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Algebraic Varieties with Boundaries

A boundary of an algebraic variety is a divisor with real coefficients. In this chapter, we introduce basic concepts of algebraic varieties with boundaries. Using the language of numerical geometry, we define cones of curves and divisors. According to the Hironaka desingularization theorem, it is possible to use birational morphisms to make algebraic varieties smooth and divisors normal crossing. We focus on log canonical divisors of algebraic varieties with boundaries, and define concepts of KLT (Kawamata log terminal) pairs and DLT (divisorially log terminal) pairs. The Kodaira vanishing theorem for smooth projective varieties can be extended to KLT or DLT pairs by constructing covering spaces using the covering trick. We also discuss the classification of algebraic varieties and singularities in lower dimensions.

1.1 Q-divisors and R-divisors

The linear equivalence class of a divisor determines a coherent sheaf which is called a divisorial sheaf. Algebraic geometry often deals with coherent sheaves, but this book focuses on the language of divisors. It is like dealing with differential forms themselves instead of cohomology classes of differential forms in differential geometry.

Fix a base field k. An *algebraic variety* X is an irreducible reduced separated scheme of finite type over k.

An algebraic variety X is attached with the structure sheaf $\mathcal{O}_X$ and a local ring $\mathcal{O}_{X,P}$ at each point P. If the local ring $\mathcal{O}_{X,P}$ is a regular local ring, then X is said to be *nonsingular* at P. In this book, we mostly work over characteristic 0, so we will use the word *smooth* instead of nonsingular which sounds better.

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When $\dim X = n$, X is smooth if and only if for every closed point P on X, the maximal ideal $\mathfrak{m}_P$ of the local ring is generated by n elements $x_1, \ldots, x_n$. Such $x_1, \ldots, x_n$ is called a *regular system of parameters* or *local coordinates*. When $k = \mathbf{C}$, this is equivalent to saying that the set of closed points of X forms a complex manifold.

The set of all smooth points $\mathrm{Reg}(X)$ of an algebraic variety X is a non-empty open subset of X, and its complement set $\mathrm{Sing}(X) = X \setminus \mathrm{Reg}(X)$, which is a proper closed subset of X, is called the *singular locus* of X.

An algebraic variety X is said to be *normal* if the local ring at every point is an integrally closed domain. Since normal local rings of dimension 1 are regular, the singular locus of a normal algebraic variety is a closed subset of codimension at least 2. That is, it is the closure of several points with codimensions at least 2.

Every algebraic variety X can be easily modified into a normal one: There is a unique finite morphism $f : X^\nu \to X$ from a normal algebraic variety which is isomorphic over $\mathrm{Reg}(X)$. This is called the *normalization* of X.

Normality can be checked by Serre's criterion ([94]):

Theorem 1.1.1 *An algebraic variety X is normal if and only if the following two conditions are satisfied:*

(1) (R_1) Its singular locus is a closed subset of codimension at least 2.
(2) (S_2) For any open subset U and any closed subset Z of codimension at least 2, the restriction map $\Gamma(U, \mathcal{O}_X) \to \Gamma(U \setminus Z, \mathcal{O}_X)$ is bijective.

From now on, we will always assume that X is a normal algebraic variety. A *prime divisor* on X is a closed subvariety of codimension 1. A *divisor* is a formal finite sum of prime divisors $D = \sum d_i D_i$. Unless otherwise stated, the coefficients d_i are integers and D_i are distinct prime divisors. In other words, divisors are elements in the free Abelian group $Z^1(X)$ generated by all prime divisors on X.

D is said to be *effective* if all coefficients d_i are nonnegative. For two divisors D, D', we write the inequality $D \geq D'$ if $D - D'$ is effective. D is said to be *reduced* if $d_i = 1$ for all i.

Let D be a prime divisor on a normal algebraic variety X and let P be the generic point of D, then the local ring $\mathcal{O}_{X,P}$ is a *discrete valuation ring* with the function field $k(X)$ as its quotient field.

For a rational function $h \in k(X)^*$, its divisor $\mathrm{div}(h)$ is defined as

$$\mathrm{div}(h) = \sum v_D(h) D.$$

Here the sum runs over all prime divisors D, and ν_D is the discrete valuation of the local ring at the generic point of D. It is known that the right-hand side

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is a finite sum. Any divisor defined by a nonzero rational function is called a *principal divisor*.

For a divisor D, the corresponding *divisorial sheaf* $\mathcal{O}_X(D)$ is defined as the following: for any open subset U of X,

$$\Gamma(U, \mathcal{O}_X(D)) = \{h \in k(X)^* \mid \operatorname{div}(h)|_U + D|_U \geq 0\} \cup \{0\}.$$

Also we define

$$H^0(X, D) = H^0(X, \mathcal{O}_X(D)).$$

If a nonzero global section s of $\mathcal{O}_X(D)$ corresponds to a rational function h, we define the divisor of s by

$$\operatorname{div}(s) = \operatorname{div}(h) + D,$$

which is effective. Generally we can also define the divisor $\operatorname{div}(s)$ of a rational section s of $\mathcal{O}_X(D)$ by the corresponding rational function h as the above equation, but in this case $\operatorname{div}(s)$ is not necessarily effective. For example, if we take s_1 to be the rational section corresponding to the rational function $h = 1$, then the corresponding divisor is just D.

There is an isomorphism $(\mathcal{O}_X(D))_\eta \cong \mathcal{O}_{X,\eta}$ on the generic point η of X. Moreover, by taking the dual, we have

$$\mathcal{O}_X(D)^* := \operatorname{Hom}(\mathcal{O}_X(D), \mathcal{O}_X) \cong \mathcal{O}_X(-D),$$

hence the divisorial sheaf $\mathcal{O}_X(D)$ is a *reflexive sheaf of rank 1*. Here a *reflexive sheaf* is a coherent sheaf which is isomorphic to its double dual: $F^{**} \cong F$.

A divisor D is called a *Cartier divisor* if its divisorial sheaf $\mathcal{O}_X(D)$ is invertible. In other words, this is to say that, in a neighborhood of each point P, this divisor is a principal divisor defined by some rational function depending on P. To distinguish from Cartier divisors, we call this divisor a *Weil divisor* or an *integral divisor*. Denote by $\operatorname{Div}(X)$ the set of all Cartier divisors. There is an inclusion $\operatorname{Div}(X) \subset Z^1(X)$, and they coincide when X is smooth.

Two divisors D, D' on an algebraic variety X are said to be *linearly equivalent*, and denoted by $D \sim D'$, if $D - D'$ is a principal divisor. Note that $D \sim D'$ if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$. In other words, divisorial sheaves can be viewed as linear equivalence classes of divisors. Here D, D' are not necessarily Cartier divisors.

The relative version is as follows. Given a *morphism* $f: X \to S$ between algebraic varieties, two divisors D, D' on X are said to be *relatively linearly equivalent* over S, and denoted by $D \sim_S D'$, if there exists an open covering $\{S_i\}$ of S such that $D|_{S_i} \sim D'|_{S_i}$ after restriction over each S_i. Here we remark that in some other references, D, D' are defined to be relatively

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linearly equivalent over S if there exists a Cartier divisor B on S such that $D \sim D' + f^*B$. In general these two definitions are not the same and the assumption in our definition is weaker. But under certain conditions, for example, when f is proper surjective with connected geometric fibers, it is easy to see that these two definitions coincide.

A closed subset B on a smooth algebraic variety X is called a *normal crossing divisor* if at each closed point P there is a regular system of parameters $z_1, \ldots, z_n$ of the local ring $\mathcal{O}_{X,P}$ and an integer $1 \le r \le n$ such that the defining equation of B is of the form $z_1 \cdots z_r = 0$ locally around P. In this case, every irreducible component of B is smooth. Also, the union of several irreducible components of B is again a normal crossing divisor.

For an algebraic variety X and a closed subset B, the set of points at which X is smooth and B is a normal crossing divisor is an open subset of X, which is denoted by $\mathrm{Reg}(X, B)$. The complement set $\mathrm{Sing}(X, B) = X \setminus \mathrm{Reg}(X, B)$ is called the *singular locus* of (X, B).

Remark 1.1.2 A normal crossing divisor defined above is also called a *simple normal crossing divisor* in many references.

If X is a complex algebraic manifold and $z_1, \ldots, z_n$ are regular local coordinates on the complex manifold associated to X, then a normal crossing divisor B satisfying the same condition as above is not necessarily a simple normal crossing divisor in the algebraic setting. In fact, irreducible components of B may have self-intersection. So, we use the term "simple" in the algebraic setting in order to distinguish with the analytic setting.

For example, in the affine plane $\mathbf{C}^2$ with coordinates x, y, the closed subset defined by the equation $x^2+y^2+y^3 = 0$ is irreducible but has self-intersection at the point $(0,0)$, therefore it is a normal crossing divisor on the complex manifold, but not a simple normal crossing divisor.

One feature of this book is to consider divisors which don't necessarily have integral coefficients. If the coefficients d_i in $D = \sum d_i D_i$ are rational numbers (respectively, real numbers), then D is called a **Q**-*divisor* (respectively, an **R**-*divisor*). Note that a **Q**-divisor is also an **R**-divisor. Those are elements in $Z^1(X) \otimes \mathbf{Q}$ or $Z^1(X) \otimes \mathbf{R}$, respectively, and these vector spaces are usually denoted by $Z^1(X)_{\mathbf{Q}}$ and $Z^1(X)_{\mathbf{R}}$. We will see soon that the range of discussions is expanded widely by considering **Q**-divisors and **R**-divisors.

Let $D = \sum d_i D_i$ be an **R**-divisor on X, where D_i are distinct prime divisors. D is said to be *effective* if all coefficients d_i are nonnegative. For two **R**-divisors D, D', we write the inequality $D \ge D'$ if $D - D'$ is effective. D is said to be *reduced* if $d_i = 1$ for all i. The *support* of D is the union

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of all D_i with $d_i \neq 0$, and is denoted by $\mathrm{Supp}(D)$. Set $D^+ = \sum_{d_i>0} d_i D_i$ and $D^- = \sum_{d_i<0}(-d_i)D_i$, then D^+ and D^- are effective **R**-divisors with no common irreducible component and the equality $D = D^+ - D^-$ holds. D^+, D^- are called the *positive part* and the *negative part* of D, respectively.

For two **R**-divisors $D = \sum_i d_i D_i$ and $D' = \sum_i d_i' D_i$, define their maximum to be $\max\{D, D'\} = \sum_i \max\{d_i, d_i'\}D_i$. For example, $D^+ = \max\{D, 0\}$, $D^- = \max\{-D, 0\}$. Similarly we can define $\min\{D, D'\} = \sum_i \min\{d_i, d_i'\}D_i$.

The *round up* (respectively, *round down*) of an **R**-divisor is defined via the round up (respectively, round down) of coefficients:

$$\ulcorner D \urcorner = \sum \ulcorner d_i \urcorner D_i, \quad \llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i.$$

A **Q**-divisor (respectively, an **R**-divisor) is said to be a **Q**-*Cartier divisor* (respectively, an **R**-*Cartier divisor*) if it is an element of $\mathrm{Div}(X) \otimes \mathbf{Q}$ (respectively, $\mathrm{Div}(X) \otimes \mathbf{R}$). Note that if a **Q**-divisor is an **R**-Cartier divisor, then it is automatically a **Q**-Cartier divisor. For a **Q**-Cartier divisor D, there exists a positive integer m such that mD is a Cartier divisor. However, in general there might not be a nonzero multiple to make an **R**-Cartier divisor Cartier. X is said to be *factorial* (respectively, **Q**-*factorial*), if all Weil divisors on X are Cartier divisors (respectively, **Q**-Cartier divisors).

Two **R**-divisors D, D' are said to be **R**-*linearly equivalent*, denoted by $D \sim_{\mathbf{R}} D'$, if $D - D'$ can be written as an **R**-linear combination of principal divisors. The relative version and **Q**-linear equivalence can be defined similarly.

Remark 1.1.3 Considering **R**-divisors is now essential to the development of the minimal model theory. This book can be viewed as a revised version of [76], in which only **Q**-divisors are treated. Later in [61], **R**-divisors already played a central role. The *divisorial Zariski decomposition* (which is called the sectional decomposition in [61]) is defined via limits of **Q**-divisors, so **R**-divisors appear naturally. Moreover, it is proved in [60] that the existence of Zariski decomposition (in a good sense, not only in codimension 1) into **R**-divisors implies the finite generation of canonical rings.

Example 1.1.4 We give examples for a **Q**-Cartier Weil divisor which is not Cartier and a Weil divisor which is not **Q**-Cartier.

(1) Let X be the hypersurface defined by the equation $xy = z^2$ in the 3-dimensional affine space $\mathbf{A}^3$ with coordinates x, y, z, which is an algebraic surface with an ordinary double point at the origin $(0, 0, 0)$. The line D defined by the equation $x = z = 0$ is a prime divisor on X. At lease two

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equations are needed to define D in X, so D is not a Cartier divisor. On the other hand, $\operatorname{div}(x) = 2D$ on X, so D is $\mathbf{Q}$-Cartier.

(2) Let X be the hypersurface defined by the equation $xy = zw$ in $\mathbf{A}^4$ with coordinates x, y, z, w, which is a 3-fold with an ordinary double point at the origin $(0, 0, 0, 0)$. The 2-dimensional linear subspace D_1 defined by the equation $x = z = 0$ is a prime divisor on X, which is not a $\mathbf{Q}$-Cartier divisor (see Example 1.2.4 for the reason). It is the same for D_2 defined by $x = w = 0$. However, the sum $D_1 + D_2 = \operatorname{div}(x)$ is a Cartier divisor. See Example 2.5.4(2) for related discussions.

1.2 Rational Maps and Birational Maps

A *rational map* $f: X \dashrightarrow Y$ between algebraic varieties is a morphism $f: U \to Y$ from a non-empty open subset U of X. Since f might not be defined on the whole X, such a map is denoted by a dashed arrow in this book. If there is another non-empty open subset U' and a morphism $f': U' \to Y$ which coincides with f on $U \cap U'$, then we consider $f = f'$ as the same rational map. The *domain of definition* of a rational map f is defined to be the largest U such that there is a morphism $f: U \to Y$ representing f.

The *graph* Γ_f of a rational map $f: X \dashrightarrow Y$ is defined to be the closure of the graph $\Gamma \subset U \times Y$ of the morphism $f: U \to Y$ in $X \times Y$.

A rational map $f: X \dashrightarrow Y$ is said to be a *birational map* if there exist non-empty open subsets U, V on X, Y such that f induces an isomorphism $U \cong V$. In this situation, the inverse map $f^{-1}: Y \dashrightarrow X$ is also a birational map.

A morphism $f: X \to Y$ is said to be a *birational morphism* if it is a birational map. If U is the largest open subset of X on which f induces an isomorphism $U \cong V$, then $\operatorname{Exc}(f) = X \setminus U$ is called the *exceptional set* of f. In this situation, V is the domain of definition of f^{-1}. A prime divisor contained in the exceptional set is called an *exceptional divisor* over Y or an *f-exceptional divisor*. Generally, a divisor whose support is contained in the exceptional set is also called an exceptional divisor over Y or an f-exceptional divisor.

X and Y are said to be *birationally equivalent* if there exists a birational map $f: X \dashrightarrow Y$. In this case, we also say that one is a *birational model* to the other.

For a morphism $f: Y \to X$ and a closed subset D of X, the inverse image $f^{-1}(D)$ is a closed subset of Y. In this book, $f^{-1}(D)$ only means the set-theoretic inverse image, and we do not consider its scheme structure. However, for a divisor we can define its direct image and inverse image as the following.

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First, we define the *inverse image* or *pullback* of a Cartier divisor. For a morphism $f: Y \to X$ and an invertible sheaf L on X, we can always define the pullback f^*L which is an invertible sheaf on Y. On the other hand, for a Cartier divisor D on X, we can define its pullback only if the image $f(Y)$ is not contained in the support of D. In this situation, the pullback f^*D is defined by pulling back the local equation of D. If D is given by a rational section s of the invertible sheaf $\mathcal{O}_X(D)$ by $\operatorname{div}(s) = D$, then the pullback f^*D is given by the rational section f^*s of the invertible sheaf $f^*\mathcal{O}_X(D)$.

For an **R**-Cartier divisor D, if we write it as an **R**-linear combination of Cartier divisors $D = \sum d_i D_i$, then we can define the pullback by $f^*D = \sum d_i f^*D_i$. Here D_i are Cartier divisors, not prime divisors. In other words, the pullback of **R**-Cartier divisors can be defined by extending the coefficients of the pullback map $f^*: \operatorname{Div}(X) \to \operatorname{Div}(Y)$ of Cartier divisors. Note that this definition does not depend on the expression of D. The pullback f^*D is also called the *total transform* of D.

On the other hand, we cannot define the pullback for a general divisor which is not an **R**-Cartier divisor. However, if the morphism $f: Y \to X$ is a birational map, we can define another form of "pullback" (the strict transform by the inverse map f^{-1}) as the following.

Let $f: X \dashrightarrow Y$ be a birational map and let D be a prime divisor on X. For the domain of definition U, if $D \cap U \neq \emptyset$, then the image $(f|_U)(D \cap U)$ is a locally closed subvariety of Y. If its closure is a prime divisor on Y, then we denote the closure by f_*D; if $D \cap U = \emptyset$ or the image $(f|_U)(D \cap U)$ has codimension at least 2, then we set $f_*D = 0$. Here f_*D is called the *strict transform* or *birational transform* of D. Generally for **R**-divisors, the definition can be extended by linearity $f_*(\sum d_i D_i) = \sum d_i f_*(D_i)$ and we consider the linear map $f_*: Z^1(X)_{\mathbf{R}} \to Z^1(Y)_{\mathbf{R}}$ by extending the coefficients.

Example 1.2.1 For a birational projective morphism $f: Y \to X$ and any prime divisor D on X, the strict transform $f_*^{-1}D$ on Y is again a prime divisor, which is not 0.

In fact, the inverse map f^{-1} is well defined at the generic point of D, and there is no prime divisor contracted by f^{-1}, hence the strict transform is a prime divisor.

Remark 1.2.2 A birational map $f: X \dashrightarrow Y$ between normal algebraic varieties induces an isomorphism between function fields $k(X) \cong k(Y)$. For a prime divisor D on X whose strict transform f_*D is nonzero, this isomorphism identifies the local rings at generic points of D and f_*D.

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When we consider birationally equivalent algebraic varieties as a whole, we identify the divisors defining the same discrete valuation ring, which is equivalent to identifying prime divisors connected by strict transforms.

Similarly, for a subvariety Z of higher codimension, we can define the strict transform f_*Z in a similar way: If $Z \cap U \neq \emptyset$ and $f|_U$ induces an isomorphism at the generic point of Z, then we define f_*Z to be the closure of $(f|_U)(Z \cap U)$ in Y. We refer to Section 1.4 for the definition in general case.

A birational map $f: X \dashrightarrow Y$ is said to be *surjective in codimension* 1 if the map $f_*: Z^1(X) \to Z^1(Y)$ is surjective, that is, for any prime divisor $E \subset Y$ there is a prime divisor D on X such that $E = f_*D$. Moreover, it is said to be *isomorphic in codimension* 1 if $f_*: Z^1(X) \to Z^1(Y)$ is bijective. The minimal model theory mainly deals with the phenomenon in codimension 1, so these maps play important roles.

Example 1.2.3 A classical example of birational maps is a *blowup*. In this book, blowing up along a smooth *center* is important. A blowup is obtained by gluing the following local construction.

(1) Define the rational map $f: X = \mathbf{A}^n \dashrightarrow Y = \mathbf{P}^{r-1}$ from the n-dimensional affine space to a projective space by $f(x_1, \ldots, x_n) = [x_1 : \ldots : x_r]$. Let Z be the linear subspace of X defined by $x_1 = \cdots = x_r = 0$, then the domain of definition of f is $U = X \setminus Z$.

The graph $X' \subset X \times Y$ of f is defined by $x_i y_j = x_j y_i$ $(1 \leq i, j \leq r)$, where $y_1, \ldots, y_r$ are the homogeneous coordinates of Y. The first projection $p: X' \to X$ is the blowup along center Z. $E = p^{-1}(Z)$ is the exceptional set of the birational morphism p, which is a prime divisor. p induces an isomorphism $X' \setminus E \to X \setminus Z$. Moreover, $E \cong Z \times \mathbf{P}^{r-1}$.

In this case, p is surjective in codimension 1, but p^{-1} is not.

(2) Let X_1 be a subvariety of X which is not contained in Z. The strict transform $X'_1 = p_*^{-1}(X_1)$ of X_1 is the closure of $p^{-1}(X_1 \setminus Z)$. In this case, $p_1 = p|_{X'_1}: X'_1 \to X_1$ is the blowup of X_1 along center $Z \cap X_1$. In particular, the case $Z \subset X_1$ is important. If $X_1 \subset Z$, we can think $X'_1 = \emptyset$, in other words, the variety disappears after the blowing up.

If $X_1 \not\subset Z$, p_1 is a birational morphism. However, the exceptional set $\mathrm{Exc}(p_1)$ does not necessarily coincide with $E \cap X'_1$. For example, consider $n = 4$, $r = 2$, $X_1 \subset \mathbf{A}^4$ is the subvariety defined by the equation $x_1x_3 + x_2x_4 = 0$. This is the situation in Example 1.1.4(2). In this case, $Z \subset X_1$, the exceptional set C of $p_1: X'_1 \to X_1$ is isomorphic to $\mathbf{P}^1$, and $p_1(C)$ is the origin. Hence p_1 is isomorphic in codimension 1, and so is p_1^{-1}.

Example 1.2.4 Consider the situations in Example 1.1.4.

(1) For a **Q**-Cartier Weil divisor which is not Cartier, the pullback might not be a Weil divisor but only a **Q**-divisor.

The blowup $f: X' \to X$ of X along the origin $Z = (0,0,0)$ as the center gives a resolution of singularity. The exceptional set $C \subset X'$ is isomorphic to $\mathbf{P}^1$. We have $f^*D = f_*^{-1}D + \frac{1}{2}C$.

The projection formula $(f^*D \cdot C) = (D \cdot f_*C)$ stated later (before Proposition 1.4.3) can be confirmed by the following facts: $(f_*^{-1}D \cdot C) = 1$, $(C^2) = -2$, and $f_*C = 0$.

(2) Non-**Q**-Cartier divisors cannot be pulled back according to the projection formula.

Consider the blowup $p_1: X_1' \to X_1$ at the end of Example 1.2.3(2). We change the notation by $f: X' \to X$. Then X' is smooth. As the exceptional set $C \subset X'$ is isomorphic to $\mathbf{P}^1$ which is only 1-dimensional, p_1 is isomorphic in codimension 1.

If the pullbacks f^*D_1, f^*D_2 of D_1, D_2 exist, they would have to coincide with the strict transforms $f_*^{-1}D_1, f_*^{-1}D_2$ since there are no other prime divisors in the supports of $f^{-1}(D_1), f^{-1}(D_2)$. However, intersecting with C, we have $(f_*^{-1}D_1 \cdot C) = -1$ and $(f_*^{-1}D_2 \cdot C) = 1$. Since $f_*C = 0$, this violates the projection formula $(f^*D \cdot C) = (D \cdot f_*C)$ which holds for pullbacks.

A coherent sheaf F on an algebraic variety X is said to be *generated by global sections* if the natural homomorphism $H^0(X,F) \otimes \mathcal{O}_X \to F$ is surjective.

For a Cartier divisor D, its *complete linear system* is defined by $|D| = \{D' \mid D \sim D' \geq 0\}$, and its *base locus* is defined by $\operatorname{Bs}|D| = \bigcap_{D' \in |D|} \operatorname{Supp}(D')$. When $\operatorname{Bs}|D| = \emptyset$, $|D|$ is said to be *free*, which is equivalent to that the corresponding coherent sheaf $\mathcal{O}_X(D)$ is generated by global sections.

D is also said to be free if $|D|$ is free, and D is said to be *semi-ample* if there exists a positive integer m such that mD is free.

More generally, a finite-dimensional linear subspace $V \subset H^0(X,D)$ corresponds to a (not necessarily complete) *linear system* $\Lambda = \{\operatorname{div}(s) \mid s \in V \setminus \{0\}\}$. As an algebraic variety, Λ is isomorphic to the projective space $\mathbf{P}(V^*) := (V \setminus \{0\})/k^*$. The base locus of Λ is defined similarly by $\operatorname{Bs}\Lambda = \bigcap_{D' \in \Lambda} \operatorname{Supp}(D')$, and Λ is said to be free if $\operatorname{Bs}\Lambda$ is empty, which is equivalent to that the natural homomorphism $V \otimes \mathcal{O}_X \to \mathcal{O}_X(D)$ is surjective.

The *fixed part* of a linear system Λ is the effective divisor $F = \min_{D' \in \Lambda} D'$. In other words, F is the maximal divisor such that $F \leq D'$ for all $D' \in \Lambda$.

In this case, the image of the natural injection $H^0(X, D - F) \to H^0(X, D)$ contains V. Being viewed as a subspace of $H^0(X, D - F)$, V corresponds to the linear system $\Lambda' = \{D' - F \mid D' \in \Lambda\}$, which is called the *movable part* of Λ. We write $\Lambda = \Lambda' + F$. By definition, the support of F coincides with the codimension 1 components of $\mathrm{Bs}\,\Lambda$.

