy$, then $\int F(x, y) \, dy$ is strongly absolutely continuous with respect to x and

$$\frac{d}{dx} \int F(x, y) \, dy = \int \frac{\partial F}{\partial x} (x, y) \, dy.$$

Our problem then boils down to showing that

$$\int_{-\infty}^y d\eta \left(\int_0^\infty iR(x + u, \eta + u) \, du \right) f'(y)$$

is an absolutely continuous function of x and that its derivative with respect to x is integrable over every bounded square.

Clearly it suffices to show these properties for

$$I = \int_{-\infty}^{y} d\eta \int_{0}^{\infty} iR(x + u, \eta + u)\, du$$
$$= \int_{S_{x,y}} iR(\xi, \eta)\, d\xi\, d\eta,$$

where $S_{x,y}$ is the union of the regions

$$S_1 = \{(\xi, \eta) : \eta \geqq y, \xi \geqq x - y + \eta\}$$

and

$$S_2 = \{(\xi, \eta) : \eta \leqq y, \xi \geqq x\}.$$

Since $\int_{S_{x,y}} = \int_{S_1} + \int_{S_2}$,

$$I = \int_{x}^{\infty} d\xi \int_{0}^{\infty} iR(\xi + u, y + u)\, du + \int_{x}^{\infty} d\xi \int_{-\infty}^{y} iR(\xi, \eta)\, d\eta.$$

It is obvious that the above function is absolutely continuous with respect to x and that the derivative is

$$-\int_{0}^{\infty} iR(x + u, y + u)\, du - \int_{-\infty}^{y} iR(x, \eta)\, d\eta \quad \text{for almost every pair } (x, y).$$

We conclude that

$$\left[\int_{-\infty}^{y} d\eta \int_{0}^{\infty} (-iR(x + u, \eta + u))\, du\right] f'(y)$$

is absolutely continuous and has derivative

$$\left[\int_{0}^{\infty} iR(x + u, y + u)\, du + \int_{-\infty}^{y} iR(x, \eta)\, d\eta\right] f'(y).$$

It is easy to see that this expression is locally integrable with respect to $dx\, dy$ because $f'(t)$ is bounded and R has integrable norm.

It follows therefore that

$$(\Gamma R) f(x) = \int dy \left(\int_{0}^{\infty} (-iR(x + u, y + u))\, du\right) f(y)$$

is square integrable and absolutely continuous with derivative

$$\int dy \left(\int_{0}^{\infty} (-iR(x + u, y + u))\, du + \int_{-\infty}^{y} (-iR(x, \eta))\, d\eta\right) f'(y).$$

We have already shown that

$$\int dy \left(\int_{0}^{\infty} (-iR(x + u, y + u))\, du\right) f'(y)$$

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is square integrable and by assumption $\int R(x, y)f(y)\,dy$ is square integrable.

We note that

$$\int dy\left(\int_{-\infty}^{y} (-iR(x,\eta))\,d\eta\right)f'(y)$$

is norm bounded for almost every x and integrable. Hence we can integrate this expression by parts to obtain

$$\int iR(x, y)f(y)\,dy \quad \text{for almost every } x.$$

Finally we combine these facts to deduce that $(\Gamma Rf)(x)$ is square integrable and that

$$(\Gamma R)Af(s) - A(\Gamma R)f(s) = i\int (-iR(x, y)f(y))\,dy = \int R(x, y)f(y)\,dy,$$

completing the proof of the lemma.

In view of the fact that we have only proved Lemma 6 under the hypothesis $f \in C_0^1(\mathscr{R}, \mathrm{N})$, we shall need Lemmas 7 and 8 to make an extension.