A non-empty linear system Λ induces a rational map $\Phi_\Lambda \colon X \dashrightarrow \mathbf{P}(V) := (V^* \setminus \{0\})/k^*$ to its dual projective space. The domain of definition of Φ_Λ contains $U = X \setminus \mathrm{Bs}\,\Lambda$; for $P \in U$, $\Phi_\Lambda(P)$ is the point in $\mathbf{P}(V)$ corresponding to the hyperplane $\{s \in V \mid s(P) = 0\}$ of V. In other words, if we take a basis $s_1, s_2, \ldots, s_m \in V$, then we can define $\Phi_\Lambda(P) = [s_1(P) \colon s_2(P) \colon \cdots \colon s_m(P)] \in \mathbf{P}(V)$. Note that here $s_i(P)$ is not a well-defined value, but $[s_1(P) \colon s_2(P) \colon \cdots \colon s_m(P)]$ is a well-defined point as long as $P \in U$. In particular, when Λ is free, Φ_Λ is a morphism. The rational map given by the movable part of a linear system coincides with the rational map given by the original linear system.

In general, for a morphism $f \colon Y \to X$ and a linear system Λ on X, the *pullback* is defined by $f^*\Lambda = \{f^*D' \mid D' \in \Lambda\}$. If there is a morphism to a projective space, a free linear system can be obtained by pulling back the linear system consisting of all hyperplanes.

The base locus of a linear system can be removed in the following sense:

Proposition 1.2.5 *Let Λ be a linear system of Cartier divisors on a normal algebraic variety X. Then there exists a birational projective morphism $f \colon Y \to X$ from a normal algebraic variety Y such that the pullback has the form $f^*\Lambda = \Lambda_1 + F$, where F is the fixed part of $f^*\Lambda$ and the linear system Λ_1 is free.*

Proof Let $V \subset H^0(X, D)$ be the linear subspace corresponding to Λ. The image of the natural map $V \otimes \mathcal{O}_X \to \mathcal{O}_X(D)$ can be written as $I\mathcal{O}_X(D)$, where I is an ideal sheaf on X. Take f to be the normalization of the blowup along I, then the inverse image ideal sheaf $I\mathcal{O}_Y$ is an invertible sheaf on Y. Then $I\mathcal{O}_Y(f^*D)$ is the image of the natural map $V \otimes \mathcal{O}_Y \to \mathcal{O}_Y(f^*D)$, so it can be written as the form $\mathcal{O}_Y(f^*D - F)$ for some effective divisor F. Since the natural map $V \otimes \mathcal{O}_Y \to \mathcal{O}_Y(f^*D - F)$ is surjective, the linear system $\Lambda_1 = f^*\Lambda - F$ is free and F is the fixed part of $f^*\Lambda$. □

For an $\mathbf{R}$-divisor D on a normal proper algebraic variety X, the set of global sections $H^0(X, \llcorner D \lrcorner)$ is a finite-dimensional k-linear space. Considering all positive integer multiples mD of D and taking a direct sum, we define the *section ring* of D by

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$$R(X, D) = \bigoplus_{m=0}^{\infty} H^0(X, \llcorner mD \lrcorner).$$

Here m runs over all nonnegative integers. It admits a graded k-algebra structure defined by

$$H^0(X, \llcorner mD \lrcorner) \otimes H^0(X, \llcorner m'D \lrcorner) \to H^0(X, \llcorner (m + m')D \lrcorner)$$

since

$$\llcorner mD \lrcorner + \llcorner m'D \lrcorner \leq \llcorner (m + m')D \lrcorner.$$

The *Iitaka–Kodaira dimension* $\kappa(X, D)$ of an **R**-divisor can be defined by the transcendental degree of the section ring:

$$\kappa(X, D) = \begin{cases} \text{tr.deg}_k\, R(X, D) - 1 & \text{if } R(X, D) \neq k; \\ -\infty & \text{otherwise.} \end{cases}$$

The Iitaka–Kodaira dimension takes value among $-\infty, 0, 1, \ldots, n = \dim X$. In particular, when it takes the maximal value, that is, when $\kappa(X, D) = \dim X$, D is said to be *big*. For example, ample divisors are big.

If $R(X, D) = k$, that is, $H^0(X, \llcorner mD \lrcorner) = 0$ for any $m > 0$, then $\kappa(X, D)$ is defined to be $-\infty$ instead of -1. The reason is the following lemma:

Lemma 1.2.6 ([47, Theorem 10.2], [116, Theorem II.3.7]) *There exist positive real numbers c_1, c_2 such that for any sufficiently large and sufficiently divisible integer m,*

$$c_1 m^{\kappa(X,D)} \leq \dim H^0(X, \llcorner mD \lrcorner) \leq c_2 m^{\kappa(X,D)}.$$

Remark 1.2.7 The canonical ring is the section ring of the canonical divisor, which is proved to be finitely generated for smooth projective varieties ([16]), and one of the main goals of this book is to explain the proof. However, in general, the section ring $R(X, D)$ of a divisor D is not necessarily finitely generated. There exist examples such that the *anti-canonical ring* (i.e. the section ring of the *anti-canonical divisor* $-K_X$) of a 2-dimensional variety is not finitely generated ([125], see also Example 2.4.8). Also, the anti-canonical ring $R(X, -K_X)$ is not a birational invariant.

The relative version is as follows. Let $f: X \to S$ be a proper morphism from a normal algebraic variety. The *relative global sections* of a coherent sheaf F on X are given by the direct image sheaf $f_* F$. F is said to be *generated by relative global sections* if the natural homomorphism $f^* f_* F \to F$ is surjective.

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A Cartier divisor D on X is said to be *relatively free* if the corresponding coherent sheaf $\mathcal{O}_X(D)$ is generated by relative global sections. D is said to be *relatively semi-ample* if there exists a positive integer m such that the multiple mD is relatively free.

For an $\mathbf{R}$-divisor D on X, the direct image sheaf $f_*(\mathcal{O}_X(\llcorner D \lrcorner))$ is a coherent $\mathcal{O}_S$-module. Considering all positive integer multiples mD of D and taking a direct sum, we define the *relative section ring* of D by

$$R(X/S, D) = \bigoplus_{m=0}^{\infty} f_*(\mathcal{O}_X(\llcorner mD \lrcorner)),$$

which is a graded $\mathcal{O}_S$-algebra.

The *relative Iitaka–Kodaira dimension* is defined by the Iitaka–Kodaira dimension of the generic fiber. Here we always assume that f is surjective with irreducible geometric generic fiber, and define

$$\kappa(X/S, D) = \kappa(X_{\bar{\eta}}, D|_{X_{\bar{\eta}}}).$$

Here X_η is the *generic fiber* which is the fiber of f over the generic point η of S and $X_{\bar{\eta}}$ is the *geometric generic fiber* which is the base change of X_η to the algebraic closure of $k(S)$. D is said to be *relatively big* or *f-big* if $\kappa(X/S, D) = \dim X_{\bar{\eta}}$. In Section 1.5.1, we will give an equivalent definition for (relative) bigness using Kodaira's lemma (Corollary 1.5.10).

1.3 Canonical Divisors

A normal algebraic variety X is automatically associated with a Weil divisor K_X which is called the canonical divisor. K_X is the key player of this book. The canonical ring is the section ring of the canonical divisor. The minimal model program (MMP) is a sequence of operations that "minimizes" the canonical divisor.

As X is normal, the singular locus $\mathrm{Sing}(X)$ is a closed subset of X of codimension at least 2. Since the complement set $U = X \setminus \mathrm{Sing}(X)$ is smooth, the sheaf of differentials $\Omega^1_{X/k}$ is a locally free sheaf of rank $n = \dim X$ over U. The determinant $\omega_U = \det(\Omega^1_{X/k}|_U)$ is an invertible sheaf on U. Taking a nonzero rational section θ_U of ω_U, we get a canonical divisor $K_U = \mathrm{div}(\theta_U)$ of U. Since $X \setminus U$ contains no prime divisors of X, the restriction map of divisors $Z^1(X) \to Z^1(U)$ is bijective. Denote by $K_X \in Z^1(X)$ the corresponding divisor of $K_U \in Z^1(U)$, which is called the *canonical divisor* of X.

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Remark 1.3.1 (1) By construction, K_X depends on the choice of θ_U. However, traditionally our discussions proceed as if the canonical divisor is a fixed one. Nevertheless, in this book, all discussions are independent of the choice of θ_U.

On the other hand, the corresponding divisorial sheaf $\omega_X = \mathcal{O}_X(K_X)$ is uniquely determined. It is called the *canonical sheaf*. The canonical sheaf ω_X is a natural subject. However, since we consider the pair with an $\mathbf{R}$-divisor called the "boundary divisor," the canonical divisor is easier to handle.

(2) In this book, the following situation appears frequently: Let $f\colon Y \to X$ be a birational morphism between normal algebraic varieties and let B be an $\mathbf{R}$-divisor on X such that $K_X + B$ is $\mathbf{R}$-Cartier. Consider the pullback $f^*(K_X + B)$. By using the isomorphism between function fields $f^*\colon k(X) \to k(Y)$, we can take the same rational differential form θ which defines K_X and K_Y (in particular, $K_X = f_*K_Y$), then the $\mathbf{R}$-divisor C can be defined by the equation $f^*(K_X + B) = K_Y + C$. Here C is uniquely determined as the sum of the strict transform $f_*^{-1}B$ and an $\mathbf{R}$-divisor supported on the exceptional set of f.

We will discuss general boundary divisors later, here we first consider the case when X is a smooth algebraic variety and $B = \sum B_i$ is a normal crossing divisor. Denote $n = \dim X$. The sheaf of differentials $\Omega_X^1(\log B)$ with at most *logarithmic poles* along B is naturally defined as a locally free sheaf of rank n with the following property. For any closed point $P \in X$, choose a regular system of parameters $x_1, \ldots, x_n$ of the local ring $\mathcal{O}_{X,P}$ such that the local equation of B is $x_1 \cdots x_r = 0$ for some integer r. In this case, the stalk $\Omega_X^1(\log B)_P$ is a free $\mathcal{O}_{X,P}$-module with basis $dx_1/x_1, \ldots, dx_r/x_r, dx_{r+1}, \ldots, dx_n$.

The determinant $\Omega_X^n(\log B)$ of $\Omega_X^1(\log B)$ is isomorphic to $\mathcal{O}_X(K_X + B)$. Therefore, $K_X + B$ is called the logarithmic canonical divisor or just *log canonical divisor*. This is the origin of the terminology "log."

In general, a log canonical divisor $K_X + B$ is a sum of the canonical divisor and an effective $\mathbf{R}$-divisor. Usually certain conditions on singularities will be imposed on the pair (X, B), which will be discussed in Sections 1.10 and 1.11. The *log canonical ring* is defined to be $R(X, K_X + B)$, and the *log Kodaira dimension* is defined to be $\kappa(X, K_X + B)$.

Let X be a smooth projective variety. $R(X) = R(X, K_X)$ is the *canonical ring* of X. $P_m(X) = \dim H^0(X, mK_X)$ is called the *m-genus*, which is an important birational invariant having been studied for a long time. Its growth order $\kappa(X, K_X)$ is called the *Kodaira dimension*, sometimes simply denoted by $\kappa(X)$. X is said to be of *general type* if K_X is big.

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When working with induction on dimensions, one key is the adjunction formula.

Let D be a smooth prime divisor on a smooth algebraic variety X. Then the log canonical divisor and the canonical divisor of the prime divisor satisfy the following *adjunction formula*:

$$(K_X + D)|_D = K_D.$$

In this formula, $K_X|_D$ and $D|_D$ have no natural meaning, but the adjunction itself is given by the map

$$\mathrm{Res}_D \colon \Omega_X^n(\log D) \to \Omega_D^{n-1},$$

which is induced by the residue map

$$\mathrm{Res}_D \colon \Omega_X^1(\log D) \to \mathcal{O}_D.$$

The residue map is a natural map which is independent of the choice of coordinates. Therefore, the adjunction formula is also a natural formula. Note that this adjunction formula still holds if D is normal and $D \cap \mathrm{Sing}(X)$ has codimension at least 2 in D, since we can first apply the above adjunction formula to $D \setminus \mathrm{Sing}(X) \subset X \setminus \mathrm{Sing}(X)$, then extend it to D by the normality.

When D is not a prime divisor but a normal crossing divisor, if we take an irreducible component D_1 of D and write $E = (D - D_1)|_{D_1}$, then we have the adjunction formula

$$(K_X + D)|_{D_1} = K_{D_1} + E.$$

Here the restriction E is well defined since the intersection of $D - D_1$ and D_1 is of codimension 1 on D_1.

More generally, we can consider the adjunction formula as a relation between canonical divisors of relevant varieties. For example, consider a surjective finite morphism $f \colon Y \to X$ between smooth algebraic varieties, which is a ramified cover whose ramification locus is a smooth prime divisor D on X with ramification index m. The set-theoretic inverse image $E = f^{-1}(D)$ is a smooth prime divisor on Y and $f^*D = mE$. In this case, the *ramification formula* or the adjunction formula with respect to the ramification is the following:

$$K_Y = f^*K_X + (m-1)E.$$

If written as

$$K_Y = f^*\left(K_X + \frac{m-1}{m}D\right),$$

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then it looks like the adjunction formula for subvarieties. The latter formula is the origin of considering log canonical divisors with boundary divisors with rational coefficients. Also, if you write

$$K_Y + E = f^*(K_X + D),$$

you will find that "ramification is killed by log setting."

As another example of the adjunction formula, consider the blowup of an n-dimensional smooth algebraic variety X along an r-codimensional smooth subvariety Z. The blowup $f: Y \to X$ is a birational morphism with exceptional set E, which is a prime divisor isomorphic to a $\mathbf{P}^{r-1}$-bundle over Z. The changing of canonical divisors is given by

$$K_Y = f^*K_X + (r-1)E.$$

As shown in the following example, if X is a singular normal algebraic variety and D is a prime divisor on X intersecting $\mathrm{Sing}(X)$ such that $D \cap \mathrm{Sing}(X)$ contains an irreducible component of codimension 1 on D, then the singularities contribute to the adjunction formula. This phenomenon is called the *subadjunction formula*, which is very important.

Example 1.3.2 Let X be the quadric surface defined by the equation $xy + z^2 = 0$ in the projective space $\mathbf{P}^3$ with homogeneous coordinates x, y, z, w. X has a singularity at $[0 : 0 : 0 : 1]$. Let H be a hyperplane section of X, then $K_X \sim -2H$.

The projective line L defined by $x = z = 0$ is a prime divisor on X. We have $\mathrm{div}(x) = 2L$ on X, hence $L \sim_{\mathbf{Q}} \frac{1}{2}H$. Therefore, $(K_X + L)|_L \sim_{\mathbf{Q}} -\frac{3}{2}H|_L$ since $L|_L \sim_{\mathbf{Q}} \frac{1}{2}H|_L$. On the other hand, $K_L \sim -2H|_L$. Therefore, we have the subadjunction formula $(K_X + L)|_L = K_L + \frac{1}{2}H|_L$ (see Remark 1.11.14).

1.4 Intersection Numbers and Numerical Geometry

Problems in algebraic geometry are equivalent to solving systems of polynomial equations, which are highly nonlinear. *Numerical geometry* attempts to linearize those problems using intersection numbers. In the following two sections, we explain basic definitions in numerical geometry. In Chapter 2, we will explain the basepoint-free theorem and the cone theorem which are important theorems in numerical geometry. The explanation here is according to Kleiman ([80]). We refer to the original paper for the proof of the ampleness criterion.

All definitions here will be for a proper morphism $f: X \to S$ between algebraic varieties over a field k. In the case $S = \operatorname{Spec} k$, the definitions are for a proper algebraic variety X. We use words "relative" or "over S" for all definitions in this section. In the case $S = \operatorname{Spec} k$, those words can be removed. For simplicity, one can just consider $S = \operatorname{Spec} k$ and ignore the word "relative," the context will be almost the same. However, it is indispensable to consider the relative version in applications.

In the following definition, k is an arbitrary field, and X is of finite type over k, not necessarily irreducible or reduced. However, in this book when considering Cartier divisors, X is always assumed to be a normal algebraic variety.

A closed subvariety Z on X is called a *relative subvariety* over S if $f(Z)$ is a closed point of S. In particular, if $\dim Z = 1$, it is called a *relative curve* over S. Denote $\dim Z = t$ and take t invertible sheaves $L_1, \ldots, L_t$ on X. Then the *intersection number* $(L_1 \cdots L_t \cdot Z)$ is defined as the coefficient of the following polynomial ([80, p. 296])

$$\chi(Z, L_1^{\otimes m_1} \otimes \cdots \otimes L_t^{\otimes m_t} \otimes \mathcal{O}_Z) = (L_1 \cdots L_t \cdot Z) m_1 \cdots m_t + (\text{other terms}).$$

Here $m_1, \ldots, m_t$ are variables with integer values and

$$\chi(Z, \bullet) = \sum (-1)^p \dim_k H^p(Z, \bullet)$$

is the *Euler–Poincaré characteristic*. Here X itself is not necessarily proper, but Z is proper as $f(Z)$ is a point, hence the cohomology groups are finite-dimensional.

The intersection number $(L_1 \cdots L_t \cdot Z)$ takes integer value and it is a symmetric t-linear form with respect to $L_1, \ldots, L_t$ ([80, p. 296]). That is, it is independent of the order of L_i and

$$((L_1^{\otimes n_1} \otimes L_1'^{\otimes n_1'}) \cdots L_t \cdot Z) = n_1 (L_1 \cdots L_t \cdot Z) + n_1' (L_1' \cdots L_t \cdot Z).$$

For Cartier divisors $D_1, \ldots, D_t$, define

$$(D_1 \cdots D_t \cdot Z) = (\mathcal{O}_X(D_1) \cdots \mathcal{O}_X(D_t) \cdot Z).$$

In particular, when $\dim Z = 1$, taking $\nu: Z^\nu \to Z$ to be the normalization where Z^ν is a smooth projective curve, then by the Riemann–Roch theorem,

$$(D_1 \cdot Z) = \deg_{Z^\nu}(\nu^*(\mathcal{O}_X(D_1)|_Z)).$$

When $Z = X$, we simply write $(D_1 \cdots D_t) = (D_1 \cdots D_t \cdot X)$. If, moreover, all D_i are the same D, then write $(D_1 \cdots D_t) = (D^t)$.

By multi-linearity, the definition of $(D_1 \cdots D_t \cdot Z)$ can be extended to the case when D_i are **R**-Cartier divisors, which takes value in real numbers.

Remark 1.4.1 (1) Here we use Euler–Poincaré characteristic to give a simple definition for intersection numbers, but the correct geometric definition of intersection numbers is by adding up local intersection numbers to get the global intersection number. This is how the number of "intersection points" is defined originally. Using the geometric definition, for effective **R**-Cartier divisors D_i and a t-dimensional relative subvariety Z, if the intersection $\bigcap_{i=1}^t \operatorname{Supp}(D_i) \cap Z$ is non-empty and of dimension 0, then the intersection number is positive, and if the intersection is empty, then the intersection number is 0. These two definitions of intersection numbers coincide.

(2) By using the definition of intersection numbers of divisorial sheaves, we can define the *self-intersection number* of a divisor, which seems to be a weird name. For example, for an effective Cartier divisor D on an n-dimensional algebraic variety, the self-intersection number (D^n) can be either positive or nonpositive.

(3) In this book, a *curve* is an irreducible reduced projective variety of dimension 1. The intersection number considered in this book is mainly the intersection number of a Cartier divisor with a curve.

Among all curves, *rational curve* plays a very important role in the minimal model theory (see Sections 2.7 and 2.8). A rational curve is a curve whose normalization is isomorphic to $\mathbf{P}^1$. In general a rational curve might have singularities and not necessarily be isomorphic to $\mathbf{P}^1$ itself.

Example 1.4.2 The intersection number of a divisor and a curve can be defined if this divisor is a **Q**-Cartier divisor. However, the intersection number is not necessarily an integer if the divisor is not Cartier. In general it cannot be defined if the divisor is not **Q**-Cartier.

Consider X, as in Example 1.1.4 or 1.2.4, and let $\bar{X}$ be its compactification in the projective space $\mathbf{P}^3$ or $\mathbf{P}^4$.

(1) $\bar{X}$ is defined by the equation $xy = z^2$ in $\mathbf{P}^3$ with homogeneous coordinates u, x, y, z. The compactification $\bar{D}$ of D is a prime divisor defined by $x = z = 0$. In this case, $(\bar{D}^2) = \frac{1}{2}$.

In fact, take a plane $\bar{H}$, then $\bar{H}|_{\bar{X}} \sim \operatorname{div}(x) = 2\bar{D}$ and $(\bar{H} \cdot \bar{D}) = 1$.

(2) $\bar{X}$ is defined by the equation $xy = zw$ in $\mathbf{P}^4$ with homogeneous coordinates u, x, y, z, w. The compactifications $\bar{D}_1, \bar{D}_2$ of D_1, D_2 are prime divisors defined by $x = z = 0$, $x = w = 0$. Take the curve C defined by $y = z = w = 0$. $\bar{D}_1 + \bar{D}_2$ is a Cartier divisor and $((\bar{D}_1 + \bar{D}_2) \cdot C) = 1$. The blowup $f_1 \colon Y_1 \to \bar{X}$ is isomorphic in codimension 1. If intersection numbers $(\bar{D}_i \cdot C)$ $(i = 1, 2)$ could be defined, by the projection formula stated later (before Proposition 1.4.3), $(\bar{D}_i \cdot C) = (f_{1*}^{-1}\bar{D}_i \cdot f_{1*}^{-1}C)$.

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The right-hand side can be calculated to be $1, 0$ for $i = 1, 2$. This is absurd since the relations between $\bar{D}_1, \bar{D}_2$ and C are symmetric.

Two invertible sheaves L, L' are said to be *relatively numerically equivalent*, denoted by $L \equiv_S L'$, if $(L \cdot C) = (L' \cdot C)$ for any relative curve C. When the base is clear, we just write $L \equiv L'$. The Abelian group consisting of isomorphism classes of all invertible sheaves is denoted by $\mathrm{Pic}(X)$ and the subgroup consisting of all invertible sheaves relatively numerically equivalent to $\mathcal{O}_X$ is denoted by $\mathrm{Pic}^\tau(X/S)$. The quotient group $\mathrm{Pic}(X)/\mathrm{Pic}^\tau(X/S)$ is a finitely generated Abelian group ([80, p. 323]), which is called the *relative Neron–Severi group*, and is denoted by $\mathrm{NS}(X/S)$. $\rho(X/S) = \operatorname{rank} \mathrm{NS}(X/S)$ is called the *relative Picard number*. When $S = \operatorname{Spec} k$, it is just called the *Picard number* and is denoted by $\rho(X)$.

If $L_1 \equiv_S \mathcal{O}_X$, then the equality $(L_1 \cdot L_2 \cdots L_t \cdot Z) = 0$ holds for arbitrary $L_2, \ldots, L_t, Z$ ([80, p. 304]). Also, for any coherent sheaf F on a relative subvariety Z, $\chi(Z, F) = \chi(Z, F \otimes L_1)$ holds ([80, p. 311]).

Two **R**-Cartier divisors D, D' are said to be *relatively numerically equivalent*, denoted by $D \equiv_S D'$ or $D \equiv D'$, if $(D \cdot C) = (D' \cdot C)$ for any relative curve C. The numerical equivalence class of D is denoted by $[D]$. The set of all numerical equivalence classes of **R**-Cartier divisors coincides with $\mathrm{NS}(X/S) \otimes \mathbf{R}$, which is a $\rho(X/S)$-dimensional real vector space and is denoted by $N^1(X/S)$.

If X is a smooth complete complex manifold, $D \equiv D'$ is equivalent to having the same cohomology class $[D] = [D'] \in H^2(X, \mathbf{R})$.

Fix an integer t, a finite formal linear sum of t-dimensional relative subvarieties $Z = \sum a_j Z_j$ is called a *relative t-cycle*. The coefficients a_i can be integers, rational numbers, or real numbers depending on the situation. By linearity, intersection numbers can be defined for relative t-cycles. In this book, we only consider the case $t = 1$ or $\dim X - 1$.

Relative 1-cycles C, C' are said to be *numerically equivalent*, denoted by $C \equiv_S C'$, if $(D \cdot C) = (D \cdot C')$ for any Cartier divisor D. The set $N_1(X/S)$ of all numerical equivalence classes of relative 1-cycles with real coefficients is a finite-dimensional real vector space. $N_1(X/S)$ and $N^1(X/S)$ are dual linear spaces to each other.

Let $g\colon Y \to X$ be a proper morphism from another algebraic variety. For a relative subvariety Z on Y over S, the *direct image* g_*Z as an algebraic cycle is defined as the following: if $\dim g(Z) = \dim Z$, then $g_*Z = [k(Z)\colon k(g(Z))]g(Z)$; if $\dim g(Z) < \dim Z$, then $g_*Z = 0$. Here $g(Z)$ is the set-theoretic image of Z, and $[k(Z)\colon k(g(Z))]$ is the extension degree of function fields. If g is a birational morphism, then g_*Z coincides with the strict

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transform defined before. Also, for a relative t-cycle $Z = \sum a_j Z_j$, its direct image can be defined as $g_* Z = \sum a_j g_* Z_j$ by linearity.

For a relative t-cycle Z and invertible sheaves $L_1, \ldots, L_t$ on X, the *projection formula*

$$(g^* L_1 \cdots g^* L_t \cdot Z) = (L_1 \cdots L_t \cdot g_* Z)$$

holds ([80, p. 299]). In this book, we often use this formula for $t = 1$ in which case

$$(g^* L \cdot C) = (L \cdot g_* C).$$

Proposition 1.4.3 ([80, p. 304]) *Let $f: X \to S$ and $g: Y \to X$ be two proper morphisms. Consider the pullback $g^* L$ of an invertible sheaf L on X.*

(1) If $L \equiv_S 0$, then $g^* L \equiv_S 0$. Therefore, g induces a natural linear map $g^*: N^1(X/S) \to N^1(Y/S)$.
(2) When g is surjective, conversely, if $g^* L \equiv_S 0$, then $L \equiv_S 0$, hence the pullback map g^* is injective.