LEMMA 7. *Let A and B be closed (this is certainly true if they are self-adjoint) operators in $\mathscr{H}$ and let U be an isometry. Suppose that the manifold $\mathscr{M}$ is dense in $\mathscr{D}(A)$ with respect to the graph norm corresponding to A and that $BU = UA$ on $\mathscr{M}$. Then, if $f \in \mathscr{D}(A)$, we have $Uf \in \mathscr{D}(B)$ and $BUf = UAf$.*

Proof. Let $f \in \mathscr{D}(A)$ and $\{f_n\} \subset \mathscr{M}$ such that

$$\|f - f_n\| + \|A(f - f_n)\| \to 0.$$

Then $UAf_n \to UAf$ implying $BUf_n \to UAf$. Since we also have that $Uf_n \to Uf$, we appeal to the assumption that B is closed to conclude

$$Uf \in \mathscr{D}(B) \qquad \text{and} \qquad BUf = UBf.$$

The next lemma shows that $\mathscr{M} = C_0^1$ is dense in $\mathscr{D}(A)$.

LEMMA 8. *C_0^∞ is dense in $\mathscr{D}(A)$ under the graph norm.*

Proof. Let $f \in \mathscr{D}(A)$ and let $\tilde{f}$ denote the L_2 Fourier transform of f. That is,

$$\tilde{f}(\lambda) = \text{l.i.m.}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-i\lambda x}f(x)\,dx,$$

where l.i.m. denotes limit in the L_2-sense. The graph norm of f is given by

$$|f|_G = \int (|f|^2 + |f'|^2) = \int (1 + \lambda^2)|\tilde{f}(\lambda)|^2\,d\lambda = |\tilde{f}|_G.$$

We can choose an interval $J = (\alpha, \beta)$ large enough that $\tilde{g} = \chi_I\tilde{f}$ satisfies $|\tilde{f} - \tilde{g}|_G < \frac{1}{2}\epsilon$. χ_I here denotes the characteristic function of I. Then g, the inverse transform of $\tilde{g}$, belongs to C^∞.

Let σ be an infinitely differentiable real-valued function with support in $(\alpha - n - 1, \beta + n + 1)$ which assumes the value 1 on J and satisfies $0 \leqq \sigma' \leqq 1/n$. Then $h = \sigma g \subset C_0^\infty(\mathscr{R}, \mathrm{N})$. Furthermore,

$$g - h = (1 - \sigma)g \qquad \text{and} \qquad g' - h' = (1 - \sigma)g' - \sigma' g.$$

It follows that $|h - g|_2^2 \leqq \int_{J^c} |g|^2$ (J^c is the complement of J) and

$$|h' - g'|_2^2 \leqq 2\int |\sigma'|^2 |g|^2 + 2\int_{J^c} |g'|^2$$

$$\leqq 2 \cdot \frac{2(n+1)}{n^2} \max |g|^2 + 2\int_{J^c} |g'|^2$$

from which

$$|h - g|_G \leqq \frac{4(n+1)}{n^2} \max |g|^2 + 2\int_{J^c} |g'|^2 + \int_{J^c} |g|^2$$

so that the right-hand side is less than $\frac{1}{2}\epsilon$ if J and n are large enough. Finally,

$$|f - h|_G \leqq |f - g|_G + |g - h|_G < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon,$$

completing the proof of the lemma.

Proof of Theorem 1. If K satisfies the hypothesis of the theorem, let R be the unique solution of $R = K + K(\Gamma R)$ (Lemma 3). Let U denote the isometry $I + \Gamma R$ (Lemmas 4 and 5) and let $f \in C_0^\infty(\mathscr{R}, \mathbf{N})$.

We now show that

$$(A + K)Uf = UAf. \tag{12}$$

Since $(\Gamma R)f \in L_1 \cap L_\infty$, it follows that $(\Gamma R)f \in \mathscr{D}(K)$, and from Lemma 6 we conclude that $(\Gamma R)f \in \mathscr{D}(A)$. Hence $Uf \in \mathscr{D}(A + K)$, and

$$(A + K)Uf = Af + A(\Gamma R)f + Kf + K(\Gamma R)f.$$

Likewise, since $Af \in C_0$, it follows that $Af \in \mathscr{D}(\Gamma R) = \mathscr{D}(U)$ and

$$UAf = Af + (\Gamma R)Af.$$

Then

$$\begin{aligned} UAf - (A + K)Uf &= (\Gamma R)Af - A(\Gamma R)f - Kf - K(\Gamma R)f \\ &= Rf - Kf - K(\Gamma R)f \quad \text{(by Lemma 6)} \\ &= \{R - K - K(\Gamma R)\}f \\ &= 0. \end{aligned}$$

By Lemmas 7 and 8 we extend (12) to all $f \in \mathscr{D}(A)$.