Proof (1) For any relative curve C' on Y,

$$\begin{aligned}&(g^* L \cdot C') = (L \cdot g_* C') \\ = &\begin{cases} [k(C') : k(g(C'))](L \cdot g(C')) & \text{if } \dim g(C') = 1, \\ 0 & \text{if } \dim g(C') = 0, \end{cases}\end{aligned}$$

which implies the assertion.

(2) If g is surjective, for any relative curve C on X, there exists a relative curve C' on Y such that $C = g(C')$, which proves the assertion. □

On the other hand, let $h: S \to T$ be a proper morphism, then the identity map on $\mathrm{Div}(X)$ induces a surjective linear map $(1/h)^*: N^1(X/T) \to N^1(X/S)$. By taking the dual, $(1/h)_*: N_1(X/S) \to N_1(X/T)$ is injective.

For proper morphisms $g: Y \to X$ and $f: X \to S$, the composition of $g^*: N^1(X/S) \to N^1(Y/S)$ and $(1/f)^*: N^1(Y/S) \to N^1(Y/X)$ is 0.

1.5 Cones of Curves and Cones of Divisors

Cones and polytopes in finite-dimensional vector spaces play important roles in this book. In Chapter 2, morphisms from algebraic varieties can be constructed by using faces of convex cones (the cone theorem). Also, in Chapter 3, a sequence of rational maps can be analyzed from the behavior of a sequence of polytopes.

1.5.1 Pseudo-Effective Cones and Nef Cones

We will define the closed convex cone generated by numerical equivalence classes of curves in the real vector space $N_1(X/S)$ and the closed convex cones generated by numerical equivalence classes of effective divisors and nef divisors in the dual space $N^1(X/S)$.

A subset $\mathscr{C}$ in a finite-dimensional vector space V is called a *convex cone* if for any $a, a' \in \mathscr{C}$ and $r > 0$, $a + a' \in \mathscr{C}$ and $ra \in \mathscr{C}$ hold. It is called a *closed convex cone* if moreover it is a closed subset.

For an element $u \in V^*$ in the dual space, define $\mathscr{C}_{u \geq 0} = \{v \in \mathscr{C} \mid (u \cdot v) \geq 0\}$. $\mathscr{C}_{u=0}$ and $\mathscr{C}_{u<0}$ can be defined similarly. The *dual closed convex cone* of a closed convex cone $\mathscr{C}$ is defined by

$$\mathscr{C}^* = \bigcap_{v \in \mathscr{C}} V^*_{v \geq 0} = \{u \in V^* \mid (u \cdot v) \geq 0 \text{ for any } v \in \mathscr{C}\}.$$

As $\mathscr{C}$ is a closed convex cone, $v \in \mathscr{C}$ is equivalent to $(u \cdot v) \geq 0$ for all $u \in \mathscr{C}^*$. That is, $\mathscr{C} = \mathscr{C}^{**}$.

For a morphism $f \colon X \to S$, an invertible sheaf L on X is said to be *relatively ample*, or *ample over* S, or f*-ample*, if there exists an open covering $\{S_i\}$ of S, positive integers m, N, and *locally closed* immersions $g_i \colon X_i = f^{-1}(S_i) \to \mathbf{P}^N \times S_i$ such that $L^{\otimes m}|_{X_i} \cong g_i^* p_1^* \mathcal{O}_{\mathbf{P}^N}(1)$, where $p_1 \colon \mathbf{P}^N \times S_i \to \mathbf{P}^N$ is the first projection. Here the left-hand side is the mth tensor power of L, and the right-hand side is the pullback of the invertible sheaf corresponding to a hyperplane section by the first projection and g_i. A Cartier divisor D is said to be *relatively ample* if its divisorial sheaf $\mathcal{O}_X(D)$ is relatively ample. A morphism admitting a relatively ample invertible sheaf is said to be *quasi-projective*. In particular, if all immersions g_i are closed immersions, then the morphism is said to be *projective*.

Here we recall the following useful fact. Let $f \colon X \to S$ and $g \colon Y \to X$ be two projective morphisms, let A be an f-ample Cartier divisor on X, and let B be a g-ample Cartier divisor on Y. Then $ng^*A + B$ is ample over S for sufficiently large n ([44, II.7.10]).

In the following, X is assumed to be normal and the morphism $f \colon X \to S$ is assumed to be projective.

In general, the convex cone consisting of numerical equivalence classes of all effective **R**-Cartier divisors is neither closed nor open. This is because there might be infinitely many prime divisors showing up when considering a limit of effective divisors in $N^1(X/S)$. The closure of this cone is denoted by $\overline{\mathrm{Eff}}(X/S)$, which is called the *relative pseudo-effective cone*. An **R**-Cartier

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divisor D is said to be *relatively pseudo-effective* if its numerical equivalence class $[D]$ is contained in $\overline{\mathrm{Eff}}(X/S)$.

The set of interior points of the closed convex cone $\overline{\mathrm{Eff}}(X/S)$ is called the *relative big cone* and is denoted by $\mathrm{Big}(X/S)$. Recall that in Section 1.2, we introduced the definition of an **R**-Cartier divisor D being *relatively big* or *f-big*. By Kodaira's lemma later (Corollary 1.5.8), it can be shown that an **R**-Cartier divisor D is relatively big if and only if its numerical equivalence class $[D]$ is contained in $\mathrm{Big}(X/S)$.

An **R**-Cartier divisor D is said to be *relatively nef* or *f-nef* if $(D \cdot C) \geq 0$ for any relative curve C. This is also called *relatively numerically effective*. "Nef" is an abbreviation, but it is commonly used now. The set of numerical equivalence classes of all nef **R**-Cartier divisors is a closed convex cone in $N^1(X/S)$, which is denoted by $\overline{\mathrm{Amp}}(X/S)$ and called the *relative nef cone*.

The set of interior points of the relative nef cone is called the *relative ample cone* and is denoted by $\mathrm{Amp}(X/S)$. An **R**-Cartier divisor D is said to be *relatively ample* or *f-ample* if its numerical equivalence class is contained in $\mathrm{Amp}(X/S)$. This definition will be justified by Kleiman's theorem discussed later (Theorem 1.5.4): For a Cartier divisor D, being f-ample in this sense is equivalent to being f-ample in the original sense.

By definition, the sum of a relatively ample **R**-Cartier divisor and a relatively nef **R**-Cartier divisor is again a relatively ample **R**-Cartier divisor.

In the dual space $N_1(X/S)$, the cone of relative curves is the convex cone generated by numerical equivalence classes of all relative curves, which is in general neither open nor closed. Its closure is called the *closed cone of relative curves*, which is denoted by $\overline{\mathrm{NE}}(X/S)$. By definition, the latter is the dual closed convex cone of the relative nef cone and the relative ample cone:

$$\overline{\mathrm{Amp}}(X/S) = \{u \in N^1(X/S) \mid (u \cdot v) \geq 0 \text{ for all } v \in \overline{\mathrm{NE}}(X/S)\} \text{ and}$$
$$\mathrm{Amp}(X/S) = \{u \in N^1(X/S) \mid (u \cdot v) > 0 \text{ for all } v \in \overline{\mathrm{NE}}(X/S) \setminus \{0\}\}.$$

Remark 1.5.1 The cones $\overline{\mathrm{Amp}}(X/S)$ and $\overline{\mathrm{NE}}(X/S)$ considered here contain interior points, but contain no linear subspaces. This is because $f: X \to S$ is assumed to be a projective morphism. For example, $\overline{\mathrm{NE}}(X/S)$ contains no lines since the intersection number of a relatively ample divisor with a nonzero element in $\overline{\mathrm{NE}}(X/S)$ is always positive by Theorem 1.5.4. A relatively ample divisor is also called a *polarization* as it gives the positive direction.

The structures of relative nef cones and closed cones of relative curves are important themes of this book.

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Proposition 1.5.2 ([80, p. 337]) *Let $f\colon X \to S$ and $g\colon Y \to X$ be two projective morphisms, and let L be an invertible sheaf on X.*

(1) If L is f-nef, then the pullback g^*L is $(f \circ g)$-nef.
(2) If g is surjective and g^*L is $(f \circ g)$-nef, then L is f-nef.
(3) If g is surjective, then

$$g^*\overline{\mathrm{Amp}}(X/S) = \overline{\mathrm{Amp}}(Y/S) \cap g^*N^1(X/S).$$

(4) Assume that g is surjective. If moreover g is a finite morphism*, then*

$$g^*\mathrm{Amp}(X/S) = \mathrm{Amp}(Y/S) \cap g^*N^1(X/S),$$

otherwise

$$g^*\overline{\mathrm{Amp}}(X/S) = \partial\overline{\mathrm{Amp}}(Y/S) \cap g^*N^1(X/S).$$

Here ∂ is the boundary of the closed convex cone.

Proof The proof of (1) and (2) is similar to that of Proposition 1.4.3. (3) follows from (2).

(4) When g is a finite morphism, the pullback of a relatively ample invertible sheaf is again a relatively ample invertible sheaf, hence the former assertion follows. On the other hand, when g is not a finite morphism, the pullback of a relatively ample invertible sheaf is never a relatively ample invertible sheaf, hence the latter assertion follows from (3). □

It was shown that a nonfinite morphism gives a face of the relative nef cone. Conversely, sometimes it is possible to construct a nonfinite morphism from a face of the relative nef cone; this is the contraction theorem in the minimal model theory.

Example 1.5.3 (1) Let X be a smooth projective *complex algebraic surface* and let C be a curve on X with negative self-intersection $(C^2) < 0$.

For any curve C' different from C, the intersection number is always nonnegative: $(C \cdot C') \geq 0$. Denote by $\mathscr{C}' \subset N_1(X)$ the closed convex cone generated by the numerical equivalence classes of all curves C' different from C, then the closed cone of curves $\overline{\mathrm{NE}}(X)$ is generated by $\mathscr{C}'$ and $[C]$.

Since $(C \cdot C') \geq 0$ for all $C' \in \mathscr{C}'$, $[C] \notin \mathscr{C}'$. Therefore, one can see that $[C]$ generates an *extremal ray* of $\overline{\mathrm{NE}}(X)$. Taking the dual, we get a face $F = \overline{\mathrm{Amp}}(X)_{C=0}$ of $\overline{\mathrm{Amp}}(X)$.

According to a result of Grauert ([33]), there exists a compact complex analytic surface Y with only normal singularities and a birational morphism $f\colon X \to Y$ between complex analytic surfaces such that C is contracted to a point. That is, $f(C)$ is a point and there is an isomorphism

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$f: X \setminus C \to Y \setminus f(C)$. However, Y is in general not an algebraic variety. But according to a result of Artin ([6]), if $C \cong \mathbf{P}^1$, then Y is a projective algebraic surface and f becomes a birational morphism between algebraic varieties.

In this sense, it may or may not be possible to construct a morphism from a face of the nef cone.

(2) Let X be an *Abelian variety*, that is, a smooth projective algebraic variety with an algebraic group structure. In this case, any prime divisor D on X is nef, and

$$\mathrm{Amp}(X) = \{v \in N^1(X) \mid (v^n) > 0\}^0.$$

Here $n = \dim X$ and ^{0} on the right-hand side means one of the connected components.

1.5.2 Kleiman's Criterion and Kodaira's Lemma

In this subsection, we introduce Kleiman's ampleness criterion. We also prove Kodaira's lemma, which characterizes big divisors.

Theorem 1.5.4 (*Kleiman's criterion* [80]) *For a projective morphism $f: X \to S$ between algebraic varieties, a Cartier divisor D on X is relatively ample if and only if its numerical equivalence class is contained in the relative ample cone* $\mathrm{Amp}(X/S)$.

Remark 1.5.5 Kleiman's criterion is a paraphrase of Nakai's criterion for projectivity and ampleness using the language of cones of divisors. In Kleiman's criterion as well as Nakai's criterion, X is not necessarily assumed to be irreducible or reduced. It is also not necessarily assumed to be projective, and whether a proper scheme is projective can be determined by whether $\mathrm{Amp}(X)$ is not empty.

As ampleness is an algebro-geometric property which is nonlinear, we can say that it is linearized by Kleiman's criterion using conditions in numerical geometry. This is a typical example of numerical geometry.

An invertible sheaf L on a projective algebraic variety X induces a functional h_L on the dual space $N_1(X)$. By Kleiman's criterion, L is ample if and only if h_L is positive on the closed cone of curves $\overline{\mathrm{NE}}(X)$.

This condition is strictly stronger than the condition that $h_L(C) = (L \cdot C) > 0$ for any curve C. We explain this by the following example:

Example 1.5.6 (Mumford's example) Let Γ be a smooth projective complex algebraic curve of genus at least 2 and let F be a locally free sheaf on Γ of rank 2 and of degree 0. The last condition means that $\bigwedge^2 F \equiv \mathcal{O}_\Gamma$. Assume that F

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is *stable*, that is, $\deg(M) < 0$ for any invertible subsheaf M of F. Such F can be constructed by using unitary representations of the fundamental group $\pi_1(\Gamma)$. In this case, for any surjective morphism $f\colon C \to \Gamma$ from a smooth projective curve, f^*F is also stable.

Let $X = \mathbf{P}(F)$ be the corresponding $\mathbf{P}^1$-bundle over Γ and $L = \mathcal{O}_{\mathbf{P}(F)}(1)$. Let C_0 be a curve on X. If it is not a fiber of f, take $f\colon C \to \Gamma$ to be the composition of the normalization $g\colon C \to C_0$ and the projection $C_0 \to \Gamma$. In this case, g^*L is an invertible sheaf which is a quotient of f^*F, hence its degree is positive. If C_0 is a fiber of f, then $(L \cdot C_0) = 1$. That is, the inequality $(L \cdot C_0) > 0$ holds for any curve C_0 on X. On the other hand, $(L^2) = 0$ since $\deg(F) = 0$, which means that L is not ample.

The following Kodaira's lemma gives a characterization of big divisors.

Theorem 1.5.7 (*Kodaira's lemma*) *(1) A Cartier divisor D on a normal projective algebraic variety X is big if and only if there exists a positive integer m, an ample Cartier divisor A, and an effective Cartier divisor E such that $mD = A + E$.*

(2) For a surjective projective morphism $f\colon X \to S$ from a normal algebraic variety to a quasi-projective algebraic variety, a Cartier divisor D on X is relatively big if and only if there exists a positive integer m, a relatively ample Cartier divisor A, and an effective Cartier divisor E such that $mD = A + E$.

In other words, big divisors are divisors bigger than ample divisors.

Proof (1) As ample divisors are big, the condition is sufficient.

Conversely, assume that D is big. Denote $n = \dim X$. Take a very ample Cartier divisor A and a general element in its complete linear system $Y \in |A|$. Consider the following exact sequence:

$$0 \to \mathcal{O}_X(mD - Y) \to \mathcal{O}_X(mD) \to \mathcal{O}_Y(mD|_Y) \to 0.$$

Look at the first part of the corresponding long exact sequence

$$0 \to H^0(X, mD - Y) \to H^0(X, mD) \to H^0(Y, mD|_Y),$$

as $\dim Y = n-1$, the dimension of the last term is bounded by cm^{n-1} for some constant c. But by the bigness, the central term goes much larger, so the first term is not 0 for sufficiently large m. Hence there exists an effective divisor E with linear equivalence $mD - Y \sim E$. In this case, $mD - E \sim Y$ is ample and the proof is completed.

(2) As the restriction of relatively ample (respectively, effective) divisors on the generic fiber are ample (respectively, effective), the condition is sufficient.

Conversely, assume that D is relatively big. By the argument of (1), for a relatively ample Cartier divisor A, there exists a sufficiently large m such that the direct image sheaf $f_*(\mathcal{O}_X(mD-A)) \neq 0$. Take a sufficiently ample Cartier divisor B on S such that

$$H^0(X, mD - A + f^*B) = H^0(S, f_*(\mathcal{O}_X(mD - A)) \otimes \mathcal{O}_S(B)) \neq 0.$$

Then there exists an effective Cartier divisor E with linear equivalence $mD - A + f^*B \sim E$. In this case, $mD - E \sim A - f^*B$ is relatively ample and the proof is completed. □

As a corollary, together with Kleiman's criterion, the definition of relative big cones is justified:

Corollary 1.5.8 *For a surjective projective morphism $f: X \to S$ from a normal algebraic variety to a quasi-projective algebraic variety, a Cartier divisor D on X is relatively big if and only if the numerical equivalence class $[D]$ is contained in the relative big cone* $\text{Big}(X/S)$.

Proof By Kleiman's criterion and Kodaira's lemma, D is relatively big if and only if $[D]$ is an interior point of the closed convex cone generated by effective divisors. □

Corollary 1.5.9 $\overline{\text{Amp}}(X/S) \subset \overline{\text{Eff}}(X/S)$.

Proof As ample divisors are big, we have an inclusion $\text{Amp}(X/S) \subset \text{Big}(X/S)$ between open cones. The conclusion follows by taking closures. □

Kodaira's lemma can be generalized as the following:

Corollary 1.5.10 *(1) An* **R***-Cartier divisor D on a normal projective algebraic variety X is big if and only if there exists an ample* **R***-Cartier divisor A, and an effective* **R***-Cartier divisor E such that $D = A + E$.*

(2) For a surjective projective morphism $f: X \to S$ from a normal algebraic variety to a quasi-projective algebraic variety, an **R***-Cartier divisor D on X is relatively big if and only if there exists a relatively ample* **R***-Cartier divisor A and an effective* **R***-Cartier divisor E such that $D = A + E$.*

Proof (1) Assume that $D = A + E$. Then there exists an ample **Q**-Cartier divisor A' and an effective **R**-Cartier divisor E' such that we can write $A = A' + E'$, hence D is big.

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Conversely, assume that D is big. By the proof of Kodaira's lemma, for sufficiently large m, there exists an ample Cartier divisor A and an effective divisor E such that $\llcorner mD \lrcorner = A + E$. Since $mD - \llcorner mD \lrcorner$ is effective, the assertion is proved.

(2) It is similarly deduced from the relative version of Kodaira's lemma. □

Proposition 1.5.11 *Let $f : Y \to X$ be a birational morphism between normal projective algebraic varieties and let D be an* **R***-Cartier divisor on X. Then D is big if and only if the pullback f^*D is big.*

Proof For a rational function $h \in k(X) \cong k(Y)$, $\operatorname{div}_X(h) + \llcorner mD \lrcorner \geq 0$ is equivalent to $\operatorname{div}_X(h) + mD \geq 0$. Here the subscript X means taking the corresponding divisor on X. The latter is equivalent to $\operatorname{div}_Y(h) + mf^*D \geq 0$, which is then equivalent to $\operatorname{div}_Y(h) + \llcorner mf^*D \lrcorner \geq 0$. Therefore, the natural homomorphism $H^0(X, \llcorner mD \lrcorner) \to H^0(Y, \llcorner mf^*D \lrcorner)$ is bijective, and the assertion is concluded. □

Theorem 1.5.12 ([92, Theorem 2.2.16]) *Let X be an n-dimensional projective algebraic variety and let D be a nef* **R***-Cartier divisor. Then D is big if and only if $(D^n) > 0$.*

Proof If D is big, then we can write $D = A + E$ for some ample **Q**-divisor A and some effective **R**-divisor E. In this case, since D and A are nef,

$$\begin{aligned}(D^n) &= (D^{n-1} \cdot A) + (D^{n-1} \cdot E) \geq (D^{n-1} \cdot A) \\ &= (D^{n-2} \cdot A^2) + (D^{n-2} \cdot A \cdot E) \geq \cdots \geq (A^n) > 0.\end{aligned}$$

Here we use the fact that if $D_1, \ldots, D_n$ are **R**-divisors on X such that $D_1, \ldots, D_{n-1}$ are nef and D_n is either effective or nef, then $(D_1 \cdots D_n) \geq 0$.

Conversely, to show that D is big provided that D is nef and $(D^n) > 0$, we will show the following slightly generalized statement: If for two nef **R**-Cartier divisors L, M we have $(L^n) > n(L^{n-1} \cdot M)$, then $L - M$ is big. The theorem follows by taking $M = 0$.

First, we assume that L, M are ample **Q**-Cartier divisors. We may assume that they are both very ample by taking a common multiple. Taking m general elements $M_i \in |M|$ $(1 \leq i \leq m)$, by the exact sequence

$$0 \to \mathcal{O}_X(m(L - M)) \to \mathcal{O}_X(mL) \to \bigoplus_i \mathcal{O}_{M_i}(mL),$$

the Riemann–Roch theorem, and the Serre vanishing theorem, when $m \to \infty$, we have

$$\begin{aligned}
&\dim H^0(X, m(L-M)) \\
&\geq \dim H^0(X, mL) - \sum_{i=1}^{m} \dim H^0(M_i, mL|_{M_i}) \\
&= \frac{(L^n)}{n!} m^n - \sum_{i=1}^{m} \frac{(L^{n-1} \cdot M_i)}{(n-1)!} m^{n-1} + O(m^{n-1}) \\
&= \frac{(L^n) - n(L^{n-1} \cdot M)}{n!} m^n + O(m^{n-1}).
\end{aligned}$$

Here note that for each M_i, the dimension of $H^0(M_i, mL|_{M_i})$ is independent of the choice of M_i, and it can be estimated by $O(m^{n-2})$. Therefore, $L-M$ is big.

Then we consider the general case. We may take two sufficiently small ample **R**-Cartier divisors H, H' such that $H' - H$ is big and $L + H$, $M + H'$ are ample **Q**-Cartier divisors. Here H, H' can be taken sufficiently small in the sense that $((L+H)^n) > n((L+H)^{n-1} \cdot (M + H'))$ holds. Then we already showed that $L + H - M - H'$ is big, which implies that $L - M$ is big. ☐

We can investigate how cones of divisors change under birational maps:

Lemma 1.5.13 *Let $\alpha\colon X \dashrightarrow X'$ be a birational map between normal* **Q***-factorial varieties which is isomorphic in codimension* 1 *and let $f\colon X \to S$ and $f'\colon X' \to S$ be projective morphisms with $f = f' \circ \alpha$.*

(1) α induces an isomorphism $\alpha_*\colon N^1(X/S) \to N^1(X'/S)$ between real linear spaces.
(2) $\alpha_*(\overline{\mathrm{Eff}}(X/S)) = \overline{\mathrm{Eff}}(X'/S)$.
(3) If α is not an isomorphism, then $\alpha_*(\mathrm{Amp}(X/S)) \cap \mathrm{Amp}(X'/S) = \emptyset$.

Proof (1) Since α is isomorphic in codimension 1, there is a 1–1 correspondence between prime divisors on X and X'. Hence $Z^1(X) \cong Z^1(X')$. Take a divisor D on X and take its strict transform $D' = \alpha_* D$. Applying the desingularization theorem discussed in Section 1.6, there exists a smooth algebraic variety W and birational projective morphisms $g\colon W \to X$ and $g'\colon W \to X'$ such that we can write $g^*D = (g')^*D' + E$, where $g_*E = 0$ and $g'_*E = 0$. Assume that $D \equiv_S 0$, then $g^*D \equiv_S 0$.

In the following we will show that $D' \equiv_S 0$. We may assume that $E \neq 0$ otherwise it is obvious. Write $E = E^+ - E^-$ into the positive part and the negative part. If $E^+ \neq 0$, then by the negativity lemma (Lemma 1.6.3), there exists a curve C contracted by g' such that $(E^+ \cdot C) < 0$ and $(E^- \cdot C) \geq 0$. On the other hand, $((g')^*D' \cdot C) = (D' \cdot g'_*C) = 0$ and $(g^*D \cdot C) = 0$, a contradiction. We can get a contradiction similarly if $E^- \neq 0$.

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(2) follows from (1) as the strict transform of an effective divisor is again effective.

(3) As the intersection is an open cone, if the intersection is non-empty, then there exists a relatively ample divisor D on X such that $\alpha_* D$ is a relatively ample divisor on X'. Since α is isomorphic in codimension 1, for any integer m, $\alpha_*\colon f_*\mathcal{O}_X(mD) \to f'_*\mathcal{O}_{X'}(mD')$ is an isomorphism. Therefore,

$$X = \operatorname{Proj}_S\left(\bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mD)\right) \cong \operatorname{Proj}_S\left(\bigoplus_{m=0}^{\infty} f'_*\mathcal{O}_{X'}(mD')\right) = X'$$

and α is an isomorphism. □

1.6 The Hironaka Desingularization Theorem

The desingularization theorem was established by Hironaka for algebraic varieties in characteristic 0. Although it is expected that the same theorem holds for positive characteristics and mixed characteristics, it is only proved in dimension 2 and for positive characteristics in dimension 3, while it remains open in the general case. Together with the Kodaira vanishing theorem, they are very important theorems in characteristic 0. Here we introduce the desingularization theorem ([45]) without proof.

Theorem 1.6.1 (*Hironaka desingularization theorem*) *(1) For any algebraic variety X defined over a field of characteristic* 0*, there exists a smooth algebraic variety Y and a birational projective morphism $f\colon Y \to X$.*

(2) For any algebraic variety X defined over a field of characteristic 0 *and a proper closed subset B of X, there exists a smooth algebraic variety Y, a normal crossing divisor C on Y, and a birational projective morphism $f\colon Y \to X$ with the following properties:*

(a) If B is non-empty, then the set-theoretic inverse image $f^{-1}(B)$ is a union of several irreducible components of C.

(b) The exceptional set $\operatorname{Exc}(f)$ *is a union of several irreducible components of C.*

For each statement, we can assume further the following properties hold:

(1') f is isomorphic over the smooth locus $\operatorname{Reg}(X) = X \setminus \operatorname{Sing}(X)$*, and the exceptional set* $\operatorname{Exc}(f)$ *coincides with the set-theoretic inverse image of the singular locus* $f^{-1}(\operatorname{Sing}(X))$.

(2') f is isomorphic over $\operatorname{Reg}(X, B)$ *and the exceptional set* $\operatorname{Exc}(f)$ *coincides with the set-theoretic inverse image* $f^{-1}(\operatorname{Sing}(X, B))$.

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A birational morphism with the property in (1) is called a *resolution of singularities* of the algebraic variety X. A birational morphism with the property in (2) is called a *log resolution* of the pair (X, B). For the definition of normal crossing divisors please refer to Section 1.1.

Remark 1.6.2 (1) If replacing the two conditions for the log resolution by the condition that $f^{-1}(B) \cup \text{Exc}(f)$ is a normal crossing divisor, we call it a *log resolution in weak sense*. This is called a log resolution in some literature. On the other hand, if we assume furthermore that $\text{Exc}(f)$ is the support of an f-ample divisor in condition (b), we call it a *log resolution in strong sense*. In this case, the f-ample divisor supported on $\text{Exc}(f)$ has negative coefficients according to Lemma 1.6.3 below.

(2) Hironaka's desingularization can be obtained by blowing up along smooth centers finitely many times. Since there exists a relatively ample divisor supported on the exceptional divisor with negative coefficient for a blowup along a smooth center, Hironaka's desingularization obtained in this way is a log resolution in strong sense.

By Theorem 1.6.4, starting from any log resolution, one can construct a log resolution in strong sense by further taking blowups along the exceptional set.

(3) In the latter part of the above theorem, a normal crossing divisor is in the sense of the Zariski topology, which is a "simple normal crossing divisor." It does not hold for normal crossing divisors in the complex analytic sense. For example, take the divisor B defined by the equation $x^2 + y^2 z = 0$ in $X = \mathbf{C}^3$. The singular locus of B is the line defined by $x = y = 0$ and B is a normal crossing divisor in the complex analytic sense if $z \neq 0$. However, the origin $P = (0, 0, 0)$ has the so-called *pinch point* singularity, no blowup which is isomorphic outside P can make B a normal crossing divisor.

(4) The above theorem is proved in Hironaka's original paper ([45]), but it has been shown that there exists a more precise "canonical resolution" in subsequent developments. The canonical resolution admits strong functoriality such that any local isomorphism (isomorphism between two open subsets) of the pair (X, B) lifts to a local isomorphism of (Y, C). However, the canonical resolution is not unique, it is only shown that there exists a universal choice ([11, 46, 140, 142]).

Lemma 1.6.3 (Negativity lemma) *Let $f : X \to Y$ be a birational projective morphism between normal algebraic varieties and let D be an* $\mathbf{R}$*-Cartier divisor on X supported in the exceptional set* $\text{Exc}(f)$.

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(1) If D is nonzero and effective, then there exists a curve C which is contracted by f and passes through a general point of an irreducible component of D such that $(D \cdot C) < 0$.

(2) If D is f-nef and nonzero, then the coefficients of D are all negative. Furthermore, the support of D coincides with the set-theoretic inverse image $f^{-1}(f(\mathrm{Supp}(D)))$.

(3) If D is f-ample, then the support of D coincides with $\mathrm{Exc}(f)$.

Proof We may assume that Y is affine. Consider $0 \le i \le \dim f(\mathrm{Supp}(D))$ and $j = \dim X - 2 - i$. Take Y_i by cutting Y by general hyperplane sections i times and take X_{ij} by cutting $f^{-1}(Y_i)$ by general hyperplane sections j times. Since $i + j = \dim X - 2$, X_{ij} is a normal algebraic surface. Let Y_{ij} be the normalization of $f(X_{ij})$, then f induces a birational projective morphism $f_{ij} : X_{ij} \to Y_{ij}$. Note that $D_{ij} = D|_{X_{ij}}$ is an $\mathbf{R}$-Cartier divisor supported in the exceptional set $\mathrm{Exc}(f_{ij})$.

(1) Since D is nonzero and effective, so is D_{ij} for some i, j. By the Hodge index theorem, applying Corollary 1.13.2 to $\pi : \tilde{X}_{ij} \to Y_{ij}$ and $\pi^* D_{ij}$, where $\tilde{X}_{ij}$ is a resolution of X_{ij}, we get $(D_{ij})^2 < 0$. In particular, there exists an irreducible component C of D_{ij} such that $(D_{ij} \cdot C) < 0$. View C as a curve in X, we have $(D \cdot C) < 0$. Note that by construction, C comes from cutting an irreducible component of D by hyperplane sections, so such C passes through a general point of an irreducible component of D.

(2) We may write $D_{ij} = D_{ij}^+ - D_{ij}^-$ in terms of its positive and negative parts. Since D_{ij} is f_{ij}-nef, $(D_{ij}^+)^2 \ge (D_{ij}^+ \cdot D_{ij}) \ge 0$. By the Hodge index theorem (Corollary 1.13.2), $D_{ij}^+ = 0$. Hence the coefficients of D_{ij} are negative. As i, j varies, any coefficient of D appears as the coefficient of some D_{ij}. So, the coefficients of D are all negative. If the support of D does not coincide with $f^{-1}(f(\mathrm{Supp}(D)))$, then there is a curve C intersecting $\mathrm{Supp}(D)$ properly such that $f(C)$ is a point. Then $(D \cdot C) < 0$, a contradiction.

(3) By (2), all coefficients of D are negative. If the support of D does not coincide with $\mathrm{Exc}(f)$, then there is a curve C not contained in $\mathrm{Supp}(D)$ such that $f(C)$ is a point. Then $(D \cdot C) \le 0$, a contradiction. □

Let X be a smooth algebraic variety and let B be a normal crossing divisor on X. A smooth subvariety Z is called a *permissible center* with respect to the pair (X, B) if the following is satisfied: For the local ring $\mathcal{O}_{X,P}$ at every point $P \in X$, there exists a regular system of parameters $z_1, \ldots, z_n$ and integers r, s, t such that the equations of B, Z are $z_1 \cdots z_r = 0$, $z_s = \cdots = z_t = 0$, respectively. Here, $0 \le r \le n$ and $0 \le s \le t \le n$, but there is no specific relation between r and s, t.

The blowup $f: Y \to X$ along a permissible center Z with respect to (X, B) is called a *permissible blowup*. In this case, the exceptional set E is a smooth prime divisor on Y and coincides with the set-theoretic inverse image $f^{-1}(Z)$. The sum $C = f_*^{-1}B + E$ with the strict transform is a normal crossing divisor on Y. We have $K_Y = f^*K_X + (t-s)E$ and $f^*B = f_*^{-1}B + \max\{r-s+1, 0\}E$.

The desingularization theorem also contains the following statement:

Theorem 1.6.4 ([45]) *Let X be a smooth algebraic variety defined over a field of characteristic* 0*, let B be a normal crossing divisor on X, and let $f: Y \to X$ be a proper birational morphism from another smooth algebraic variety Y. Then there exists a sequence of blowups $f_i: X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ and a birational morphism $g: X_n \to Y$ with the following properties:*

(1) $X = X_0$ and $f \circ g = f_1 \circ \cdots \circ f_n$.
(2) f_i is a permissible blowup with respect to (X_{i-1}, B_{i-1}). Here $B = B_0$ and the normal crossing divisor B_i on X_i is defined inductively by $B_i = f_{i*}^{-1}B_{i-1} + \mathrm{Exc}(f_i)$.

1.7 The Kodaira Vanishing Theorem

The Kodaira vanishing theorem holds only in characteristic 0. There are counterexamples in positive characteristics ([118]). The vanishing theorem and its generalizations are indispensable tools for the minimal model. Here we introduce the Kodaira vanishing theorem ([82]) without proof.

Theorem 1.7.1 (*Kodaira vanishing theorem*) *Let X be a smooth projective complex algebraic variety and let D be an ample divisor on X. Then for any positive integer $p > 0$, $H^p(X, K_X + D) = 0$. Here K_X is the canonical divisor of X.*

The Kodaira vanishing theorem is a theorem in complex differential geometry established for a compact complex manifold X. Let L be a holomorphic line bundle on a compact complex manifold X. L is always endowed with a C^∞ Hermitian metric h. The curvature of the corresponding connection of h determines a C^∞ (1, 1)-form on X. In this case, the following assertion holds by the *Kodaira embedding theorem*:

Theorem 1.7.2 ([83]) *Let X be a compact complex manifold and let L be a line bundle with a Hermitian metric h. If the curvature $\sqrt{-1}\Theta$ is positive definite everywhere, then X has a projective complex algebraic variety structure and L is the line bundle corresponding to an ample divisor.*

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We have the following implications:

Algebraic geometry	$\Rightarrow$	Complex differential geometry	$\Rightarrow$	Numerical geometry
Ample divisors	$\Rightarrow$	Line bundles with positive curvatures	$\Rightarrow$	Numerically positive divisors

The feature of the Kodaira vanishing theorem is that the canonical divisor appears in the statement and it provides a more accurate vanishing compared to the Serre vanishing theorem below. This paves the way for geometric applications. To be applied in higher dimensional algebraic geometry, the Kodaira vanishing theorem is greatly generalized and used in many directions, as will be discussed in Section 1.9.

Remark 1.7.3 The Kodaira vanishing theorem is originally proved for algebraic varieties defined over complex numbers, but it holds also for algebraic varieties defined over any field in characteristic 0, since a field in characteristic 0 which is finitely generated over the prime field $\mathbf{Q}$ can be always embedded into $\mathbf{C}$.

Theorem 1.7.4 (*Serre vanishing theorem* [126], [44, III.5.2]) *Let X be a projective scheme over a field k, let L be an ample sheaf on X, and let F be a coherent sheaf on X. Then there exists a positive integer m_0 such that for any integer $m \geq m_0$, the following assertions hold:*

(1) $F \otimes L^{\otimes m}$ is generated by global sections.
(2) For any positive integer $p > 0$, $H^p(X, F \otimes L^{\otimes m}) = 0$.

The Serre vanishing theorem holds without conditions on characteristics of the field k and singularities of X. It has much more applicability than the Kodaira vanishing theorem, but it is weaker.

The log version of the Kodaira vanishing theorem can be proved by the adjunction formula ([117]):

Corollary 1.7.5 *Let X be a smooth projective algebraic variety defined over a field of characteristic 0, let B be a normal crossing divisor on X, and let D be an ample divisor on X. Then for any positive integer $p > 0$, $H^p(X, K_X + B + D) = 0$.*

Proof We do induction on the dimension n of X and the number r of prime divisors of B. If $r = 0$, this is just the Kodaira vanishing theorem. If $r > 0$, take a prime divisor B_1 of B, denote $B' = B - B_1$ and $C = B'|_{B_1}$. By the adjunction formula, we get an exact sequence

$$0 \to \mathcal{O}_X(K_X + B' + D) \to \mathcal{O}_X(K_X + B + D) \to \mathcal{O}_{B_1}(K_{B_1} + C + D|_{B_1}) \to 0.$$

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By the inductive hypothesis, for any positive integer $p > 0$, $H^p(X, K_X + B' + D) = H^p(B_1, K_{B_1} + C + D|_{B_1}) = 0$. This concludes the assertion. □

1.8 The Covering Trick

The *covering trick* is a classical method to construct new algebraic varieties from a given one by using *cyclic coverings*. However, in this method, the new algebraic variety may have singularities even if the given algebraic variety is smooth. Therefore, we describe how to construct a covering without creating new singularities.

First, we describe the construction of cyclic coverings. Let X be an algebraic variety over an algebraically closed field k, let h be a nonzero rational function on X, and let m be a positive integer coprime to the characteristic of k. When $k = \mathbf{C}$, m can be taken arbitrarily. Consider the function field extension $K = k(X)[h^{1/m}]$, take Y to be the normalization of X in K with the natural morphism $f\colon Y \to X$. The extension Y/X is a Galois extension with a cyclic Galois group, and the extension degree $m' = [k(Y)\colon k(X)]$ is a divisor of m.

Y can be constructed as the following. Assume that X is covered by affine open subsets $U_i = \mathrm{Spec}(A_i)$. The fractional field of A_i is the function field $k(X)$. Take B_i to be the normalization of A_i in K, then Y is obtained by gluing affine varieties $\mathrm{Spec}(B_i)$.

Example 1.8.1 Let X be a smooth complex algebraic variety, let D be a divisor on X, and let s be a global section of $\mathcal{O}_X(mD)$. The zero divisor $\mathrm{div}(s)$ of s and the divisor $\mathrm{div}(h)$ of the rational function h corresponding to s is related by

$$\mathrm{div}(s) = \mathrm{div}(h) + mD.$$

Here $\mathrm{div}(s)$ is an effective divisor but $\mathrm{div}(h)$ is not necessarily effective and might have poles along D in general.

Assume that $B = \mathrm{div}(s)$ is reduced and is a smooth subvariety of X. Consider Y to be the cyclic covering of X induced by h. In this case, Y is smooth and $f\colon Y \to X$ is a finite morphism branched along B. Here D is not contained in the branch locus. Indeed, for any point P in B, take a regular system of parameters $z_1, \ldots, z_n$ such that $B = \mathrm{div}(z_1)$, then the regular system of parameters of any point Q over P can be taken as $z_1^{1/m}, z_2, \ldots, z_n$.

One should be careful that if $B = \mathrm{div}(s)$ has singularities, then Y has singularities correspondingly. When the support of B is a normal crossing divisor, Y has at worst *toric singularities*, which is easier to handle. This will be discussed in Section 3.7.

We can produce a more useful covering by considering the *Kummer covering*, a generalization of cyclic covering.

Theorem 1.8.2 ([52]) *Let X be a smooth projective algebraic variety defined over an algebraically closed field of characteristic* 0 *and let B be a normal crossing divisor on X. Fix a positive integer m_i for each irreducible component B_i of B. Then there exists a smooth projective algebraic variety Y and a finite morphism $f : Y \to X$ with the following properties:*

(1) The set-theoretic inverse image $C = f^{-1}(B)$ is a normal crossing divisor.
(2) For each i, there exists a reduced divisor C_i such that the pullback of B_i as a divisor can be written as $f^*B_i = m_iC_i$. Here a reduced divisor is a divisor with all coefficients equal to 1.
(3) f is a Galois covering and the Galois group G is an Abelian group.

One feature of this covering is that it is a finite morphism branched along a normal crossing divisor such that the covering space is again smooth. Note that the branch locus of f is a normal crossing divisor containing B, but they do not coincide in general. Moreover, since X is smooth, f is a *flat morphism*.

Proof Denote $n = \dim X$. Take a very ample divisor A such that $m_iA - B_i$ is very ample for all i. For each i, take n general global sections s_{ij} $(j = 1, \ldots, n)$ in $H^0(X, m_iA - B_i)$. We may assume that for each i, j, $M_{ij} = \operatorname{div}(s_{ij})$ is smooth and $\sum_{i,j} M_{ij} + \sum_i B_i$ is a normal crossing divisor.

Take the rational function h_{ij} corresponding to s_{ij} and take $f_{ij} : Y_{ij} \to X$ to be the normalization of X in $k(X)[h_{ij}^{1/m_i}]$. It is easy to see that the branch locus is $M_{ij} + B_i$ and the ramification index is m_i.

Take $f : Y \to X$ to be the normalization of the fiber product of all $f_{ij} : Y_{ij} \to X$ over X. In other words, Y is just the normalization of X in the field $k(X)[h_{ij}^{1/m_i}]_{ij}$. We will check that this Y satisfies the required properties.

For any point P in X, denote by B_{i_l} $(l = 1, \ldots, r)$ and $M_{j_qk_q}$ $(q = 1, \ldots, s)$ the irreducible components of $\sum_{i,j} M_{ij} + \sum_i B_i$ containing P. Note that $r + s \leq \dim X = n$.

If $r = 0$, that is, P is not contained in the support of B, then by construction, Y_{ij} is smooth over a neighborhood of P, and there is nothing to prove. So we may assume that $r \geq 1$.

By the numbers of M_{ij}, for each i_l, there exists at least one p_l such that $M_{i_lp_l}$ does not contain P. Denote $\bar{h}_{j_qk_q} = h_{j_qk_q}/h_{i_lp_l}$ if $j_q = i_l$; otherwise $\bar{h}_{j_qk_q} = h_{j_qk_q}h_A^{m_{j_q}}$, where h_A is a local equation of the divisor A. In this case,

$$h_{i_1p_1}h_A^{m_{i_1}}, \ldots, h_{i_rp_r}h_A^{m_{i_r}}, \bar{h}_{j_1k_1}, \ldots, \bar{h}_{j_sk_s}$$

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is a part of a regular system of parameters of $\mathcal{O}_{X,P}$. Indeed, in a neighborhood of P, these functions are exactly the defining equations of

$$B_{i_1}, \ldots, B_{i_r}, M_{j_1 k_1}, \ldots, M_{j_s k_s},$$

which form a normal crossing divisor in a neighborhood of P. The localization $Y \times_X \operatorname{Spec} \mathcal{O}_{X,P}$ is étale over the normalization of $\operatorname{Spec} \mathcal{O}_{X,P}$ in

$$\begin{aligned} & k(X)\left[h_{i_1 p_1}^{1/m_{i_1}}, \ldots, h_{i_r p_r}^{1/m_{i_r}}, h_{j_1 k_1}^{1/m_{j_1}}, \ldots, h_{j_s k_s}^{1/m_{j_s}}\right] \\ = {} & k(X)\left[h_{i_1 p_1}^{1/m_{i_1}} h_A, \ldots, h_{i_r p_r}^{1/m_{i_r}} h_A, \bar{h}_{j_1 k_1}^{1/m_{j_1}}, \ldots, \bar{h}_{j_s k_s}^{1/m_{j_s}}\right]. \end{aligned}$$

Therefore, Y is smooth. The properties on C and C_i can be checked similarly. □

The covering in the above theorem preserves smoothness by adding branch locus artificially. The covering below is a natural construction for a **Q**-Cartier Weil divisor which is not Cartier:

Proposition 1.8.3 *Let X be a normal algebraic variety defined over an algebraically closed field of characteristic* 0 *and let D be a divisor on X. Assume that for some positive integer r, rD is Cartier and moreover $\mathcal{O}_X(rD) \cong \mathcal{O}_X$. Take r to be the minimal one, then there exists a Galois finite morphism $f: Y \to X$ from a normal algebraic variety whose Galois group is the cyclic group of order r such that f is étale in codimension* 1 *and f^*D is a Cartier divisor on Y.*

Proof Fix an everywhere nonzero global section s of $\mathcal{O}_X(rD)$. The corresponding rational function h satisfies $\operatorname{div}_X(h) = -rD$. Take Y to be the normalization of X in the function field extension $L = k(X)[h^{1/r}]$. L is a field as r is minimal. Then $-f^*(D) = \operatorname{div}_Y(h^{1/r})$ is Cartier. It is easy to see that f is étale over the locally free locus of $\mathcal{O}_X(D)$, and in particular, f is étale over $X \setminus \operatorname{Sing}(X)$. □

Such $f: Y \to X$ is called the *index* 1 *cover* of the divisor D. In particular, if $D = K_X$, it is called the *canonical cover*.

Remark 1.8.4 (1) This covering is not unique, it depends on the choice of s. Take another global section s', there is a nowhere 0 function u such that $s' = us$. The normalization of X in $k(X)[u^{1/r}]$ gives an étale covering $X' \to X$, and the base change to X' gives an isomorphism $Y \times_X X' \cong Y' \times_X X'$. Here Y' is the cyclic covering obtained by s'. Therefore, this covering is unique up to étale base changes.

(2) Fix a point $P \in X$ and take r_P to be the minimal positive integer such that $r_P D$ is Cartier in a neighborhood of P, then $f^{-1}(P)$ consists of r/r_P points by construction. In particular, f is étale over the points where D is Cartier.

1.9 Generalizations of the Kodaira Vanishing Theorem

According to [76], we generalize the Kodaira vanishing theorem to different directions in order to apply it to higher dimensional algebraic geometry. The generalized vanishing theorems will be used as the key point of proofs in each part of this book.

In this section, we always assume that the base field is of characteristic 0.

First, we extend the Kodaira vanishing theorem to **R**-divisors:

Theorem 1.9.1 *Let X be a smooth projective algebraic variety and let D be an ample **R**-divisor on X such that the support of $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then for any positive integer $p > 0$, $H^p(X, K_X + \ulcorner D \urcorner) = 0$.*

Here we prove the following equivalent theorem:

Theorem 1.9.2 *Let X be a smooth projective algebraic variety, let B be an **R**-divisor on X with coefficients in $(0, 1)$ and supported on a normal crossing divisor, and let D be an integral divisor on X. Assume that $D - (K_X + B)$ is an ample **R**-divisor. Then for any positive integer $p > 0$, $H^p(X, D) = 0$.*

Proof Write $B = \sum b_i B_i$. Here B_i are prime divisors and $\sum B_i$ is a normal crossing divisor. As ampleness is an open condition, for each i we can take a fraction n_i/m_i $(0 < n_i < m_i)$ sufficiently close to b_i such that $D - (K_X + \sum (n_i/m_i) B_i)$ is an ample **Q**-divisor. In the following we may assume that $B = \sum (n_i/m_i) B_i$.

Take the covering $f : Y \to X$ as in Theorem 1.8.2 for irreducible components B_i of B with positive integers m_i. By construction, f^*B is a divisor with integral coefficients. The Galois group G acts on the invertible sheaf $\mathcal{O}_Y(K_Y - f^*(K_X + B))$ equivariantly in the following way. The action of G on the tangent sheaf T_Y induces the action on the canonical sheaf $\mathcal{O}_Y(K_Y)$, and the action of G on $\mathcal{O}_Y(-f^*(K_X + B))$ is induced from that on $\mathcal{O}_Y$ as $-f^*(K_X + B)$ is a G-invariant divisor. Since f is flat, the direct image sheaf $f_*\mathcal{O}_Y(K_Y - f^*(K_X + B))$ is a locally free sheaf with a G-action and the G-invariant part $L = (f_*\mathcal{O}_Y(K_Y - f^*(K_X + B)))^G$ is an invertible sheaf. Here since G is Abelian, $f_*\mathcal{O}_Y$ decomposes to a direct sum of invertible sheaves corresponding the G-eigenspaces, and hence L is invertible.

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L can be written as the form of a divisorial sheaf $\mathcal{O}_X(E)$. In order to determine E, we only need to look at the generic points of the branched divisor of f. First, any prime divisor not contained in B is not an irreducible component of E. Indeed, for any finite Galois covering $g \colon W \to Z$ between smooth varieties with Galois group G, we have a natural isomorphism $(g_*\omega_W)^G \cong \omega_Z$, which means that over $U = X \setminus \mathrm{Supp}(B)$, $L|_U = (f_*\mathcal{O}_Y(K_Y - f^*(K_X + B)))^G|_U = (f_*\mathcal{O}_Y(K_Y - f^*(K_X)))^G|_U \simeq \mathcal{O}_U$.

For the generic point P of B_i, set x_1 to be the regular parameter of the discrete valuation ring $\mathcal{O}_{X,P}$. Then for a point Q on Y over P, $y_1 = f^*x_1^{1/m_i}$ is a regular parameter and the invertible sheaf $\mathcal{O}_Y(K_Y - f^*(K_X + B))$ is generated by the section $y_1^{-(m_i-1)+n_i}$. Since $0 < n_i < m_i$, G-invariant sections are generated by 1. Therefore, it turns out that $E = 0$. In summary, $L = (f_*\mathcal{O}_Y(K_Y - f^*(K_X + B)))^G = \mathcal{O}_X$.

As the pullback of an ample divisor by a finite morphism is ample, the pullback $f^*(D-(K_X+B))$ is again ample. By the Kodaira vanishing theorem, for any positive integer $p > 0$, $H^p(Y, K_Y + f^*(D - (K_X + B))) = 0$. As f is finite, there is no higher direct image, hence $H^p(X, f_*\mathcal{O}_Y(K_Y + f^*(D - (K_X+B)))) = 0$. As the G-invariant part is a direct summand, $H^p(X, D) = 0$. □

Next, we prove the relative version of the vanishing theorem:

Theorem 1.9.3 *Let X be a smooth algebraic variety, let B be an* **R***-divisor on X with coefficients in* $(0, 1)$ *and supported on a normal crossing divisor, let D be an integral divisor on X, and let $f \colon X \to S$ be a projective morphism to another algebraic variety. Assume that $D - (K_X + B)$ is a relatively ample* **R***-divisor. Then for any positive integer $p > 0$,*

$$R^p f_*(\mathcal{O}_X(D)) = 0.$$

We will prove the following equivalent theorem:

Theorem 1.9.4 *Let X be a smooth algebraic variety, let $f \colon X \to S$ be a projective morphism to another algebraic variety and let D be a relatively ample* **R***-divisor on X such that the support of $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then for any positive integer $p > 0$,*

$$R^p f_*(\mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0.$$

Proof As the assertion is local on S, we may assume that S is affine. Replacing the integral part of D by a linearly equivalent one while keeping $\ulcorner D \urcorner - D$ unchanged, we may assume that the support of D is a normal crossing divisor.

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However, D is not necessarily effective. We may assume that D is a $\mathbf{Q}$-divisor as ampleness is an open condition.

Shrinking S if necessary, we can find a sufficiently large integer m such that mD is an integral divisor and there exists a closed immersion $g\colon X \to \mathbf{P}^N \times S$ such that $\mathcal{O}_X(mD) \cong g^* p_1^* \mathcal{O}_{\mathbf{P}^N}(1)$, where p_1 is the first projection.

Next, take a projective algebraic variety $\bar{S}$ to be the compactification of S, and take $\bar{X}$ to be the normalization of the closure of X in $\mathbf{P}^N \times \bar{S}$. The projective morphism $\bar{f}\colon \bar{X} \to \bar{S}$ and the finite morphism $\bar{g}\colon \bar{X} \to \mathbf{P}^N \times \bar{S}$ are naturally induced.

Here $\bar{X}$ is possibly singular and the extension of D is a $\mathbf{Q}$-Cartier divisor $\bar{D}$ defined by $\mathcal{O}_{\bar{X}}(m\bar{D}) \cong \bar{g}^* p_1^* \mathcal{O}_{\mathbf{P}^N}(1)$. Since $\bar{D}$ is relatively ample over $\bar{S}$, we can choose an ample Cartier divisor A_1 on $\bar{S}$ such that $\bar{D} + \bar{f}^* A_1$ is ample. As S is affine, we may assume that the support of A_1 is contained in $\bar{S} \setminus S$.