This completes the proof of the theorem if we remark that the projection P is simply $P = UU^*$, in the case when $\frac{1}{2} \leqq |K|_1 < 1$. We have already noted following Lemma 4 that U is unitary when $|K|_1 < \frac{1}{2}$, implying that in this case $P = I$.

It turns out that in applications the following corollary is useful.

COROLLARY 1. *Suppose that the hypotheses of Theorem* 1 *are satisfied and in addition that* K *maps a reducing subspace* $\mathscr{H}'$ *of* A *into itself. Let* P' *be the projection of* $\mathscr{H}$ *onto* $\mathscr{H}' \subset \mathscr{H}$. *Then the isometry* U *and the projection* P *in the proof of Theorem* 1 *commute with* P'. *In particular, if* $|K|_1 < \frac{1}{2}$ *and* A', K'

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are the restrictions of A, K, *respectively, to* $\mathscr{H}'$, *then* A' *and* $A' + K'$ *are unitarily equivalent.*

Proof. We begin by observing that for a dense set of f,

$$\begin{aligned}\Gamma Rf(s) &= -i\int dt\int dx\, R(s+x, t+x)f(t)\\ &= -i\int dx\int dt\, R(s+x, t)f(t-x)\\ &= -iT_{-x}RT_x f(s),\end{aligned}$$

where T_x is the operator of translation through a distance x.

Now it is well known that the translation operator is a function of the simple differentiation operator and therefore must commute with P'. Thus, if R commutes with P', so does ΓR. If we check the proof of Lemma 3 we find that if K commutes with P' then so do all the R_n, and therefore so does the limit R. From equation (8) it is clear that U commutes with P', and from the definition of P in the proof of Theorem 1 we conclude that P commutes with P'. The conclusion for $|K|_1 < \frac{1}{2}$ follows immediately from these facts.

Scattering for the pair A, $A + K$. We have shown in the proof of Theorem 1 that when $|K|_1 < 1/2$, $A + K = UAU^*$. To compute the wave operators we consider

$$W(t) = e^{it(A+K)}e^{-itA} = Ue^{itA}U^*e^{-itA}, \tag{13}$$

and observe that any questions concerning the existence of $s\text{-}\lim_{t\to\pm\infty} W(t)$ can be settled by investigating $s\text{-}\lim_{t\to\pm\infty} e^{itA}U^*e^{-itA}$. If we now use the well-known fact that

$$(e^{itA}f)(x) = f(x-t),$$

it follows that

$$\begin{aligned}[(\Gamma R^*)e^{-itA}f](y) &= i\int dx\left[\int_0^\infty \overline{R(x+u, y+u)}\, du\right]f(x+t)\\ &= i\int dx\left[\int_0^\infty \overline{R(x-t+u, y+u)}\, du\right]f(x)\end{aligned}$$

and that

$$\begin{aligned}e^{itA}(\Gamma R^*)e^{-itA}f(y) &= -\int dx\left[\int_0^\infty \overline{R(x-t+u, y-t+u)}\, du\right]f(x)\\ &= -i\int dx\left[\int_{-t}^\infty \overline{R(x+u, y+u)}\, du\right]f(x).\end{aligned} \tag{14}$$

We define Γ_σ $(-\infty \leqq \sigma < \infty)$ by the equation

$$\Gamma_\sigma X(x, y) = -\int_\sigma^\infty X(x+u, y+u)\, du$$

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and note that $\Gamma_0 = \Gamma$. Using equations (8) and (14), we obtain

$$(e^{itA}U^*e^{-itA}f)(y) = f(y) + \int \Gamma_{-t}\overline{R(x,y)}f(x)\,dx. \tag{15}$$

The next lemma enables us to calculate the limit as $t \to \pm\infty$ in (15).