Take $h\colon Y \to \bar{X}$ to be a log resolution of the pair $(\bar{X}, \bar{D} + \bar{f}^* A_1)$ in strong sense. As X is smooth and the support of D is a normal crossing divisor, h can be assumed to be the identity over X. We may choose a $\mathbf{Q}$-Cartier divisor A_2 supported in the exceptional set of h such that $\bar{D}' = h^* \bar{D} + h^* \bar{f}^* A_1 + A_2$ is ample. By construction, the support of $\bar{D}'$ is a normal crossing divisor, and by Theorem 1.9.1, for any positive integer p, $H^p(Y, K_Y + \ulcorner \bar{D}' \urcorner) = 0$. Note that the support of $h^* \bar{f}^* A_1 + A_2$ is contained in $Y \setminus X$.

Consider the following spectral sequence:

$$E_2^{p,q} = H^p(\bar{S}, R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner))) \Rightarrow H^{p+q}(Y, K_Y + \ulcorner \bar{D}' \urcorner).$$

For any positive integer m_1, replacing A_1 by $m_1 A_1$, the above argument still works. When m_1 is sufficiently large, by the Serre vanishing theorem, for any positive integer p and any integer q,

$$H^p(\bar{S}, R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner))) = 0.$$

Also the coherent sheaf $R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner))$ on $\bar{S}$ is generated by global sections.

By the spectral sequence, when $q > 0$, $H^0(\bar{S}, R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner))) = 0$. Therefore, $R^q(\bar{f} \circ h)_*(\mathcal{O}_Y(K_Y + \ulcorner \bar{D}' \urcorner)) = 0$. We conclude the theorem by restricting on S. □

The next lemma shows that the conditions in the definitions of KLT and LC (log canonical) defined in Sections 1.10 and 1.11 are birational properties:

Lemma 1.9.5 *Let $f\colon Y \to X$ be a proper birational morphism between smooth algebraic varieties and let B, C be $\mathbf{R}$-divisors on X, Y supported on normal crossing divisors such that $f^*(K_X + B) = K_Y + C$. Then the* *coefficients of B are all contained in the open interval $(-\infty, 1)$ if and only if so are the coefficients of C.*

The same also holds for the condition that the coefficients are contained in the half-open interval $(-\infty, 1]$. Moreover, in this case, assume that the irreducible components of B with coefficients exactly 1 are disjoint, then the coefficients of $C - f_*^{-1}B$ are all contained in the open interval $(-\infty, 1)$.

Proof As $B = f_*C$, if the coefficients of C are all contained in the open interval $(-\infty, 1)$, then the coefficients of B are all contained in the open interval $(-\infty, 1)$.

Conversely, assume that the coefficients of B are all contained in the open interval $(-\infty, 1)$. First, we consider the case that f is a permissible blowup with respect to the pair (X, B). Set $B = \sum b_i B_i$. Suppose that the center Z of the blowup is of codimension r and contained in $B_1, \ldots, B_s$. Note that $r \geq s$. The coefficient e of the exceptional divisor E of f in C is given by

$$e = \sum_{j=1}^{s} b_j + 1 - r.$$

As $b_j < 1$, we have $e < 1$. Since the coefficients of other prime divisors of C coincide with those of B, the coefficients of C are all contained in the open interval $(-\infty, 1)$.

The general case can be reduced to the above case by applying Theorem 1.6.4. The later part can be proved similarly. □

We can also prove the following lemma which will be used in Section 1.11:

Lemma 1.9.6 *Fix an n-dimensional pair (X, B) of a normal algebraic variety and an effective $\mathbf{R}$-divisor such that $K_X + B$ is $\mathbf{R}$-Cartier and let P be a point on X. Take effective Cartier divisors $D_1, \ldots, D_n$ passing through P such that P is an irreducible component of $\bigcap D_i$. Then there exists a log resolution $f : Y \to (X, B + \sum D_i)$ such that if we write $K_Y + C = f^*(K_X + B + \sum D_i)$, then there exists an irreducible component C_1 of C with coefficient at least 1 and $f(C_1) = \{P\}$.*

Proof We may assume that X is affine by shrinking X. Write $D_i = \mathrm{div}(h_i)$, where h_i are regular functions on X. Define the morphism $h : X \to Z = \mathbf{A}^n$ by $h = (h_1, \ldots, h_n)$. By assumption, h is quasi-finite in a neighborhood of P. Take $E_1, \ldots, E_n$ to be coordinate hyperplanes of Z, and $h^*E_i = D_i$ by construction. Take $g : Z' \to Z$ to be the blowup at the origin and take F to be the exceptional divisor, we get $g^*(K_Z + \sum E_i) = K_{Z'} + F + \sum g_*^{-1} E_i$. As differential forms on Z with poles along $\sum E_i$ can be pulled back by h,

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$h^*(K_Z + \sum E_i) \leq K_X + B + \sum D_i$. By taking a log resolution $f\colon Y \to (X, B + \sum D_i)$ which factors through $X \times_Z Z'$, we may assume that the exceptional set contains a prime divisor C_1 mapping onto F, and this satisfies the requirements. □

Using the relative version of the vanishing theorem, it is easy to show the following generalization:

Theorem 1.9.7 ([76, Theorem 1.2.3]) *Let X be a smooth algebraic variety, let $f\colon X \to S$ be a projective morphism to another algebraic variety, and let D be a relatively nef and relatively big* **R***-divisor on X such that the support of $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then for any positive integer $p > 0$,*

$$R^p f_*(\mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0.$$

Proof Since the assertion is local on S, we may assume that S is affine. By Kodaira's lemma, we can write $D = A + E$ for some relatively ample **R**-Cartier divisor A and some effective **R**-Cartier divisor E. If $0 < \epsilon < 1$, then $D - \epsilon E = (1-\epsilon)D + \epsilon A$ is relatively ample.

Take $g\colon Y \to X$ to be a log resolution of $(X, D + E)$ in strong sense, and take $h\colon Y \to S$ to be the composition with f. We can choose a sufficiently small effective **R**-divisor A' supported on the exceptional set of g such that $-A'$ is g-ample and $D' = g^*(D - \epsilon E) - A'$ is h-ample. By Theorem 1.9.4, for any positive integer p,

$$R^p h_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner)) = R^p g_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner)) = 0.$$

By the spectral sequence

$$E_2^{p,q} = R^p f_*(R^q g_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner))) \Rightarrow R^{p+q} h_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner)),$$

$R^p f_*(g_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner))) = 0$ holds for $p > 0$.

Take ϵ and A' to be sufficiently small, then $\ulcorner D' \urcorner = \ulcorner g^* D \urcorner$. Take $B = \ulcorner D \urcorner - D$ and $g^*(K_X + B) = K_Y + C$, by Lemma 1.9.5, the coefficients of C are less than 1. Therefore, by

$$g^*(K_X + \ulcorner D \urcorner) = g^*(K_X + B + D) = K_Y + C + g^* D \leq K_Y + \ulcorner g^* D \urcorner$$

(here note that $C + g^* D$ is an integral divisor) and $g_*(K_Y + \ulcorner g^* D \urcorner) = K_X + \ulcorner D \urcorner$, we have

$$g_*(\mathcal{O}_Y(K_Y + \ulcorner D' \urcorner)) = \mathcal{O}_X(K_X + \ulcorner D \urcorner),$$

which proves the theorem. □

Higher dimensional algebraic geometry became greatly developed since the following result was proved:

Corollary 1.9.8 (*Kawamata–Viehweg vanishing theorem* [53, 139]) *Let X be a smooth projective algebraic variety and let D be a nef and big* **R***-divisor on X such that the support of $\ulcorner D \urcorner - D$ is a normal crossing divisor. Then for any positive integer $p > 0$,*

$$H^p(X, K_X + \ulcorner D \urcorner) = 0.$$

1.10 KLT Singularities for Pairs

We can define various singularities for a pair (X, B), where X is a normal algebraic variety and B is an **R**-divisor on X. B is called the *boundary* of the pair for historical reasons. These singularities appear naturally in the minimal model theory. Vanishing theorems can be also generalized to these singularities. The characteristic of the base field is always assumed to be 0 if not specified.

First, we define the KLT condition. This is a very natural condition corresponding to the L^2-condition in complex analysis. It does not depend on the choice of log resolutions. Furthermore, it is easy to handle since it satisfies the so-called "open condition" in the sense that it is stable under *perturbation* of the divisors. The KLT condition defines a category in which the minimal model theory works most naturally and easily.

For simplicity, sometimes we denote a pair (X, B) and a morphism $f: X \to S$ together by a morphism $f: (X, B) \to S$.

Definition 1.10.1 A pair (X, B) is *KLT* if it satisfies the following conditions:

(1) $K_X + B$ is **R**-Cartier.
(2) The coefficients of B are contained in the open interval $(0, 1)$.
(3) There exists a log resolution $f: Y \to (X, B)$ such that if we write $f^*(K_X + B) = K_Y + C$, then the coefficients c_j of $C = \sum c_j C_j$ are contained in $(-\infty, 1)$. Here C_j are distinct prime divisors.

Condition (1) is necessary in order to define the **R**-divisor C in condition (3). The support of C is contained in the union of the set-theoretic inverse image of the support of B and the exceptional set of f, which is a normal crossing divisor. The coefficients c_j of C play an important role in higher dimensional algebraic geometry. Further, $-c_j$ is called the *discrepancy coefficient* and $1-c_j$ is called the *log discrepancy coefficient*.

Historically, KLT singularity is just called *log terminal singularity* in [54].

Condition (3) in the definition of KLT does not depend on the choice of log resolutions:

Proposition 1.10.2 *Assume that (X, B) satisfies conditions (1) and (2) in Definition 1.10.1 and there exists a log resolution $f: Y \to (X, B)$ in weak sense satisfying condition (3). Then (X, B) is KLT. Moreover, for any log resolution $f': Y' \to (X, B)$ in weak sense, condition (3) in Definition 1.10.1 holds.*

Proof For two log resolutions $f_1: Y_1 \to X$, $f_2: Y_2 \to X$, there exists a third log resolution $f_3: Y_3 \to X$ dominating them. That is, there exist morphisms $g_i: Y_3 \to Y_i$ $(i = 1, 2)$ such that $f_3 = f_i \circ g_i$. Therefore, the assertion follows from Lemma 1.9.5. □

The following proposition is obvious:

Proposition 1.10.3 *(1) A pair (X, B) is KLT if and only if there exists an open covering $\{X_i\}$ of X such that the pairs $(X_i, B|_{X_i})$ are all KLT.*

(2) Let (X, B) be a KLT pair and let B' be another effective $\mathbf{R}$-divisor such that $B \geq B'$ and $B - B'$ is $\mathbf{R}$-Cartier, then (X, B') is again KLT.

(3) When X is a normal complex analytic variety, we can define the KLT condition similarly by using complex analytic resolution of singularities. When X is a complex algebraic variety, for a pair (X, B), the algebraic KLT condition and the analytic KLT condition are equivalent.

Remark 1.10.4 Take regular functions $h_1, \ldots, h_r$ on the polydisk $X = \Delta^n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid |z_i| < 1\}$ and write the corresponding divisors by $B_i = \mathrm{div}(h_i)$. Take real numbers $b_i \in (0, 1)$. Then $(X, B = \sum b_i B_i)$ is KLT if and only if the function $h = \prod |h_i|^{-b_i}$ is locally L^2 everywhere.

Indeed, the L^2-condition on integrability can be studied via resolutions of singularities. When the support of B is a normal crossing divisor, the absolute value of a regular function with poles along B satisfies the L^2-condition if and only if the coefficients of B are in $(-\infty, 1)$, which is exactly the KLT condition.

We introduce quotient singularities as an important example of KLT pairs.

An algebraic variety X is said to have *quotient singularities* if it is a quotient variety of a smooth algebraic variety in an étale neighborhood of each point P. That is, there exists a neighborhood U of P, an étale morphism $g: V \to U$ such that $P \in g(V)$, and a smooth algebraic variety $\tilde{V}$ with a finite group action G such that $V \cong \tilde{V}/G$.

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Example 1.10.5 Fix a positive integer r and integers $a_1, \ldots, a_n$. Define the action of the cyclic group $G = \mathbf{Z}/(r)$ on the affine space $\tilde{X} = \mathbf{A}^n$ by $z_i \mapsto \zeta^{a_i} z_i$. Here $(z_1, \ldots, z_n)$ are coordinates of $\tilde{X}$ and ζ is a primitive rth root of 1. Then the quotient space $X = \tilde{X}/G$ has only quotient singularities. The image P_0 of the origin might or might not be an isolated singularity, depending on the values of a_i. X is said to have a cyclic quotient singularity of *type* $\frac{1}{r}(a_1, \ldots, a_n)$ at P_0.

Proposition 1.10.6 *For an algebraic variety X with quotient singularities, the pair $(X, 0)$ with divisor $B = 0$ is KLT.*

Proof As discrepancy coefficients remain unchanged under étale morphisms, we may assume that X is a global quotient variety. That is, there is a smooth algebraic variety $\tilde{X}$ and a finite group G such that $X = \tilde{X}/G$. It is not hard to see that K_X is $\mathbf{Q}$-Cartier, indeed, $|G| \cdot K_X$ is Cartier.

Take a log resolution $f: Y \to X$ and write $f^*K_X = K_Y + C$. Take $\tilde{Y}$ to be the normalization of Y in the function field $k(\tilde{X})$ and take $\tilde{f}: \tilde{Y} \to \tilde{X}$ and $\pi_Y: \tilde{Y} \to Y$ to be the induced morphisms, write $\tilde{f}^*K_{\tilde{X}} = K_{\tilde{Y}} + \tilde{C}$. Take a prime divisor E on Y contained in the exceptional set of f and take a prime divisor $\tilde{E}$ on $\tilde{Y}$ such that $\pi_Y(\tilde{E}) = E$. Denote the coefficients of $E, \tilde{E}$ in $C, \tilde{C}$ by $c, \tilde{c}$, respectively, denote the ramification index of $\tilde{E}$ with respect to π_Y by e, then we have

$$ce = \tilde{c} + e - 1.$$

Here $\tilde{c} \le 0$ as $\tilde{X}$ is smooth, hence $c < 1$. □

A KLT pair admits the following special log resolution. We call it a *very log resolution* in this book.

Proposition 1.10.7 *Let (X, B) be a KLT pair consisting of a normal algebraic variety and an $\mathbf{R}$-divisor. Then there exists a log resolution $f: Y \to (X, B)$ such that if we write $f^*(K_X + B) = K_Y + C$, then the support of the $\mathbf{R}$-divisor $C' = \max\{C, 0\}$ is a disjoint union of smooth prime divisors.*

Proof Fix a log resolution $f_0: Y_0 \to (X, B)$ and write $f_0^*(K_X + B) = K_{Y_0} + C_0$. Choose two prime divisors in C_0 and blowup along their intersection, we get $g_1: Y_1 \to Y_0$. The composition with f_0 gives a new log resolution $f_1: Y_1 \to X$. We will show that a very log resolution can be constructed by repeating this operation.

Write $C_0 = \sum c_{0j} C_{0j}$. Take a positive number n such that the inequality $c_{0j} \le 1 - \frac{1}{n}$ holds for all j. n will be fixed in the following process.

For any log resolution $f\colon Y \to (X, B)$, write $f^*(K_X + B) = K_Y + C$ and $C = \sum c_j C_j$. Note that $c_j \le 1 - \frac{1}{n}$ for all j by the proof of Lemma 1.9.5. We define a sequence of integers $r(f) = (r_3(f), \ldots, r_{2n}(f))$ by the formula

$$r_i(f) = \#\left\{(j_1, j_2) \mid j_1 < j_2, C_{j_1} \cap C_{j_2} \neq \emptyset, 2 - \frac{i}{n} < c_{j_1} + c_{j_2} \le 2 - \frac{i-1}{n}\right\}.$$

For two sequences $(r_3, \ldots, r_{2n})$ and $(r'_3, \ldots, r'_{2n})$, we consider the lexicographical order. As $r_i \ge 0$, the set of sequences $(r_3, \ldots, r_{2n})$ satisfies the *DCC* (short for *descending chain condition*). That is, there is no infinite strictly decreasing chain.

For a given f, take the minimal i such that $r_i(f) \neq 0$ and take a pair (j_1, j_2) realizing it. That is, $j_1 < j_2$, $C_{j_1} \cap C_{j_2} \neq \emptyset$, and $2 - \frac{i}{n} < c_{j_1} + c_{j_2} \le 2 - \frac{i-1}{n}$. Take $g\colon Y' \to Y$ to be the blowup along center $Z = C_{j_1} \cap C_{j_2}$, denote $f' = f \circ g$, and write $(f')^*(K_X + B) = K_{Y'} + C'$. The coefficient e of the exceptional divisor $E = \mathrm{Exc}(g)$ in C' satisfies $1 - \frac{i}{n} < e \le 1 - \frac{i-1}{n}$.

The construction of Y' kills the intersection of C_{j_1} and C_{j_2}, and produces the intersections of E with the strict transforms of C_{j_1}, C_{j_2}, and C_j which intersect with $C_{j_1} \cap C_{j_2}$. Note that $e + c_j \le 2 - \frac{i}{n}$ as $c_j \le 1 - \frac{1}{n}$. So these new intersections do not contribute to $r_k(f')$ for $k \le i$. Therefore, $r_k(f') = r_k(f) = 0$ for $k < i$ and $r_i(f') = r_i(f) - 1$, which means that $r(f') < r(f)$. Since there is no infinite strictly decreasing chain for the sequence $r(f)$, eventually we can get a log resolution f such that $r_i(f) = 0$ for all i. This concludes the proof. □

Note that the log resolution in the above proposition is obtained by blowing up repeatedly, it does not satisfy condition (2') in Theorem 1.6.1. Also, the proposition cannot be extended to DLT pairs.

We can generalize the vanishing theorem to KLT pairs:

Theorem 1.10.8 ([76, 1.2.5]) *Let X be a normal algebraic variety, let $f\colon X \to S$ be a projective morphism, let B be an $\mathbf{R}$-divisor on X, and let D be a $\mathbf{Q}$-Cartier integral divisor on X. Assume that (X, B) is KLT and $D - (K_X + B)$ is relatively nef and relatively big. Then for any positive integer p, $R^p f_*(\mathcal{O}_X(D)) = 0$.*

Proof Take a log resolution $g\colon Y \to (X, B)$, denote $h = f \circ g$, and write $g^*(K_X + B) = K_Y + C$. Note that $g^*D - (K_Y + C)$ is h-nef and h-big. Here note that the coefficients of g^*D are not necessarily integers.

By Theorem 1.9.7, for any positive integer p, $R^p g_*(\mathcal{O}_Y(\ulcorner g^*D - C \urcorner)) = R^p h_*(\mathcal{O}_Y(\ulcorner g^*D - C \urcorner)) = 0$. Hence $R^p f_*(g_*(\mathcal{O}_Y(\ulcorner g^*D - C \urcorner))) = 0$.

For a rational function $r \in k(X) \cong k(Y)$, if $\mathrm{div}_X(r) + D \ge 0$, then $\mathrm{div}_Y(r) + g^*D \ge 0$. In this case, $\mathrm{div}_Y(r) + \llcorner g^*D \lrcorner \ge 0$ and then $\mathrm{div}_Y(r) +$

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$\ulcorner g^* D - C \urcorner \geq 0$ since the coefficients of C are contained in the open interval $(-\infty, 1)$. This shows that the natural inclusion

$$g_*(\mathcal{O}_Y(\ulcorner g^* D - C \urcorner)) \subset g_*(\mathcal{O}_Y(\ulcorner g^* D \urcorner)) \simeq \mathcal{O}_X(D)$$

is indeed an isomorphism and the proof is finished. □

Remark 1.10.9 In a KLT pair (X, B), X has only *rational singularities*, and hence is *Cohen–Macaulay* ([76, 1.3.6]).

This asserts that KLT is a "good" singularity. On the other hand, LC to be introduced in Section 1.11 is not "good" in this sense. This fact will not be used in this book.

Consider a pair (X, B) consisting of a normal algebraic variety and an effective **R**-divisor such that $K_X + B$ is **R**-Cartier. In Chapter 2, we will introduce the multiplier ideal sheaf in order to measure how far this pair is from being KLT.

The set of points $P \in X$ in whose neighborhood the pair (X, B) is not KLT is a closed subset of X. It is called the *non-KLT locus* of the pair (X, B). The cosupport of the multiplier ideal sheaf coincides with the non-KLT locus. Also, the vanishing theorem can be generalized using multiplier ideal sheaves (see Section 2.11).

1.11 LC, DLT, and PLT Singularities for Pairs

The KLT condition is easy to handle since it is an open condition with respect to changes of coefficients of divisors. However, in the minimal model theory, since it is necessary to consider the limits of divisors, it is necessary to consider the closed condition called the LC condition. Among LC pairs, we call by $\overline{\text{KLT}}$ pairs the pairs obtained by increasing boundaries of KLT pairs. The property of general LC pairs is not so good, but for $\overline{\text{KLT}}$ pairs it is possible to have similar discussions as for KLT pairs. Besides, there are conditions called DLT and PLT (purely log terminal) between KLT and LC, which are a little complicated but very useful. In this book, we develop the minimal model theory mainly for DLT pairs. The characteristic of the base field is always assumed to be 0 if not specified.

1.11.1 Various Singularities

Definition 1.11.1 A pair (X, B) is *LC* if it satisfies the following conditions:

(1) $K_X + B$ is **R**-Cartier.

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(2) The coefficients of B are contained in the half-open interval $(0, 1]$.
(3) There exists a log resolution $f : Y \to (X, B)$ such that if we write $f^*(K_X + B) = K_Y + C$, then the coefficients c_j of $C = \sum c_j C_j$ are contained in the half-open interval $(-\infty, 1]$. Here C_j are distinct prime divisors.

When (X, B) is an LC pair, (X, B) is said to have *log canonical singularities*. Same as Proposition 1.10.2, condition (3) above does not depend on the choice of log resolutions. Also, the same assertion as in Proposition 1.10.3 holds for LC pairs.

Example 1.11.2 The property of singularities of LC pairs is not always good.

Let Z be a smooth projective n-dimensional algebraic variety such that $K_Z \sim 0$, that is, $\omega_Z \cong \mathcal{O}_Z$. Take an ample invertible sheaf L and take the total space $Y = \mathrm{Spec}_Z(\bigoplus_{m=0}^{\infty} L^{\otimes m})$ of the dual sheaf L^*. Y admits an $\mathbf{A}^1$-bundle structure over Z. Denote $X = \mathrm{Spec}(\bigoplus_{m=0}^{\infty} H^0(Z, L^{\otimes m}))$, there is a natural birational morphism $f : Y \to X$ which contracts the 0-section E of $Y \to Z$ to a point $P = f(E)$.

By the adjunction formula $(K_Y + E)|_E \sim K_E \sim 0$, we have $K_Y + E \sim 0$ and $K_X \sim 0$, which implies that $f^*K_X \sim K_Y + E$. Hence $(X, 0)$ is LC.

The higher direct images of $\mathcal{O}_Y$ are supported on the singular point P of X:

$$R^p f_* \mathcal{O}_Y \cong \bigoplus_{m=0}^{\infty} H^p(Z, L^{\otimes m}) \supset H^p(Z, \mathcal{O}_Z).$$

For $p = n$, $H^n(Z, \mathcal{O}_Z) \neq 0$, hence X is not a *rational singularity*. Moreover, if Z is an Abelian variety, then for $0 < p \leq n$, the right-hand side is not 0, and X is not *Cohen–Macaulay*.

As the property of singularities of LC pairs is not always good, we consider intermediate conditions:

Definition 1.11.3 A pair (X, B) is *DLT* if it satisfies the following conditions:

(1) $K_X + B$ is $\mathbf{R}$-Cartier.
(2) The coefficients of B are contained in the half-open interval $(0, 1]$.
(3) There exists a log resolution $f : Y \to (X, B)$ such that if we write $f^*(K_X + B) = K_Y + C$, then the coefficients c_j of $C = \sum c_j C_j$ are contained in the open interval $(-\infty, 1)$ for those C_j contained in the exceptional set of f.

A pair (X, B) is *PLT* if it satisfies the above conditions (1) and (2) and the following condition (3'):

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(3') For any log resolution $f: Y \to (X, B)$, if we write $f^*(K_X + B) = K_Y + C$, then the coefficients c_j of $C = \sum c_j C_j$ are contained in the open interval $(-\infty, 1)$ for those C_j contained in the exceptional set of f.

Remark 1.11.4 (1) In [76], a condition called *WLT* (short for *weak log terminal*) is considered. The definition of WLT is by assuming further that the log resolution in condition (3) of the definition of DLT is in strong sense. By using similar argument as in Proposition 1.10.2, it can be shown that DLT and WLT are indeed equivalent ([136]). In this book, we will just use DLT rather than WLT.

(2) For a log resolution $f: Y \to X$ of (X, B), when considering the relation $f^*(K_X + B) = K_Y + C$, sometimes we just write "a morphism $f: (Y, C) \to (X, B)$."

Example 1.11.5 (1) Take the affine space $X = \mathbf{A}^n$ and coordinate hyperplanes $B_1, \ldots, B_n$, denote $B = \sum b_i B_i$. Then (X, B) is KLT (respectively, PLT, DLT) if and only if $0 \le b_i < 1$ for all i (respectively, $0 \le b_i \le 1$ for all i and $b_i < 1$ except for at most one i, $0 \le b_i \le 1$ for all i). Furthermore, DLT and LC coincide.

(2) Let $X = \mathbf{A}^2/\mathbf{Z}_2$ be the quotient of the 2-dimensional affine space $\mathbf{A}^2$ with coordinates x, y by the order 2 cyclic group $\mathbf{Z}_2$ action $(x, y) \mapsto (-x, -y)$. That is, it is a cyclic quotient singularity of type $\frac{1}{2}(1, 1)$. This singularity is the same as the ordinary double point in Example 1.1.4(1). Denote the image of the coordinate axes in X by B_1, B_2 and take $B = \sum b_i B_i$. Then (X, B) is KLT (respectively, PLT, LC) if and only if $0 \le b_i < 1$ for all i (respectively, $0 \le b_{i_1} \le 1$ for one i_1 and $0 \le b_{i_2} < 1$ for the other i_2, $0 \le b_i \le 1$ for all i). Furthermore, PLT and DLT coincide.

Indeed, the blowup $f: Y \to X$ of X along the image of the origin $(0, 0)$ is a log resolution. The exceptional set E is isomorphic to $\mathbf{P}^1$, $f^* B_i = f_*^{-1} B_i + \frac{1}{2} E$, and $f^* K_X = K_Y$. So the assertion can be checked easily.

(3) Take $X = \mathbf{A}^2$ to be the 2-dimensional affine space with coordinates x, y and a prime divisor $D = \mathrm{div}(x^2 + y^3)$. We determine the necessary and sufficient condition for the pair (X, dD) to be KLT or LC for a real number d (see Figure 1.1).

We can construct a log resolution of (X, dD) in the following way. First, take the blowup $f_1: Y_1 \to X$ along the origin $P_0 = (0, 0)$, the exceptional set E_1 is a prime divisor isomorphic to $\mathbf{P}^1$. The strict transform $D_1 = f_{1*}^{-1} D$ is smooth, E_1 and D_1 intersect at one point P_1.

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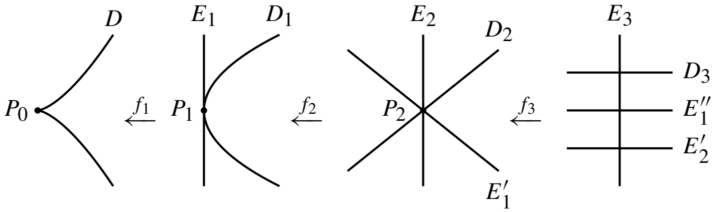

Figure 1.1 A log resolution of (X, D).

Take the blowup $f_2\colon Y_2 \to Y_1$ along P_1, the exceptional set E_2 is a prime divisor isomorphic to $\mathbf{P}^1$. Three smooth prime divisors E_2, $D_2 = f_{2*}^{-1}D_1$, and $E_1' = f_{2*}^{-1}E_1$ intersect at one point P_2.

Take the blowup $f_3\colon Y = Y_3 \to Y_2$ along P_2, the exceptional set E_3 is a prime divisor isomorphic to $\mathbf{P}^1$. The union of four prime divisors E_3, $D_3 = f_{3*}^{-1}D_2$, $E_1'' = f_{3*}^{-1}E_1'$, and $E_2' = f_{3*}^{-1}E_2$ is a normal crossing divisor.

The composition $f\colon Y \to X$ is a log resolution of (X, dD). We have $K_Y = f^*K_X + E_1'' + 2E_2' + 4E_3$ and $f^*D = D_3 + 2E_1'' + 3E_2' + 6E_3$. Therefore, the pair (X, dD) is KLT (respectively, LC) if and only if $0 \leq d < 5/6$ (respectively, $0 \leq d \leq 5/6$).

(4) Consider the example in Examples 1.1.4(2) or 1.2.4(2). In addition to the prime divisors D_1, D_2, consider prime divisors D_3, D_4 defined by $y = z = 0$ or $y = w = 0$. Note that $D_3 + D_4$ and K_X are Cartier divisors. Take $B = \sum_{i=1}^4 D_i$ and consider the pair (X, B). Take the resolution of singularities $f\colon X' \to X$ as in Example 1.2.4(2), then $B' = \sum_{i=1}^4 f_*^{-1}D_i$ is a normal crossing divisor. As f is isomorphic in codimension 1, $f^*(K_X + B) = K_{X'} + B'$.

The pair (X, B) is LC but not DLT. Here, as the exceptional set of f is not a normal crossing divisor, f is a log resolution in weak sense, but not a log resolution in the sense of Theorem 1.6.1(2). In order to obtain a log resolution, we need to do further blowups on X' along the exceptional set of f and that will induce an exceptional divisor with log discrepancy coefficient 1. However, this is not a rigorous proof of the fact that (X, B) is not DLT.

The blowup $g\colon Y \to X$ along the origin $(0, 0, 0, 0)$ of X is a log resolution. The exceptional set E is a prime divisor isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and $C = \sum_{i=1}^4 g_*^{-1}D_i + E$ is a normal crossing divisor. Since $g^*(K_X + B) = K_Y + C$, (X, B) is LC.

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(5) Take a smooth projective algebraic curve C of genus 1 and two line bundles L_1, L_2 of negative degrees. Take Y to be the total space of the vector bundle $L = L_1 \oplus L_2$ and denote by C_1, C_2, E the subvarieties of Y corresponding to subbundles $L_1 \oplus 0, 0 \oplus L_2, 0 \oplus 0$, respectively. Note that $E \cong C$. Denote $X = \mathrm{Spec}(\bigoplus_{m=0}^{\infty} H^0(C, L^{\otimes -m}))$, there is a natural birational morphism $f: Y \to X$ which contracts E to a point $P = f(E)$. Write $B_i = f(C_i)$. Then $f^*(K_X + B_1 + B_2) = K_Y + C_1 + C_2$ and the pair $(X, B_1 + B_2)$ is not DLT but LC. Indeed, X is not a rational singularity. The pairs $(B_i, 0)$ are also LC.

We introduce one more definition:

Definition 1.11.6 A pair (X, B) is $\overline{KLT}$ if it satisfies the following conditions:

(1) (X, B) is LC.
(2) There is another effective $\mathbf{R}$-divisor B' such that $B' \leq B$ and (X, B') is KLT.

In this situation, for any positive real number ϵ smaller than 1, $(X, (1-\epsilon) B + \epsilon B')$ is KLT. That is, $\overline{\mathrm{KLT}}$ is the limit of KLT. For this reason, different from general LC pairs, it shares similar properties as a KLT pair.

Toric varieties provide good examples (see [31, 79] for details):

Proposition 1.11.7 *Let T be an algebraic torus and let $T \subset X$ be a* toric variety*, that is, a T-equivariant open immersion into a normal algebraic variety with a T-action. Consider the complement set $B = X \setminus T$ as a reduced divisor. Then the following assertions hold:*

(1) The pair (X, B) is LC. Moreover, it is $\overline{KLT}$.
(2) X is $\mathbf{Q}$-factorial if and only if the corresponding fan *consists of simplicial cones.*

Proof (1) Take a T-equivariant resolution of singularities $f: Y \to X$ such that $f^{-1}(T) \cong T$ and $C = Y \setminus f^{-1}(T)$ is a normal crossing divisor.

Denote $\dim T = n$ and take coordinates $x_1, \ldots, x_n$ by pulling back from the standard embedding $T \subset \mathbf{A}^n$. The regular differential form $\theta = dx_1/x_1 \wedge \cdots \wedge dx_n/x_n$ on T can be extended to a logarithmic differential form on X and gives a global section of $K_X + B$ without zeros. Therefore, $K_X + B \sim 0$.

Similarly θ extends to a global section of $K_Y + C$ without zeros. Therefore, the equality $f^*(K_X + B) = K_Y + C$ holds, and hence (X, B) is LC.

As T is affine, there exists an effective Cartier divisor B' with support B. For a sufficiently small real number $\epsilon > 0$, $(X, B - \epsilon B')$ is KLT, and hence (X, B) is $\overline{\mathrm{KLT}}$.

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(2) We may assume that X is affine and its fan consists of a single cone. Irreducible components B_i of B correspond to points P_i on 1-dimensional rays of this cone σ. The condition for B_i becoming a **Q**-Cartier divisor is that there exists a regular function on X such that the corresponding divisor is a nonzero multiple of B_i. This is equivalent to saying that there exists a linear functional on σ which takes value 1 at P_i and 0 at all points on other rays, which is equivalent to σ being simplicial. □

The following is a corollary of Lemma 1.9.6.

Corollary 1.11.8 *Let (X, B) be an n-dimensional KLT pair and let P be a point. Take sufficiently general effective Cartier divisors $D_1, \ldots, D_n, E$ passing through P and a positive number $1 > \epsilon > 0$. Then there exists a sufficiently small number $\delta > 0$ such that the pair $(X, B + \sum(1-\delta)D_i + \epsilon E)$ is KLT in a punctured neighborhood of P, but not LC at P.*

Proof As $D_1, \ldots, D_n, E$ are general outside P, take a log resolution $\bar{f}: Y \to (X, B)$ and write $\bar{f}^*(K_X + B) = K_Y + \bar{C}$, we may assume that $\bar{C} + \bar{f}^*(\sum D_i + E)$ is normal crossing outside $\bar{f}^{-1}(P)$. The coefficients of $D_1, \ldots, D_n, E$ in the pair $(X, B + \sum(1-\delta)D_i + \epsilon E)$ are strictly smaller than 1 for $\delta > 0$, hence the pair is KLT in a punctured neighborhood of P.

On the other hand, take the log resolution f and prime divisor C_1 as in Lemma 1.9.6, then the coefficient of C_1 in f^*E is at least 1 and the coefficient of C_1 in $f^*(K_X + B + \sum D_i + \epsilon E)$ is strictly larger than 1. Hence $(X, B + \sum(1-\delta)D_i + \epsilon E)$ is not LC at P for sufficiently small $\delta > 0$. □

1.11.2 The Subadjunction Formula

We will look at the behavior of singularities when restricting a given pair to lower dimensions.

First, we show Shokurov's connectedness lemma ([91, Theorem 17.4], [128]), which is a consequence of the vanishing theorem:

Lemma 1.11.9 (*Connectedness lemma*) *Let (X, B) be a pair of a normal variety and an* **R***-divisor such that $K_X + B$ is* **R***-Cartier, and let $f: (Y, C) \to (X, B)$ be a log resolution in weak sense. Write $C = C^+ - C^-$, where C^+, C^- are effective* **R***-divisors with no common irreducible component. Then the natural homomorphism $\mathcal{O}_X \to f_*\mathcal{O}_{\llcorner C^+ \lrcorner}$ is surjective and the induced morphism* $\mathrm{Supp}(\llcorner C^+ \lrcorner) \to f(\mathrm{Supp}(\llcorner C^+ \lrcorner))$ *has connected geometric fibers.*

Proof Note that

$$-\llcorner C \lrcorner - (K_Y + C - \llcorner C \lrcorner) \equiv -f^*(K_X + B)$$

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is f-nef and f-big. As the coefficients of $C - \llcorner C \lrcorner$ are contained in the open interval $(0, 1)$, by the vanishing theorem (Theorem 1.9.7),

$$R^1 f_*(\mathcal{O}_Y(-\llcorner C \lrcorner)) = 0.$$

Since $\llcorner C \lrcorner = \llcorner C^+ \lrcorner - \ulcorner C^- \urcorner$, the natural homomorphism

$$f_*(\mathcal{O}_Y(\ulcorner C^- \urcorner)) \to f_*(\mathcal{O}_{\llcorner C^+ \lrcorner}(\ulcorner C^- \urcorner))$$

is surjective. Since the support of the effective divisor C^- is contained in the exceptional set, the natural homomorphism $f_*\mathcal{O}_Y \to f_*(\mathcal{O}_Y(\ulcorner C^- \urcorner))$ is bijective. In the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X \cong f_*\mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_{\llcorner C^+ \lrcorner} \\ \downarrow & & \downarrow \\ f_*(\mathcal{O}_Y(\ulcorner C^- \urcorner)) & \longrightarrow & f_*(\mathcal{O}_{\llcorner C^+ \lrcorner}(\ulcorner C^- \urcorner)), \end{array}$$

the left vertical arrow is bijective, the bottom horizontal arrow is surjective, and the right vertical arrow is injective, hence the top horizontal arrow is surjective. We conclude the proof. □

Corollary 1.11.10 *A DLT pair (X, B) is PLT if and only if the irreducible components of $\llcorner B \lrcorner$ are disjoint from each other.*

Proof The sufficiency is easy. In order to show the converse direction, suppose that two irreducible components B_1, B_2 of $\llcorner B \lrcorner$ intersect. Take a log resolution $f : (Y, C) \to (X, B)$ as in Lemma 1.11.9, then the strict transforms $f_*^{-1}B_1, f_*^{-1}B_2$ are contained in the same connected component of the support of $\llcorner C^+ \lrcorner$. Then there exists an irreducible component of $\llcorner C^+ \lrcorner - f_*^{-1}B_1$ intersecting $f_*^{-1}B_1$. Blowing up along the intersection, the coefficient of the exceptional divisor is 1, which means that (X, B) is not PLT. □

Corollary 1.11.11 *For a DLT pair (X, B), every irreducible component of $\llcorner B \lrcorner$ is normal.*

Proof We may assume that X is affine. Take B_1 to be an irreducible component of $\llcorner B \lrcorner$. Since there are sufficiently many regular functions on X, we can take a general effective **R**-divisor B' **R**-linearly equivalent to $B - B_1$ such that $\llcorner B' \lrcorner = 0$. Indeed, take a sufficiently large integer N and take general global sections $s_1, \ldots, s_N$ of the divisorial sheaf $\mathcal{O}_X(\llcorner B \lrcorner - B_1)$, then $\mathrm{div}(s_i) \sim \llcorner B \lrcorner - B_1$ and we may just take

$$B' = B - B_1 + \sum \mathrm{div}(s_i)/N - \llcorner B \lrcorner + B_1 = B - \llcorner B \lrcorner + \sum \mathrm{div}(s_i)/N.$$

Since s_i are taken to be general, a log resolution $f: Y \to X$ of (X, B) is also a log resolution of $(X, B_1 + B')$. So $(X, B_1 + B')$ is still DLT. Write $K_Y + C = f^*(K_X + B_1 + B')$, then $\llcorner C^+ \lrcorner = f_*^{-1} B_1$. Therefore, Lemma 1.11.9 implies that D is normal. □

Remark 1.11.12 According to this corollary, the irreducible components of $\llcorner B \lrcorner$ have no "self-intersection." For example, if X is a smooth complex algebraic variety and B is a reduced divisor normal crossing in analytic sense but not simple normal crossing, then (X, B) is not DLT. This is derived from the definition of normal crossing divisors in the definition of log resolutions.

Induction arguments on dimensions using the adjunction formula is compatible with the property of DLT. The reason is the following result:

Theorem 1.11.13 (*Subadjunction formula*) *Let (X, B) be a DLT pair and let Z be an irreducible component of $\llcorner B \lrcorner$. Then we can naturally define an effective* **R**-*divisor B_Z on Z satisfying*

$$(K_X + B)|_Z = K_Z + B_Z,$$

and the pair (Z, B_Z) is again DLT. Moreover, if (X, B) is PLT in a neighborhood of Z, then (Z, B_Z) is KLT.

Proof Take a log resolution $f: (Y, C) \to (X, B)$ such that the coefficients of exceptional prime divisors in C are less than 1. Write $W = f_*^{-1} Z$, $C_W = (C - W)|_W$ and $B_Z = (f|_W)_* C_W$. Here the coefficients of C_W are at most 1, so are those of B_Z.

Here we claim that the coefficients of B_Z are contained in the half-open interval $(0, 1]$. To see that $B_Z \geq 0$, after cutting X by general hyperplanes, we may assume that $\dim X = 2$. In this case, $f: (Y, C) \to (X, B)$ factors through the minimal resolution of X (see Proposition 1.13.8). Hence there exists a pair (Y_1, C_1) and birational morphisms $f_1: Y \to Y_1$, $f_2: Y_1 \to X$ such that $f = f_2 \circ f_1$ and $K_{Y_1} + C_1 = f_2^*(K_X + B)$, and moreover $C_1 \geq 0$. Then $B_Z \geq 0$.

As $(K_Y + C)|_W = K_W + C_W$, we get $(K_X + B)|_Z = K_Z + B_Z$. Hence $K_Z + B_Z$ is **R**-Cartier. Note that $f|_W$ is a log resolution of (Z, B_Z) and $(f|_W)^*(K_Z + B_Z) = K_W + C_W$.

Recall that every irreducible component of C with coefficient 1 is a strict transform of an irreducible component of $\llcorner B \lrcorner$. Take D to be an irreducible component of C_W with coefficient 1, then D is contained in the intersection of $f_*^{-1} \llcorner B \lrcorner - W$ and W. Since $\mathrm{Exc}(f) \cup f_*^{-1} \llcorner B \lrcorner$ is a normal crossing divisor, D is not contained in $\mathrm{Exc}(f)$. Therefore, D is not contained in the exceptional set of $f|_W$ and hence (Z, B_Z) is DLT.

The latter part is obvious. □

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Remark 1.11.14 It is possible that $B_Z \neq 0$ even if $B = Z$, that is, K_Z might be smaller than expected, and this is why we use the word "sub." For example, consider the quadric surface X defined by the equation $xy = z^2$ in the affine space $\mathbf{C}^3$ with coordinates x, y, z and the divisor Z on X defined by the equation $x = z = 0$. Then the pair (X, Z) is DLT and the subadjunction formula in this case is $(K_X + Z)|_Z = K_Z + \frac{1}{2}P$ (see Example 1.3.2).

For a pair (X, B), a subvariety Z of X is called an *LC center* if there exists a log resolution $f\colon (Y, C) \to (X, B)$ such that there is an irreducible component C_i of $\llcorner C^+ \lrcorner$ with $Z = f(C_i)$.

Lemma 1.11.15 *Fix a log resolution $f\colon (Y, C) \to (X, B)$ of an LC pair (X, B). Then the LC centers of the pair (X, B) are exactly the images of irreducible components of intersections of several irreducible components of $\llcorner C^+ \lrcorner$.*

Proof Take the blowup Y along an irreducible component of the intersection of several irreducible components of $\llcorner C^+ \lrcorner$, we get a new log resolution and the exceptional divisor has coefficient 1 in the new boundary. Hence the image is an LC center. On the other hand, by an easy computation, blowing up along other centers gives an exceptional divisor with coefficient strictly smaller than 1. By Theorem 1.6.4, any log resolution is dominated by a log resolution obtained in this way, which concludes the proof. □

In particular, when (X, B) is DLT, there exists a log resolution $f\colon (Y, C) \to (X, B)$ with $\llcorner C^+ \lrcorner = f_*^{-1} \llcorner B \lrcorner$, hence an LC center is nothing but an irreducible component of the intersection of several irreducible components of $\llcorner B \lrcorner$. In other words, the reduced part of the boundary of the DLT pairs obtained by applying the subadjunction formula several times to (X, B) are LC centers.

We extend the vanishing theorem to DLT pairs. Note that the condition "relatively ample" cannot be replaced by "relatively nef and relatively big" as DLT is not an open condition.

Theorem 1.11.16 *Let X be a normal algebraic variety, let $f\colon X \to S$ be a projective morphism, let B be an* **R***-divisor on X, and let D be a* **Q***-Cartier integral divisor on X. Assume that the pair (X, B) is DLT and $D - (K_X + B)$ is relatively ample. Then for any positive integer p, $R^p f_*(\mathcal{O}_X(D)) = 0$.*

Proof Take a log resolution $g\colon (Y, C) \to (X, B)$ in strong sense and denote $h = f \circ g$. By the definition of DLT, we may assume that the coefficients of exceptional divisors in C are strictly less than 1, note that here we use the fact

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that DLT is equivalent to WLT (see Remark 1.11.4). Take a sufficiently small effective **R**-divisor A supported on the exceptional set of g such that $-A$ is g-ample, $\llcorner C + A \lrcorner = \llcorner C \lrcorner$, and $g^*D - (K_Y + C + A)$ is h-ample. Take a sufficiently small number $\epsilon > 0$ such that $g^*D - (K_Y + (1-\epsilon)C + A)$ is again h-ample.

Write $D' - C' = g^*D - ((1-\epsilon)C + A)$, where D' is a divisor with integral coefficients and C' is an **R**-divisor with coefficients in the interval $(0,1)$, in other words, take $D' = \ulcorner g^*D - ((1-\epsilon)C + A) \urcorner$. Since the support of C' is a normal crossing divisor, by Theorem 1.9.3, for $p > 0$, $R^p g_*(\mathcal{O}_Y(D')) = R^p h_*(\mathcal{O}_Y(D')) = 0$. Therefore, for $p > 0$, $R^p f_*(g_*(\mathcal{O}_Y(D'))) = 0$. Since $g_*D' = D$ by definition and $D' \geq \llcorner g^*D \lrcorner$ as the coefficients of $(1-\epsilon)C + A$ are smaller than 1, we have $g_*(\mathcal{O}_Y(D')) = \mathcal{O}_X(D)$ and the theorem is proved. □

Here we remark that we can give an alternative proof by applying Lemma 2.1.8 to replace (X, B) by a KLT pair and then applying Theorem 1.9.3 directly.

1.11.3 Terminal and Canonical Singularities

In the end of this section, we introduce terminal and canonical singularities. These singularities are not considered in the main part of this book. However, they are important in applications and have a longer history than KLT, DLT, LC, et cetera in dimensions 3 and higher. Originally 3-dimensional algebraic geometry was successful because these singularities can be classified. However, classification of singularities is impossible in higher dimensions, and it is replaced by using log pairs and induction on dimensions.

Definition 1.11.17 A normal algebraic variety X is said to have *canonical singularities* if the following conditions are satisfied:

(1) K_X is **Q**-Cartier.
(2) For a resolution of singularities $f: Y \to X$, if write $f^*K_X = K_Y + C$, then $-C$ is effective.

Furthermore, X is said to have *terminal singularities* if the following is satisfied:

(3) The support of $-C$ coincides with the divisorial part of $\mathrm{Exc}(f)$.

In terms of discrepancy coefficients, the feature of terminal singularities (canonical singularities) is that all discrepancy coefficients are positive

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(nonnegative). It is easy to see that conditions (2) and (3) do not depend on the choice of resolutions of singularities.

The concept of terminal and canonical singularities can be also extended to pairs.

Definition 1.11.18 A pair (X, B) consisting of a normal algebraic variety X and an effective **R**-divisor B on X is said to have *canonical singularities* if the following conditions are satisfied:

(1) $K_X + B$ is **R**-Cartier.
(2) For any resolution of singularities $f: Y \to X$, if write $f^*(K_Y + B) = K_Y + C$, then $-C + f_*^{-1}B$ is effective.

Furthermore, (X, B) is said to have *terminal singularities* if the following is satisfied:

(3) The support of $-C + f_*^{-1}B$ coincides with the divisorial part of $\text{Exc}(f)$.

In conditions (2) and (3), it is not sufficient to check for only one log resolution.

As will be explained in Section 2.5, discrepancy coefficients are nondecreasing under the MMP, hence the MMP preserves types of singularities. That is, when applying a birational map in the MMP to an algebraic variety with certain singularities, we get an algebraic variety with the same type of singularities. In other words, the MMP can be considered within the category of varieties having certain singularities. In particular, when considering the MMP starting from a smooth algebraic variety, everything is within the category of terminal singularities. Note that 2-dimensional terminal singularities without boundaries are just smooth, that is the reason why it is not necessary to consider singularities in the classical 2-dimensional MMP.

1.12 Minimality and Log Minimality

The minimality in the minimal model theory is defined by the minimality of canonical divisors. A log minimal model is the log version of a minimal model, where the log canonical divisor is minimized. The MMP is a process to find a "minimal model" which is a birational model with good properties for a given algebraic variety.

First, we define "minimality" by the property of singularities and numerical property of canonical divisors:

Definition 1.12.1 (1) A projective morphism $f: X \to S$ from a normal algebraic variety to another algebraic variety is said to be relatively *minimal* over S if it satisfies the following conditions (a), (b). It is said to be relatively *minimal in weak sense* over S if it satisfies the following conditions (a'), (b).

(a) X has $\mathbf{Q}$-factorial terminal singularities.
(a') X has canonical singularities.
(b) K_X is relatively nef over S.

(2) A projective morphism $f: (X, B) \to S$ from a pair consisting of a normal algebraic variety and an $\mathbf{R}$-divisor to another algebraic variety is said to be relatively *log minimal* over S if it satisfies the following conditions (a), (b). It is said to be relatively *log minimal in weak sense* over S if it satisfies the following conditions (a'), (b).

(a) X is $\mathbf{Q}$-factorial and the pair (X, B) is DLT.
(a') The pair (X, B) is LC.
(b) $K_X + B$ is relatively nef over S.

The minimality in weak sense defined above leads to the minimality of the canonical divisor K_X and the log canonical divisor $K_X + B$:

Proposition 1.12.2 *(1) Let $f: X \to S$ be a relatively minimal morphism in weak sense. Consider a projective morphism $g: Y \to S$ from another normal algebraic variety and birational projective morphisms $f': Z \to X$ and $g': Z \to Y$ from a third normal algebraic variety with $f \circ f' = g \circ g'$. If K_Y is $\mathbf{Q}$-Cartier, then the inequality $(f')^* K_X \le (g')^* K_Y$ holds. That is, K_X is minimal in birational equivalence classes.*

(2) Let $f: (X, B) \to S$ be a relatively log minimal morphism in weak sense. Consider a projective morphism $g: (Y, C) \to S$ from another pair of a normal algebraic variety and an $\mathbf{R}$-divisor, and birational projective morphisms $f': Z \to X$ and $g': Z \to Y$ from a third normal algebraic variety with $f \circ f' = g \circ g'$. Furthermore, assume the following conditions:

(a) For each irreducible component B_i of B, its strict transform $C_i = g'_*(f')_*^{-1} B_i$ is an irreducible component of C. If we denote the coefficients of B_i, C_i by b_i, c_i, then the inequalities $b_i \le c_i$ hold.
(b) For each irreducible component C_j of C satisfying $f'_*(g')_*^{-1} C_j = 0$, its coefficient c_j is 1.