LEMMA 9. *Let $R \in L_1$ and consider $\Gamma_t R$ as an operator in L_2. Then*

$$s\text{-}\lim_{t\to\infty} \Gamma_t R = 0 \qquad \textit{and} \qquad s\text{-}\lim_{t\to-\infty} \Gamma_t R = \Gamma_{-\infty} R.$$

Proof. Let f be a function on the real line. We have

$$|(\Gamma_t Rf)(x)| \leqq \int dy \int_t^\infty du\, |R(x+u, y+u)|\, |f(y)| \tag{16}$$

which yields immediately that

$$||\Gamma_t R||_{1,1} \leqq |R|_1 \qquad \text{and} \qquad ||\Gamma_t R||_{\infty,\infty} \leqq |R|_1.$$

The Riesz convexity theorem then enables us to conclude that $||\Gamma_t R||_{2,2} \leqq |R|_1$ also.

We shall see presently that when $f \in L_1$ then (16) yields the conclusion

$$L_1\text{-}\lim_{t\to\infty} \Gamma_t Rf = 0. \tag{17}$$

To prove this, write $g(y) = \int |R(x,y)|\,dx$ and notice that from (16),

$$\int |(\Gamma_t Rf)(x)|\,dx \leqq \int_t^\infty du \int g(y+u)|f(y)|\,dy.$$

Since both f and g are integrable, so is their convolution and (17) follows.

Now let $f \in L_2$. We decompose $f = f_1 + f_2$ such that $f_1 \in L_1 \cap L_\infty$ and $||f_2||_2$ is small. Then

$$\int |(\Gamma_t Rf_1)(x)|^2\,dx \leqq |R|_1\, |f_1|_\infty \int |\Gamma_t Rf_1(x)|\,dx. \tag{18}$$

Finally we combine (16), (17), and (18) to conclude that $\lim_{t\to\infty} ||\Gamma_t Rf||_2 = 0$. If we write $\Gamma_{-\sigma}R = \Gamma_{-\infty}R + S_\sigma$, then a similar argument shows that $S_\sigma \to 0$ as $\sigma \to \infty$. This completes the proof of the lemma.

Finally we use Lemma 9 in conjunction with equation (8) to deduce Theorem 2.

THEOREM 2. *The wave limits $W_\pm = \lim_{t\to\pm\infty} W(t)$ exist in the strong operator topology and equal $U(I - \Gamma_{-\infty}R^*)$ and U, respectively. Furthermore, the scattering operator is*

$$S = W_+^* W_- = I + \Gamma_{-\infty}R.$$

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Example. Let H_0 be the self-adjoint operator on $L_2(\mathscr{R}^+)$ obtained from $-d^2/ds^2$ by imposing the boundary condition $f(0) = 0$. It is known that the operator U defined by

$$(Uh)(\lambda) = \text{l.i.m.} \frac{1}{(\pi)^{1/2}} \int_0^\infty \lambda^{-1/4} \sin \lambda^{1/2} s\, h(s)\, ds, \tag{19}$$

maps $L_2(\mathscr{R}^+)$ isometrically onto $L_2(\mathscr{R}^+)$, and that $UH_0U^* = L$, the simple multiplication operator on $L_2(\mathscr{R}^+)$.