If $K_Y + C$ is $\mathbf{R}$-Cartier, then the inequality $(f')^*(K_X + B) \le (g')^*(K_Y + C)$ holds. That is, $K_X + B$ is minimal in birational equivalence classes.

Proof (1) By the desingularization theorem we may assume that Z is smooth. Write $(f')^*K_X = K_Z + E$, $(g')^*K_Y = K_Z + F$.

Since X has canonical singularities, $-E$ is effective. That is, K_X is smaller than K_Z. So the condition on singularities guarantees the minimality locally.

In order to see the global property, we apply the negativity lemma (Lemma 1.6.3). Write $F - E = G^+ - G^-$, where G^+, G^- are effective $\mathbf{Q}$-divisors with no common irreducible component. Our goal is to show $G^- = 0$. Suppose that $G^- \neq 0$. As $-E$ is effective, the support of G^- is contained in the support of F, which is contracted by g'.

By Lemma 1.6.3, there exists a curve C contracted by g' such that $(G^- \cdot C) < 0$ and $(G^+ \cdot C) \geq 0$. Note that $((K_Z + F) \cdot C) = 0$. On the other hand, since K_X is nef,

$$0 \leq ((K_Z + E) \cdot C) = ((E - F) \cdot C) = -(G^+ \cdot C) + (G^- \cdot C) < 0,$$

which is a contradiction. Therefore, $G^- = 0$ and $F - E$ is effective.

(2) We may assume that f', g' are log resolutions. Write $(f')^*(K_X + B) = K_Z + E$ and $(g')^*(K_Y + C) = K_Z + F$.

Since (X, B) is LC, the coefficients of E are at most 1. Therefore, if we denote by $\bar{E}$ the sum of the strict transform $(f')_*^{-1}B$ and all exceptional divisors of f' with given coefficients 1, then $(f')^*(K_X + B)$ is smaller than $K_Z + \bar{E}$. So the LC condition guarantees the minimality locally.

Let us look at the global property. Write $F - E = G^+ - G^-$, where G^+, G^- are effective $\mathbf{R}$-divisors with no common irreducible component. Our goal is to show $G^- = 0$. Suppose that $G^- \neq 0$.

Once it is shown that the support of G^- is contracted by g', the conclusion follows exactly as the proof of (1). In order to show that the support of G^- is contracted by g', for any prime divisor R on Z, we are going to show that R is not an irreducible component of G^- if $g'_*R = Q$ is a prime divisor on Y.

If $f'_*R = P$ is a prime divisor on X, by assumption (a), the coefficient of P in B is not greater than that of Q in C. This holds even if P is not an irreducible component of B in which case we just formally set the coefficient to be 0. Therefore, the coefficient of R in $F - E$ is nonnegative and it is not an irreducible component of G^-.

If $f'_*R = 0$, by assumption (b), the coefficient of Q in C is 1 while that of R in E is at most 1. Therefore, the coefficient of R in $F - E$ is nonnegative and it is not an irreducible component of G^-. □

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Remark 1.12.3 (1) In the minimal model theory in classical algebraic surface theory, a minimal model is defined to be the minimal one under the following relation using birational morphisms: For two smooth projective algebraic surfaces X, Y, we define $X \leq Y$ if there exists a birational morphism $Y \to X$.

However, in dimensions 3 and higher, there are examples showing that such a definition does not work ([25, 26]). Therefore, in the minimal model theory discussed in this book, we consider projective algebraic varieties with singularities, and define the minimal model by the size of canonical divisors; the relation $X \leq Y$ between two birationally equivalent algebraic varieties is defined by the inequality $K_X \leq K_Y$. Here the inequality of divisors is by comparing the pullbacks on an appropriate birational model: We write $K_X \leq K_Y$ if $f^*K_X \leq g^*K_Y$ for birational projective morphisms $f: Z \to X$ and $g: Z \to Y$.

The relation $(X, B) \leq (Y, C)$ for log pairs is defined by $f^*(K_X + B) \leq g^*(K_Y + C)$ for birational projective morphisms $f: Z \to X$ and $g: Z \to Y$ together with two conditions of (2) of the above proposition.

Such change of viewpoint has already been observed in the log version of algebraic surfaces ([51]). The importance of considering the log version showed up at that time. Furthermore, extending to the log version is indispensable for the inductive proof of the existence of minimal models in this book.

(2) From the above proposition, one can see that the minimality in weak sense is equivalent to the minimality of canonical divisors. Furthermore, according to Corollary 3.6.10 which is derived from the main theorems of this book, minimal models are maximal among minimal models in weak sense under the relation defined by birational morphisms.

Looking at this locally, we can say that: Canonical singularities are characterized by the property that the canonical divisors are locally minimal. Furthermore, $\mathbf{Q}$-factorial terminal singularities are maximal, among those with canonical divisors locally minimal, under the relation defined by birational morphisms.

For pairs, the log minimality in weak sense is equivalent to the minimality of log canonical divisors. But as a DLT blowup can be further blown up, it is impossible to construct a "maximal minimal model." However, if the minimal model is KLT, then we can construct a maximal minimal model by Corollary 3.6.10. This is a pair with $\mathbf{Q}$-factorial terminal singularities.

Looking at this locally, we can say that: LC pairs are characterized by the property that the log canonical divisors are locally minimal. Furthermore, by looking at only KLT pairs, **Q**-factorial terminal pairs are maximal, among pairs with log canonical divisors locally minimal, under the relation defined by birational morphisms.

Therefore, the theory requiring **Q**-factorial terminal singularities can be regarded as "maximalist" and the theory requiring canonical singularities or LC singularities can be regarded as "minimalist." Models that are expected to be obtained using the minimal model program will be "maximalist."

(3) Let $\alpha\colon X \dashrightarrow Y$ be a birational map between normal algebraic varieties projective over S. X, Y are said to be *crepant* or *K-equivalent* to each other if there are birational projective morphisms $f\colon Z \to X$, $g\colon Z \to Y$ from a third normal algebraic variety with $g = \alpha \circ f$ such that $f^*K_X = g^*K_Y$. Here the comparison of canonical divisors is by using rational differential forms identified by the birational map. By the above proposition, birationally equivalent minimal models are crepant to each other.

Furthermore, given effective **R**-divisors B, C on X, Y, assume that $K_X + B$ and $K_Y + C$ are **R**-Cartier. The pairs (X, B) and (Y, C) are said to be *log crepant* or *K-equivalent* to each other if $f^*(K_X+B) = g^*(K_Y+C)$, or just *crepant* for simplicity. When considering minimal models with boundaries, only being birational is not enough, we should also pay attention to how to define the boundaries. This is settled in Section 2.5.5.

1.13 The 1-Dimensional and 2-Dimensional Cases

In this section, we describe known results including the finite generation of canonical rings in dimensions up to 2. Many of them are special phenomena which only happen in dimensions up to 2. In particular, we describe the classification of DLT pairs in dimension 2. We obtain a subadjunction formula from this, and apart from this formula, other results will not be used in subsequent sections. For a DLT pair in arbitrary dimension, its structure in codimension 2 can be considered by cutting down the dimension by general hyperplanes and reducing to the classification of DLT pairs in dimension 2.

1.13.1 The 1-Dimensional Case

First, we discuss the 1-dimensional case briefly. Take an algebraic curve X, that is, a smooth projective 1-dimensional algebraic variety. Denote its genus by g. If $g = 0$, then $X \cong \mathbf{P}^1$ and $R(X, K_X) \cong k$. If $g = 1$, then $K_X \sim 0$ and $R(X, K_X) \cong k[t]$. These cases are simple.

In the following we consider the case $g \geq 2$. This condition is equivalent to X being of general type. It is also equivalent to that the degree of the canonical divisor is positive $\deg(K_X) > 0$ since the degree of the canonical divisor K_X is $2g - 2$. The plurigenera are given by $\dim H^0(X, mK_X) = (2m-1)(g-1)$ for $m \geq 2$. As K_X is ample, the canonical ring $R(X, K_X)$ is finitely generated and

$$X = \operatorname{Proj} R(X, K_X).$$

X is called a *hyperelliptic curve* if there exists a finite morphism $\pi\colon X \to \mathbf{P}^1$ of degree 2. The canonical linear system $|K_X|$ is always free, but it is very ample if and only if X is not a hyperelliptic curve. When X is a hyperelliptic curve,

$$|K_X| = \pi^*|\mathcal{O}_{\mathbf{P}^1}(g-1)|,$$

where π is the morphism corresponding to $|K_X|$. In this case, $|3K_X|$ is very ample ([44, IV.5]).

To be more specific, if X is not a hyperelliptic curve, then the canonical ring is generated by the degree 1 part $H^0(X, K_X)$ (a theorem of Max Noether [5, p. 117]). On the other hand, if X is a hyperelliptic curve, then degree up to 3 parts are required to generate the canonical ring.

1.13.2 Minimal Models in Dimension 2

In the following, we consider the 2-dimensional case. For details please refer to [10]. Let X be an *algebraic surface*, that is, a 2-dimensional algebraic variety.

Numerical geometry is particularly effective on algebraic surfaces. This is because the intersection number becomes a symmetric bilinear form since prime divisors are the same as curves. The following powerful theorem is often used in algebraic surface theory. It can be used even for problems in higher dimensional algebraic geometry, by cutting by hyperplane sections and reducing to algebraic surfaces (see Lemma 1.6.3).

Theorem 1.13.1 (*Hodge index theorem* [44, Theorem V.1.9]) *Let A, B be Cartier divisors on a proper 2-dimensional algebraic variety X. If $(A^2) > 0$, $(A \cdot B) = 0$, and $B \not\equiv 0$, then $(B^2) < 0$.*

Corollary 1.13.2 *Let $f\colon Y \to X$ be a resolution of singularities of an algebraic surface and let D be a nonzero divisor on Y supported in the exceptional set $\operatorname{Exc}(f)$. Then $(D^2) < 0$. Therefore, if the exceptional divisors of f are $E_1, \ldots, E_r$, then the matrix of intersection numbers $[(E_i \cdot E_j)]$ is negative definite.*

Proof We may assume that X is projective. Take an ample divisor H on X, then $(f^*H \cdot f^*H) > 0$ and $(f^*H \cdot D) = 0$. If $D \geq 0$, as Y is projective, $D \neq 0$ implies $D \not\equiv 0$. Therefore, $(D^2) < 0$. In general, we can write $D = D^+ - D^-$ in terms of the positive part and the negative part, then $(D^2) \leq (D^+)^2 + (D^-)^2 < 0$. □

In general, given a resolution of singularities $f : Y \to X$, the *dual graph* Γ can be constructed from the exceptional set as the following:

(1) Take vertices $v_1, \ldots, v_r$ of Γ corresponding to prime divisors $E_1, \ldots, E_r$ in $\mathrm{Exc}(f)$.
(2) Join v_i, v_j with an edge if two distinct prime divisors E_i, E_j intersect and associate the edge with weight $(E_i \cdot E_j)$.
(3) Associate each vertex v_i with the self-intersection number (E_i^2) as its weight.

First of all, we recall the minimality of algebraic surfaces. The definition of minimal models in algbraic surface theory is different from that in higher dimensional algebraic geometry. Hence here we use "minimal in the classical sense." Given two smooth algebraic surfaces X, Y, the relation $X \geq Y$ is defined by that there is a birational projective morphism $f : X \to Y$. An algebraic surface minimal under this relation is defined to be *minimal in the classical sense*.

A curve C on X is called a (-1)-*curve* if $C \cong \mathbf{P}^1$ and the self-intersection number $(C^2) = -1$. If we blow up a smooth algebraic surface Y at a point P, then the exceptional set is a (-1)-curve. Conversely, a (-1)-curve can be contracted to a smooth point:

Theorem 1.13.3 (*Castelnuovo's contraction theorem* [44, Theorem V.5.7]) *For a smooth algebraic surface X and a (-1)-curve C on X, there exists a birational projective morphism $f : X \to Y$ to another smooth algebraic surface such that $f(C)$ is a point and f induces an isomorphism $X \setminus C \cong Y \setminus f(C)$.*

Minimality is characterized by the absence of (-1)-curve:

Theorem 1.13.4 ([44, Proposition V.5.3]) *A smooth algebraic surface X is minimal in the classical sense if and only if there is no (-1)-curve on X.*

Corollary 1.13.5 *For a smooth projective algebraic surface X, its minimal model in the classical sense always exists.*

Proof In the case that $f : X \to Y$ is a contraction of a (-1)-curve, the Picard number decreases exactly by one: $\rho(X) = \rho(Y) + 1$. As the Picard number

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is always positive, a minimal model in the classical sense can be obtained by taking contractions finitely many times. □

Minimal projective algebraic surfaces in the classical sense are classified into the following three types:

(1) A surface with K_X nef.
(2) A $\mathbf{P}^1$-bundle over a curve.
(3) $\mathbf{P}^2$.

In this book, (1) is called a *minimal model*, and (2) or (3) is called a *Mori fiber space*. In case (1), the minimal model is unique, so it is the minimum one. On the other hand, in cases (2) and (3), the minimal model (in the classical sense) is not unique, so such a model is sometimes said to be *relatively minimal*, but to avoid confusion we will not use this terminology.

Combining the existence of resolution of singularities and Castelnuovo's contraction theorem, we get the *minimal resolution of singularities* of a normal algebraic surface. It is a minimal model in the relative setting, which is obtained by considering $\rho(Y/X)$ instead of $\rho(X)$:

Corollary 1.13.6 ([44, Theorem V.5.8]) *Let X be a normal algebraic surface. Then among all birational projective morphisms $g: Y \to X$ from smooth algebraic surfaces, there exists a unique minimal one in the classical sense.*

We also have the following *minimal log resolution of singularities* which is the log version of the minimal resolution of singularities:

Proposition 1.13.7 *Let (X, B) be a pair consisting of a normal algebraic surface and a reduced divisor. Then among all birational projective morphisms $g: Y \to X$ from smooth algebraic surfaces such that the sum of $f_*^{-1}B$ and the exceptional divisor E is a normal crossing divisor, there exists a unique minimal one in the classical sense.*

For a projective algebraic curve C on a smooth algebraic surface X, the following *genus formula* holds ([44, Example V.3.9.2]):

$$(K_X \cdot C) + (C^2) = 2\bar{g} - 2 \geq -2.$$

Here $\bar{g}$ is called the *virtual genus* of C, which is a nonnegative integer. Take g to be the genus of the smooth projective curve C^ν obtained from taking the normalization of C, then $\bar{g} \geq g$. The difference $\bar{g} - g$ comes from the singularities of C. In particular, the equality holds if and only if C is smooth.

Minimal resolution of singularities is characterized by relative nefness of the canonical divisor. This coincides with the definition of minimality in this book:

Proposition 1.13.8 *(1) A birational projective morphism $f: Y \to X$ from a smooth algebraic surface to a normal algebraic surface is the minimal resolution of singularities if and only if K_Y is relatively nef.*
(2) Let $f: Y \to X$ be the minimal resolution of singularities of a normal algebraic surface. If we write $f^*K_X = K_Y + C$, then C is effective.

Proof (1) If there is a (-1)-curve C such that $f(C)$ is a point, then $(K_Y \cdot C) = -1$ and K_Y is not relatively nef.

Conversely, if K_Y is not relatively nef, then there is a curve C such that $(K_Y \cdot C) < 0$ and $f(C)$ is a point. By the Hodge index theorem (Corollary 1.13.2), $(C^2) < 0$. On the other hand, by the genus formula, $(K_Y \cdot C) + (C^2) \geq -2$. Hence we have $((K_Y + C) \cdot C) = -2$, and hence $C \cong \mathbf{P}^1$ and $(C^2) = -1$. So C is a (-1)-curve.

(2) Write $C = C^+ - C^-$, where C^+ and C^- are effective divisors with no common irreducible component. If $C^- \neq 0$, then $(K_Y \cdot C^-) = -(C^+ \cdot C^-) + (C^- \cdot C^-) < 0$, which contradicts the fact that K_Y is relatively nef. □

For the Euler characteristic $\chi(\mathcal{O}_X) = \sum(-1)^i \dim H^i(X, \mathcal{O}_X)$ of a smooth projective algebraic surface X, we have *Noether's formula*

$$\chi(\mathcal{O}_X) = \frac{1}{12}((K_X^2) + c_2(X)).$$

Here $c_2(X)$ is the *second Chern class* of the tangent bundle of X and $-K_X = c_1(X)$ is the *first Chern class*.

1.13.3 The Classification of Algebraic Surfaces

Let us consider the finite generation problem for canonical rings of smooth projective algebraic surfaces. The important thing here is that canonical rings are invariant under contractions of (-1)-curves: $f^*: R(X', K_{X'}) \cong R(X, K_X)$. Therefore, in the following we consider X to be minimal.

In the classification of minimal models in the classical sense, for a Mori fiber space in case (2) or (3), its canonical ring is just k, and the finite generation is trivial. In the following we just consider case (1). The following content is a deep result called the *Kodaira–Enriques classification theory* for algebraic surfaces. In addition, Kodaira also classified (not necessarily algebraic) compact complex surfaces, but we will not discuss them here ([9]).

The Kodaira dimension $\kappa(X)$ takes value among $0, 1, 2$. When $\kappa(X) = 0$, there exists a positive integer r such that $rK_X \sim 0$. If we take r to be the smallest one with such property, then r is among $1, 2, 3, 4, 6$. In particular, $R(X, K_X) \cong k[t^r]$.

When $\kappa(X) = 1$, there exists a surjective morphism $f: X \to Y$ to a smooth projective algebraic curve such that the generic fiber is an elliptic curve. The following *Kodaira's canonical bundle formula* holds:

$$K_X \sim_{\mathbf{Q}} f^*(K_Y + B).$$

Moreover, $\deg(K_Y + B) > 0$. Here B is a $\mathbf{Q}$-divisor on Y determined by types of singular fibers of f and $\sim_{\mathbf{Q}}$ means $\mathbf{Q}$-linearly equivalent. Singular fibers are completely classified and the corresponding coefficients of B are determined. Here the coefficients of B are not necessarily contained in the open interval $(0, 1)$. This is because it also includes a part induced from the J-function $J: Y \to \mathbf{P}^1$ coming from the fibers of f. Anyway, there exists a positive integer r such that $rK_X \sim f^*(r(K_Y+B))$ and $R(X, rK_X) \cong R(Y, r(K_Y+B))$. The latter is finitely generated as $r(K_Y + B)$ is an ample divisor, which implies that $R(X, K_X)$ is finitely generated.

Consider the case $\kappa(X) = 2$. A minimal model X is of general type if and only if the self-intersection number of the canonical divisor is positive $(K_X^2) > 0$. For $m \geq 2$, by a vanishing theorem of Kodaira type, we have the following plurigenus formula:

$$\dim H^0(X, mK_X) = \frac{1}{2}m(m-1)(K_X^2) + \chi(\mathcal{O}_X).$$

We discuss the canonical models. A curve C on X is called a (-2)-*curve* if $C \cong \mathbf{P}^1$ and $(C^2) = -2$. On a minimal surface of general type, a (-2)-curve is characterized by the condition $(K_X \cdot C) = 0$. This is because, on the one hand, $(C^2) < 0$ by the Hodge index theorem (Corollary 1.13.2) and on the other hand, $(K_X \cdot C) + (C^2) \geq -2$ by the genus formula. According to *Artin's contraction theorem* ([6] or Theorem 1.13.10), we can contract all (-2)-curves by a birational morphism; there exists a birational morphism $g: X \to Y$ to a normal algebraic surface such that the exceptional set of g coincides with the union of all (-2)-curves. Y is called the *canonical model*.

The canonical divisor K_Y of Y is a Cartier divisor and $K_X = g^*K_Y$. Therefore, there is an isomorphism $g^*: R(Y, K_Y) \cong R(X, K_X)$. Since K_Y is ample, the canonical ring $R(X, K_X)$ is finitely generated and $Y = \operatorname{Proj} R(X, K_X)$. This is the proof of the finite generation of canonical rings in dimension 2 by Mumford ([107]). More precisely, on the canonical model, $|5K_Y|$ is very ample ([17]).

1.13.4 Rational Singularities

For a minimal model X of general type, its canonical model Y has *canonical singularities* because the birational morphism $g: X \to Y$ is crepant

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($K_X = f^* K_Y$). Canonical singularities in dimension 2 are known to be the same as *rational double points*, that is, rational singularities of multiplicity 2. Such singularities were investigated in many different situations historically. They are also called *Du Val singularities*, *Klein singularities*, *simple singularities*, or *ADE singularities*. Here we summarize the classification of 2-dimensional canonical singularities:

Theorem 1.13.9 *Let $P \in X$ be a canonical singularity in dimension* 2*.*

(1) Take $f : Y \to X$ to be the minimal resolution of singularity, then the exceptional set Exc(f) *is a normal crossing divisor whose irreducible components are all* (-2)*-curves and the dual graph defined by their intersections is among the* Dynkin diagrams *of type A_n, D_n, E_6, E_7, E_8 (see Figure 1.2).*

Conversely, on a smooth algebraic surface, a normal crossing divisor whose irreducible components are all (-2)*-curves with dual graph of type A_n, D_n, E_6, E_7, E_8 can be contracted to a canonical singularity by a birational projective morphism.*

(2) When the base field is **C***, there exists an analytic neighborhood of P isomorphic to the neighborhood of the origin of the hypersurface in* $\mathbf{C}^3$ *defined by one of the following equations:*

$$A_n : \ x^2 + y^2 + z^{n+1} = 0, \quad n \geq 1;$$
$$D_n : \ x^2 + y^2 z + z^{n-1} = 0, \quad n \geq 4;$$
$$E_6 : \ x^2 + y^3 + z^4 = 0;$$

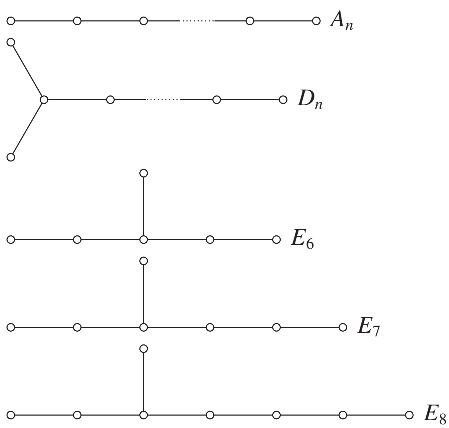

Figure 1.2 Dynkin diagrams.

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$$E_7:\ x^2 + y^3 + yz^3 = 0;$$
$$E_8:\ x^2 + y^3 + z^5 = 0.$$

Here (x, y, z) *are coordinates of* $\mathbf{C}^3$.

(3) *When the base field is* $\mathbf{C}$*, it is analytically isomorphic to the singularity of the image of the origin of the quotient space* $\mathbf{C}^2/G$ *by a finite subgroup* G *of* $\mathrm{SL}(2, \mathbf{C})$.

More generally, rational singularities on algebraic surfaces are defined by Artin ([7]). Please refer to the original paper for the proof. The theorem is characteristic free:

Theorem 1.13.10 *Let* X *be a smooth algebraic surface and let* E_i $(i = 1, \ldots, r)$ *be projective curves on* X *such that the union* $E = \bigcup E_i$ *is connected. Assume that the matrix of intersections* $[(E_i \cdot E_j)]$ *is negative definite. Then the following assertions hold:*

(1) *There exists a smallest effective integral divisor* $F = \sum e_i E_i \neq 0$ *satisfying the property that* $(F \cdot E_i) \leq 0$ *for all* i*. It is called the* fundamental cycle.
(2) *The inequality* $(K_X \cdot F) + (F^2) \geq -2$ *holds.*
(3) *If the equality* $(K_X \cdot F) + (F^2) = -2$ *holds, then there exists a birational projective morphism* $f: X \to Y$ *to a normal algebraic surface and the exceptional set* $\mathrm{Exc}(f)$ *coincides with* E*. In this case, the singularity of* Y *is called a* rational singularity.
(4) *Rational singularities are* $\mathbf{Q}$*-factorial. Moreover,* $R^1 f_* \mathcal{O}_X = 0$*. Conversely, a normal singularity on an algebraic surface* Y *with a resolution of singularity* $f: X \to Y$ *satisfying* $R^1 f_* \mathcal{O}_X = 0$ *is a rational singularity.*

The condition $R^1 f_* \mathcal{O}_X = 0$ is independent of the choice of resolutions of singularities since for $g: X' \to X$ a blowup of a smooth algebraic surface at a point, $R^1 g_* \mathcal{O}_{X'} = 0$ and $g_* \mathcal{O}_{X'} \cong \mathcal{O}_X$ hold.

Example 1.13.11 (1) On a smooth algebraic surface, a curve satisfying $C \cong \mathbf{P}^1$ and $(C^2) < 0$ can be contracted to a rational singularity.
(2) Dual graphs obtained by taking resolutions of singularities of 2-dimensional DLT pairs (see Figure 1.3) can be contracted to rational singularities.