Let $\mathscr{H}$ be the subspace of $L_2(\mathscr{R})$ consisting of boundary values of functions analytic in the lower half plane. Let A be the self-adjoint operator defined in Theorem 1. It is clear that $\mathscr{H}$ reduces A. Let A_0 be the restriction of A to $\mathscr{H}$. The one-dimensional Fourier transform defined by

$$Vg(t) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{-i\lambda t} g(\lambda)\, d\lambda \tag{20}$$

maps $L_2(\mathscr{R}^+)$ isometrically onto $\mathscr{H}$ and we have again

$$V^*A_0V = L.$$

We conclude that $U\mathscr{H}_0U^* = V^*A_0V$, or $\mathscr{H}_0 = U^*V^*A_0VU$.

We proceed to show that VU is an integral operator, and to obtain an explicit formula for its kernel. In order to accomplish this task, we observe that the multiplication operator δ_ϵ, mapping $L_2(\mathscr{R}^+)$ into itself, defined by

$$(\delta_\epsilon f)(\lambda) = e^{-\epsilon\lambda} f(\lambda), \tag{21}$$

satisfies

$$s\text{-}\lim_{\epsilon \downarrow 0} \delta_\epsilon = I.$$

Consequently, s-lim $V\delta_\epsilon U = VU$.

Using (19), (20), and (21) together with Fubini's theorem, we find that $V\delta_\epsilon U$ has the kernel

$$\frac{1}{\pi(2)^{1/2}} \int_0^\infty e^{-(\epsilon+it)\lambda} \lambda^{-1/4} \sin \lambda^{1/2} s\, d\lambda. \tag{22}$$

If we make the change of variables $\mu = s\lambda^{1/2}$, (22) becomes

$$\frac{(2)^{1/2}}{\pi} s^{-3/2} \int_0^\infty \mu^{1/2} e^{-(\epsilon+it)\mu^2/s^2} \sin \mu\, d\mu$$

which is evaluated in [**2**, p. 74, No. 24] to yield

$$\frac{s^{-3/2}}{((\epsilon + it)/s^2)^{5/4}} \Gamma\left(\frac{5}{4}\right) {}_1F_1\left(\frac{5}{4}, \frac{3}{2}; \frac{s^2}{4(\epsilon + it)}\right). \tag{23}$$

Since the hypergeometric function ${}_1F_1$ is an entire function of its argument we can take the limit as $\epsilon \downarrow 0$ in (23) to obtain

$$W(s, t) = \frac{s}{(it)^{5/4}} \Gamma\left(\frac{5}{4}\right) {}_1F_1\left(\frac{5}{4}, \frac{3}{2}; \frac{is^2}{-4t}\right) \tag{24}$$

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with uniform convergence for (t, s) in each compact subset of $(\mathscr{R} \times \mathscr{R}^+)\backslash\{0\}$. It follows that VU is an integral operator with kernel $W(s, t)$.

Estimates for W. It is known [**6**, p. 88] that the Kummer hypergeometric function ${}_1F_1$ has the integral representation

$${}_1F_1(a, c; z) = \frac{\Gamma(c)2^{1-c}}{\Gamma(a)\Gamma(c-a)} e^{z/2} \int_{-1}^{1} e^{z\tau/2}(1-\tau)^{c-a-1}(1+\tau)^{a-1}\, d\tau$$

when $0 < \operatorname{Re} a < \operatorname{Re} c$. In our case this formula yields the estimate

$$\left| {}_1F_1\left(\frac{5}{4}, \frac{3}{2}; \frac{is^2}{-4t}\right)\right| \leqq \left|\frac{\Gamma(3/2)2^{-1/2}}{\Gamma(5/4)\Gamma(1/4)}\right| \int_{-1}^{1} (1-\tau)^{-3/4}(1+\tau)^{1/4}\, d\tau$$
$$= 1,$$

after observing that the integral on the right is almost a beta function. Thus we obtain

$$|W(s, t)| \leqq \Gamma(5/4)s|t|^{-5/4}. \tag{25}$$

Furthermore, using the asymptotic formulas in [**6**, p. 87], we deduce the following formula, valid for large values of s^2/t:

$$W(s, t) = ie^{-i\pi a}\Gamma(3/2)(-4)^{1/4}\, e^{is^2/4t}\frac{s^{1/2}}{t}\left(1 + O\left(\frac{t}{s^2}\right)\right) \tag{26}$$
$$= C\, e^{is^2/4t}\frac{s^{1/2}}{t}\left(1 + O\left(\frac{t}{s^2}\right)\right),$$

where $C = ie^{-i\pi a}\Gamma(3/2)(-4)^{1/4}$.

Conditions on the potential. Suppose that $q(s)$ is a real-valued function on $\mathscr{R}^+$ satisfying the following three conditions:

(1) $\int_0^\infty s^2 q(s)\, ds < \infty$;

(2) q vanishes in a neighbourhood of the origin;

(3) The Fourier transform $\hat{q}_1$ of $q_1(s) = q(\sqrt{s})/\sqrt{s}$ satisfies $\hat{q}_1(t) \leqq C/t^\eta$ for some $\eta > \frac{1}{2}$ and t large.

We let Q denote the operator in $L_2(\mathscr{R}^+)$ defined by

$$Qf(s) = q(s)f(s).$$

The integrability of the kernel. If $f(\tau)$ is a continuous function of compact support which vanishes in a neighbourhood of the origin, then

$$(W^*QWf)(t) = \int ds\, \overline{W(s, t)}\, q(s) \int W(s, \tau)f(\tau)\, d\tau.$$

Using (25) we find that

$$|\overline{W(s, t)}q(s)W(s, \tau)f(\tau)| \leqq \frac{s^2 q(s)}{|t\tau|^{5/4}} \sup_\tau |f(\tau)|. \tag{27}$$

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Since we have assumed that

$$\int_0^\infty s^2|q(s)|\,ds < \infty,$$

it follows that the product of functions on the left is integrable with respect to $ds\,d\tau$. Thus the integrals can be reordered to deduce that W^*QW is an integral operator with kernel

$$K(t, \tau) = \int_0^\infty \overline{W(s, t)}q(s)W(s, \tau)\,ds. \tag{28}$$

We proceed to show that $K \in L_1$. From (27) it is evident that $|K|$ is integrable on the set $\{(s, t): |s| \geqq \eta, |t| \geqq \eta\} = \sigma_\eta$ for all $\eta > 0$. We need only show that K is integrable on the complement of σ_η for some $\eta > 0$. For this we use (25) to obtain the formula

$$K(t, \tau) = |C^2| \int_0^\infty \left(1 + O\left(\frac{t}{s^2}\right)\right)\left(1 + O\left(\frac{\tau}{s^2}\right)\right) \frac{e^{is^2(1/t - 1/\tau)}}{t\tau}\, sq(s)\,ds.$$

From assumptions (1) and (3) it follows that there exists $\eta > 0$ such that whenever both $|\tau| < \eta$ and $|t| < \eta$ we have

$$|K(t, \tau)| \leqq \frac{\text{const}}{|t\tau|\,|1/\tau - 1/t|^\eta} = \frac{\text{const}}{|t|^{1-\eta}|\tau|^{1-\eta}|t - \tau|^\eta}$$

which is integrable.

Similar techniques can be used to obtain estimates of the form

$$|K(t, \tau)| \leqq \begin{cases} \dfrac{\text{const}}{|t|^{1-\eta}|\tau|^{5/4}} & \text{for } |\tau| > \eta, \quad |t| < \eta, \\[2ex] \dfrac{\text{const}}{|t|^{5/4}|t|^{1-\eta}} & \text{for } |\tau| < \eta, \quad |t| > \eta. \end{cases}$$

This completes the proof of the integrability of K.

Conclusion. Once we know that $K \in L_1$ we can apply Corollary 1 to conclude that the operators $i\,d/dt$ and $i\,d/dt + K$ on $\mathscr{H}$ are unitarily equivalent whenever $|K|_1$ is small enough and that they form a scattering pair. Consequently, $-d^2/ds^2$ and $-d^2/ds + cq(s)$ are unitarily equivalent whenever c is small enough and they also form a scattering pair.

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