Proposition 1.13.12 *Let* X *be a normal algebraic surface with at most rational singularities and let* $f: Y \to X$ *be a resolution of singularities. Then prime divisors in the exceptional set of* f *are all isomorphic to* $\mathbf{P}^1$ *and the dual graph*

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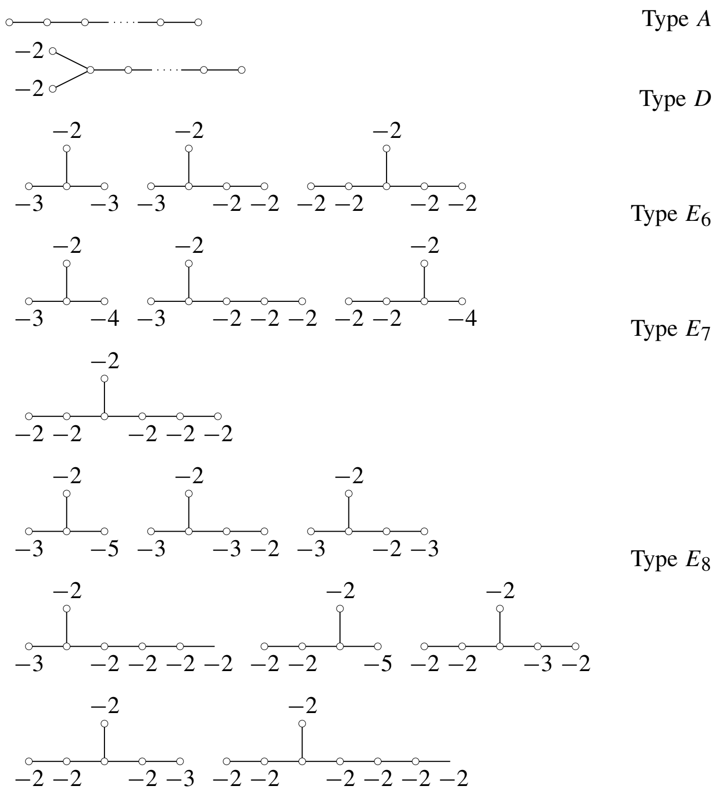

Figure 1.3 2-dimensional DLT.

is a tree. *Here a tree is a graph with all edges having weight one and with no cycles.*

Proof Since $R^1 f_* \mathcal{O}_Y = 0$, $\lim_E H^1(E, \mathcal{O}_E) = 0$ by [44, Theorem III.11.1]. Here the limit is the inverse limit for all subschemes E supported on the exceptional set of f. Since the exceptional set of f is 1-dimensional, for any effective divisor E supported in $\mathrm{Exc}(f)$, we have $H^1(E, \mathcal{O}_E) = 0$. This concludes the proof. □

Remark 1.13.13 According to a theorem of Grauert ([33]), for a smooth complex analytic surface X and projective curves E_i ($i = 1, \ldots, r$) on X such that the union $E = \bigcup E_i$ is connected and the matrix of intersections $[(E_i \cdot E_j)]$ is negative definite, there always exists a proper birational morphism $f : X \to Y$ to a normal complex analytic surface such that the exceptional set

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of f coincides with E. However, Y does not necessarily admit an algebraic structure and f is not necessarily algebraic.

1.13.5 The Classification of DLT Surface Singularities I

Numerical geometry becomes easy for normal algebraic surfaces. Even for **R**-divisors which are not **R**-Cartier, intersection numbers and pullback by a morphism can be well defined.

Let X be a normal algebraic surface and let D be an **R**-divisor on X. Take a resolution of singularities $f\colon Y \to X$ and denote by E_i $(i = 1, \ldots, r)$ the exceptional divisors. *Mumford's numerical pullback* $f^*D = f_*^{-1}D + \sum e_i E_i$ is defined as the following ([106]): The coefficients e_i are the solution of the equations $(f^*D \cdot E_i) = 0$ for all i, which are uniquely determined since $[(E_i \cdot E_j)]$ is negative definite. If moreover D is effective, we can see that f^*D is again effective.

For two **R**-divisors D and D', their intersection number can be defined by $(D \cdot D') = (f^*D \cdot f^*D')$.

From now on, we work on the classification of 2-dimensional DLT pairs. Here, in all discussions, we assume that the base field is of characteristic 0. There is also a classification in positive characteristics ([51]).

As the definition of pullback extends to all **R**-divisors, for a pair (X, B) we can define the concept such as KLT and DLT without assuming that $K_X + B$ is **R**-Cartier. Therefore, in the following, this assumption is removed. However, as will be shown later in this section, it turns out that $K_X + B$ automatically becomes **R**-Cartier.

First, we generalize the vanishing theorem slightly. For algebraic surfaces, the normal crossing condition which is important in Theorem 1.9.7 can be removed:

Proposition 1.13.14 *Let X be a smooth projective algebraic surface defined over an algebraically closed field of characteristic 0, let $f\colon X \to S$ be a projective morphism to another algebraic variety, and let D be a relatively nef and relatively big* **R***-divisor on X. Then $R^1 f_*(\mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$.*

Proof Take a log resolution $g\colon Y \to X$ of (X, D). By Theorem 1.9.7, $R^1(f \circ g)_*(\mathcal{O}_Y(K_Y + \ulcorner g^*D \urcorner)) = R^1 g_*(\mathcal{O}_Y(K_Y + \ulcorner g^*D \urcorner)) = 0$. Then, arguing by spectral sequence, we get $R^1 f_*(g_*(\mathcal{O}_Y(K_Y + \ulcorner g^*D \urcorner))) = 0$. In the exact sequence

$$0 \to g_*(\mathcal{O}_Y(K_Y + \ulcorner g^*D \urcorner)) \to \mathcal{O}_X(K_X + \ulcorner D \urcorner) \to Q \to 0,$$

the cokernal Q of the natural homomorphism has 0-dimensional support, hence it does not have higher cohomologies. Therefore, the proof is completed. □

DLT pairs have rational singularities:

Proposition 1.13.15 *Let (X, B) be a 2-dimensional DLT pair defined over an algebraically closed field of characteristic 0. Then X has rational singularities. If (X, B) is only LC, then X has rational singularities at points in the support of B.*

Proof Since (X, B) is DLT, $(X, 0)$ is again DLT. Here note that the condition $K_X + B$ being **R**-Cartier is removed in the definition of DLT. As $(X, 0)$ has no boundary, it is KLT. Take the minimal resolution of singularities $f : Y \to X$ and write $f^*K_X = K_Y + C$. As it is the minimal resolution, C is effective. Since $(X, 0)$ is KLT, $\ulcorner -C \urcorner = 0$. Applying Proposition 1.13.14 to $D = -f^*K_X$, we get $R^1 f_*\mathcal{O}_Y = R^1 f_*(\mathcal{O}_Y(\ulcorner -C \urcorner)) = 0$.

For the latter assertion, when the pair (X, B) is LC, $(X, 0)$ is KLT at points in the support of B. □

Rationality of singularities implies **Q**-factoriality:

Proposition 1.13.16 *Algebraic surfaces defined over the complex number field with only rational singularities are* **Q***-factorial.*

Proof Take a resolution of singularities $f : Y \to X$. Consider Y as a complex analytic variety, consider its sheaves in the classical topology instead of the Zariski topology. Then there exists an *exponential exact sequence*

$$0 \to \mathbf{Z}_Y \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0.$$

Here the map $\mathcal{O}_Y \to \mathcal{O}_Y^*$ is defined by the exponential function $z \mapsto e^{2\pi i z}$. Note that such kind of exact sequence does not exist in the Zariski topology.

By assumption, $R^1 f_*\mathcal{O}_Y = 0$, hence the map $R^1 f_*\mathcal{O}_Y^* \to R^2 f_*\mathbf{Z}_Y$ is injective.

For any divisor D on X, its numerical pullback f^*D is a **Q**-divisor, so we can take a positive integer m such that mf^*D is integral. Note that $\mathcal{O}_Y(mf^*D)$ determines an element in $R^1 f_*\mathcal{O}_Y^*$ whose image in $R^2 f_*\mathbf{Q}_Y$ is 0 since $(mf^*D \cdot E) = 0$ for every f-exceptional curve E. Therefore, there exists a positive integer m' such that the image of $\mathcal{O}_Y(mm'f^*D)$ in $R^2 f_*\mathbf{Z}_Y$ is 0. This induces an isomorphism

$$\mathcal{O}_Y(mm'f^*D) \cong \mathcal{O}_Y.$$

The global section of the left-hand side corresponding to 1 of the right-hand side determines a rational function h on Y such that $\mathrm{div}(h)_Y = -mm' f^* D$. Hence $\mathrm{div}(h)_X = -mm' D$ which means that $mm' D$ is Cartier. □

As 2-dimensional DLT pairs are rational singularities, they are **Q**-factorial, and hence numerical pullback is actually the same as pullback. For an LC pair, the same holds true on the support of the boundary.

Next, we show that the KLT or LC property is preserved under covering:

Lemma 1.13.17 *Let $f: Y \to X$ be a finite surjective morphism étale in codimension 1 between normal algebraic varieties defined over an algebraically closed field of characteristic 0.*

*Let B be an effective **R**-divisor on X such that $K_X + B$ is **R**-Cartier and write $f^*(K_X + B) = K_Y + C$. Then the pair (X, B) is LC if and only if the pair (Y, C) is LC. The same holds true for KLT pairs.*

Proof As f is étale in codimension 1, C is effective. Take a log resolution $g: X' \to X$ of (X, B) and take Y' to be the normalization of X' in the function field $k(Y)$. Denote the induced morphisms by $h: Y' \to Y$ and $f': Y' \to X'$. Write $g^*(K_X + B) = K_{X'} + B'$ and $h^*(K_Y + C) = K_{Y'} + C'$.

First, we show that (X, B) is LC assuming that (Y, C) is LC. Take an arbitrary prime divisor D contracted by g and denote its coefficient in B' by d. Take a prime divisor E on Y' such that $f'(E) = D$ and denote the ramification index of E with respect to f' by r. Then the coefficient of E in $(f')^* D$ and $K_{Y'} - (f')^* K_{X'}$ are r and $r - 1$, respectively. Therefore, take e to be the coefficient of E in C', we get the relation

$$dr = r - 1 + e.$$

Since $e \leq 1$ by assumption, we get $d \leq 1$. Moreover, if $e < 1$, then $d < 1$.

Conversely, we show that (Y, C) is LC assuming that (X, B) is LC. By using the result we just proved in the first part, we may replace Y by taking the Galois closure and assume that the field extension $k(Y)/k(X)$ is Galois from the beginning. As the Galois group G acts on Y, we now take $h: Y' \to Y$ to be a *G-equivariant* log resolution. For example, a canonical resolution (Remark 1.6.2(4)) is automatically G-equivariant. The quotient space $X' = Y'/G$ has quotient singularities. Denote by $g: X' \to X$ and $f': Y' \to X'$ the induced morphisms. Take a prime divisor E contracted by h and define D, e, d in the same way as the first part. Although X' is not smooth, we still have $dr = r - 1 + e$. Since $d \leq 1$ by assumption, we get $e \leq 1$. Moreover, if $d < 1$, then $e < 1$. □

Remark 1.13.18 Here we give some remarks about the topology of algebraic varieties defined over the complex number field. In general, the topology of algebraic varieties is the Zariski topology, but when the base field is the complex number field, the classical Euclidean topology is also useful. For example, the exponential exact sequence that appeared in Proposition 1.13.16 makes sense only in the latter topology.

As an open subset in the Zariski topology is large, it admits nontrivial structure itself, on the other hand, classical topology has polydisks as a base and its local structure is trivial. Since there are many open subsets, even the constant sheaf has nontrivial cohomology groups.

For algebraic varieties defined over the complex number field, many definitions and results hold both for the Zariski topology and the classical topology. Furthermore, in many cases they can be generalized to nonalgebraic complex analytic varieties. For example, the definitions of DLT pairs and LC pairs can be generalized using resolutions of complex analytic singularities. The same is true for DLT pairs having rational singularities. The fact that LC and KLT are preserved by étale in codimension 1 coverings can be also generalized since it is a consequence of the ramification formula.

The construction of *index* 1 *covers* can be also generalized. For example, for an effective divisor D on a complex analytic variety X such that there is an isomorphism $\mathcal{O}_X(rD) \cong \mathcal{O}_X$, take a regular function h such that $\mathrm{div}(h) = rD$, take the normalization of the subvariety defined by the equation $z^r = h$ in the trivial line bundle $X \times \mathbf{C}$ over X, we get the index 1 cover. Here z is the coordinate in the fiber direction. When D is not effective, we can consider a similar construction in $X \times \mathbf{P}^1$.

However, as stated in Remark 1.1.2, we should take care of the concept of normal crossing divisor. We should also take care of $\mathbf{Q}$-factoriality. A complex analytic variety X is *analytically* $\mathbf{Q}$*-factorial* if for any analytic neighborhood U of any point $P \in X$ and any codimension 1 subvariety D defined on U, there exists a neighborhood U' of P in U, a positive integer r, and a regular function h on U' such that $\mathrm{div}_{U'}(h) = r(D \cap U')$. As the algebraic $\mathbf{Q}$-factoriality is a condition for globally defined prime divisors, analytical $\mathbf{Q}$-factoriality is a stronger condition.

1.13.6 The Classification of DLT Surface Singularities II

We describe the classification of DLT pairs for algebraic surfaces. The results are established in a sufficiently small analytic neighborhood near the singularity.

First, consider the structure near points in the support of the boundary:

Theorem 1.13.19 ([61]) *Let X be an algebraic surface defined over the complex number field and let B be a reduced divisor on X. Assume that (X, B) is DLT. Then for any point $P \in X$ in the support of B, there exists an analytic neighborhood U such that one of the following assertions holds:*

(1) U is smooth and $B|_U$ is a normal crossing divisor in complex analytic sense.

(2) U has a cyclic quotient singularity of type $\frac{1}{r}(1, s)$ and $B|_U$ is irreducible. Here r, s are coprime positive integers. In more detail, there exists a neighborhood U_0 of the origin of the affine space $\mathbf{C}^2$ with coordinates x, y, a group action by $G = \mathbf{Z}/(r)$ as $x \mapsto \zeta x$, $y \mapsto \zeta^s y$ such that the pair $(U, B|_U)$ is analytically isomorphic to $(U_0/G, B_0/G)$. Here ζ is a primitive rth root of 1 and $B_0 = \operatorname{div}(x)$. In this case, $(U, B|_U)$ is PLT.

Conversely, pairs satisfying (1) or (2) are DLT.

Proof Take a sufficiently small analytic neighborhood U of P and take an analytic irreducible component B_1 of $B \cap U$. We may assume that B_1 remains irreducible when replacing U by smaller neighborhoods. Here note that it is possible that an (algebraic) irreducible component of B containing B_1 and passing P is strictly bigger than B_1 when restricting to U.

Since X has rational singularities, it is analytically $\mathbf{Q}$-factorial. Hence the divisor B_1 on U is $\mathbf{Q}$-Cartier. Take r_1 to be the smallest positive integer such that $r_1 B_1$ is Cartier. Then we may assume that $\mathcal{O}_U(r_1 B_1) \cong \mathcal{O}_U$. Take $\pi_1 \colon Y_1 \to U$ to be the index 1 cover. As π_1 is étale in codimension 1, by Lemma 1.13.17, $(Y_1, \pi_1^* B)$ is LC.

If one of the analytic irreducible components of $\pi_1^* B$ is not Cartier, note that Y_1 has again rational singularities, we can construct an index 1 cover $\pi_2 \colon Y_2 \to Y_1$ again. Therefore, we can construct a finite cover $\pi : Y \to U$ étale in codimension 1 such that any analytically irreducible component of $C = \pi^* B$ is Cartier. By construction, $Q = \pi^{-1}(P)$ is one point.

We will show that Y is smooth. Suppose not, take the minimal resolution of singularities $g \colon Z \to Y$. Take C_j to be an analytically irreducible component of C, as C_j is Cartier, $g^* C_j$ is an integral divisor. Note that the support of $g^* C_j$ contains the exceptional set of g.

Take s to be the number of such C_j. If $s \geq 2$, then any exceptional divisor of g has coefficients at least 1 in $g^* C_1$ and $g^* C_2$. Since $K_Z \leq g^* K_Y$, this contradicts the fact that (Y, C) is LC.

Now $s = 1$. Take $E_1, \ldots, E_r$ to be the exceptional divisors of g. Since Y has rational singularities, the dual graph of the exceptional divisors of g is a tree.

Since (Y, C) is LC, we get $g^*C_1 = g_*^{-1}C_1 + \sum E_i$ and $K_Z = g^*K_Y$. Since C_1 is analytically irreducible, set-theoretically $g_*^{-1}C_1$ intersects the support of $\sum E_i$ at one point. If the graph of $g_*^{-1}C_1 + \sum E_i$ is not a tree, then we need more blowups to get a log resolution of (Y, C), but this procedure will produce an exceptional divisor with log discrepancy coefficient at least 2, which is a contradiction.

On the other hand, if the graph of $g_*^{-1}C_1 + \sum E_i$ is a tree, then there exists an irreducible component E_1 intersecting $g_*^{-1}C_1 + \sum_{i \neq 1} E_i$ at just one point. But by $(K_Z \cdot E_1) = 0$ we get $(E_1^2) = -2$, which contradicts to $(g^*C_1 \cdot E_1) = 0$.

In summary, we showed that Y is smooth. By a similar argument, we can show that C is normal crossing. Note that $Y \setminus Q$ is connected and simply connected, so it coincides with the universal covering of $U \setminus P$. In particular, $\pi : Y \to U$ is a Galois covering. Take G to be the Galois group.

Embed Y into the affine space $\mathbf{C}^2$ with coordinates x, y such that Q is the origin. Since (Y, C) is LC and Q is contained in the support of C, we may assume that the equation of C is $xy = 0$ or $x = 0$. By construction, C is invariant under the action of G.

If the equation of C is $x = 0$, then $B \cap U$ is analytically irreducible, and hence G is the Galois group of an index 1 cover which is isomorphic to $\mathbf{Z}/(r_1)$. We get into case (2) by diagonalizing the generator of G. Here if r, s are not coprime, then there is a nontrivial subgroup of G with fixed locus outside Q, which contradicts the fact that $\pi : Y \to U$ is étale in codimension 1.

Consider the case that the equation of C is $xy = 0$. First, consider the case that every irreducible component of C is invariant under the action of G. By choosing coordinates properly, the log canonical form $dx/x \wedge dy/y$ is invariant under the action of G, and determines a log canonical form $\theta \in H^0(U, K_U + B)$ on the quotient space $Y/G \cong U$. Since θ has no zeros, $K_U + B$ is Cartier on U. Suppose that U is not smooth, take $h \colon V \to U$ to be the minimal resolution of singularities and write $h^*(K_U + B) = K_V + B_V$, then the coefficients of B_V are integers. Since $h^*K_U \geq K_V$, the coefficients of B_V are at least 1. This contradicts the fact that (X, B) is DLT. Hence U is smooth and we get into case (1).

Next, suppose that there exists an element in G exchanging irreducible components of C. Then $B \cap U$ is again analytically irreducible. Hence the DLT pair (U, B) is PLT. Take G' to be the subgroup of G consisting of all elements preserving irreducible components of C, then $G_1 = G/G' \cong \mathbf{Z}/(2)$ and the log canonical divisor $K_{Y'} + C'$ on $Y' = Y/G'$ is Cartier. Here C' is the image of C, which is a reduced divisor with two irreducible components. If Y' is not smooth, take $g' \colon Z' \to Y'$ to be the minimal resolution of singularities and write $(g')^*(K_{Y'} + C') = K_{Z'} + C'_Z$, then the coefficients of C'_Z all equal to 1. The action of G_1 on Y' extends to Z' and induces a birational morphism

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$h: V = Z'/G_1 \to U = Y'/G_1$. This is not necessarily the minimal resolution of singularities, but if write $h^*(K_U + B) = K_V + B_V$, then by the ramification formula, the coefficients of B_V all equal to 1, which contradicts the fact that (U, B) is PLT. Therefore, Y' is smooth. Then $G' = \{1\}$ and the action of G_1 exchanging irreducible components of C is étale in codimension 1, which is absurd. □

As an application in arbitrary dimension, we can show the subadjunction formula for DLT pairs (see Theorem 1.11.13):

Corollary 1.13.20 *Let (X, B) be a DLT pair and let Z be an irreducible component of $\llcorner B \lrcorner$. Define the* **R***-divisor B_Z on Z by $(K_X + B)|_Z = K_Z + B_Z$. Take an irreducible component P of B_Z with coefficient p. Denote by b_i the coefficients of irreducible components of B containing P. Then there exist positive integers m_i, r such that*

$$p = \frac{r - 1 + \sum b_i m_i}{r}.$$

Proof As we can check the coefficient of P on its generic point, we may assume that $\dim X = 2$ and P is a point. The coefficient remains the same when X is considered as a complex analytic variety, hence we just need to consider two cases in Theorem 1.13.19 applied to $(X, \llcorner B \lrcorner)$. Case (1) is trivial, we only consider case (2).

Let $Y = \mathbf{C}^2$, $W = \operatorname{div}(x)$, $G = \mathbf{Z}/(r)$, $X = Y/G$, and $Z = W/G$. Denote the projection by $\pi : Y \to X$. Take the origin $Q \in Y$ and denote $P = \pi(Q)$. In the DLT pair (X, B), $B = Z + \sum b_i B_i$. Take $C_i = \pi^* B_i$ and $m_i = (C_i \cdot W)_Q$ which are local intersection numbers at Q. When B_i passes through P, m_i is a positive integer.

Since the covering $\pi : Y \to X$ is étale outside the origin, $\pi^*(K_X + Z) = K_Y + W$. On the other hand, $\pi|_W : W \to Z$ is ramified over Q with index r, hence $\pi^* P = rQ$, $K_W = (\pi|_W)^* K_Z + (r - 1)Q$. On the smooth variety Y we have the usual adjunction formula $(K_Y + W)|_W = K_W$. Then the assertion follows. □

Next we consider points outside the boundary:

Theorem 1.13.21 ([61]) *Let X be an algebraic surface defined over the complex number field. Assume that the pair $(X, 0)$ is DLT. Then any point $P \in X$ is a quotient singularity. That is, there exists an analytic neighborhood U of P which is analytically isomorphic to the quotient of a neighborhood of the origin $(0, 0)$ of $\mathbf{C}^2$ by the linear action of a finite subgroup G of the general linear group* $\mathrm{GL}(2, \mathbf{C})$.

Conversely, if X has quotient singularities, then $(X, 0)$ is DLT.

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Proof Since $B = 0$, $(U, 0)$ is KLT. First, take the index 1 cover $\pi_1 : Y_1 \to U$ of K_X. Since $(Y_1, 0)$ is also KLT and K_{Y_1} is Cartier, Y_1 has canonical singularities. Therefore, $Y_1 = U_0/G_1$, where U_0 is a neighborhood of the origin of $\mathbf{C}^2$ and G_1 is a finite subgroup of $\mathrm{SL}(2, \mathbf{C})$. Now $U_0 \setminus \{0\}$ is the universal cover of $U \setminus \{P\}$ and we get the conclusion.

The converse statement follows from the ramification formula and holds for any dimension (Proposition 1.10.6). □

Birational geometry of algebraic surfaces works for arbitrary characteristics. The classification theorem of minimal models works under certain modification ([18, 19, 110]). The theory of rational singularities remains true, also the contraction theorem remains true ([6, 7]). The dual graph of the resolution of singularities of a DLT pair is completely classified, which is the same as in characteristic 0 ([51]; Figure 1.3). However, in characteristic 0 the singularity can be determined by the dual graph of the resolution of singularity, which turns out to be a quotient singularity, but on the other hand, in positive characteristics it is only known to be a rational singularity and the structure of the singularity is not determined only by the dual graph of the resolution of singularity, the classification seems to be more complicated. In addition, [51] is the origin where the author was involved in the minimal model theory.

1.13.7 The Zariski Decomposition

Finally, we state the Zariski decomposition theorem for divisors on algebraic surfaces:

Theorem 1.13.22 *Let D be an integral divisor on a smooth projective surface X. Assume that there exists a positive integer m such that $|mD| \neq \emptyset$. Then there exists an effective $\mathbf{Q}$-divisor N satisfying the following conditions:*

(1) $P = D - N$ is nef.
(2) $(P \cdot E_i) = 0$ for every i, where $E_1, \ldots, E_m$ are irreducible components of N.
(3) The matrix $[(E_i \cdot E_j)]$ is negative definite.

Moreover, N is uniquely determined by the above conditions.

Such a decomposition $D = P + N$ is called the *Zariski decomposition* of D ([144]).

Proposition 1.13.23 *Let X be a smooth projective surface and let $f : X \to Y$ be a morphism to a minimal model in the classical sense. Assume that K_Y is nef. Set $N = K_X - f^* K_Y$, then $K_X = f^* K_Y + N$ is the Zariski decomposition.*

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That is, we can say that the Zariski decomposition indeed gives the minimal model without taking a birational model. This is the reason why Zariski decomposition has drawn a lot of attention.

Example 1.13.24 We give an example of a log minimal model in dimension 2. The correspondence between Zariski decompositions and log minimal models holds in general ([51]).

Consider an irreducible curve B of degree 4 with three ordinary cusp singularities on the projective plane $X = \mathbf{P}^2$. Here an ordinary cusp singularity is a singularity analytically equivalent to the singularity given by the equation $x^2 - y^3 = 0$ at the origin. By the genus formula, B is a rational curve, that is, its normalization is isomorphic to $\mathbf{P}^1$. Let $f\colon Y \to X$ be the minimal log resolution of the pair (X, B) and let $C_0 = f_*^{-1}B$ be the strict transform. Let P_i $(i = 1,2,3)$ be the three singular points on B. Over each point there are three exceptional divisors E_{ij} $(i, j = 1,2,3)$ on Y. It is easy to calculate the intersection numbers $(C_0^2) = -2$ and $(E_{ij}^2) = -j$. $C = C_0 + \sum_{i,j} E_{ij}$ is a normal crossing divisor with all irreducible components isomorphic to $\mathbf{P}^1$. The dual graph is shown in Figure 1.4.

The Zariski decomposition $K_Y + C = P + N$ is given by

$$P = K_Y + C_0 + \sum_i \left(E_{i1} + \frac{1}{2}E_{i2} + \frac{2}{3}E_{i3} \right), \; N = \sum_i \left(\frac{1}{2}E_{i2} + \frac{1}{3}E_{i3} \right).$$

Here P is nef and big with $(P^2) = 1/2$.

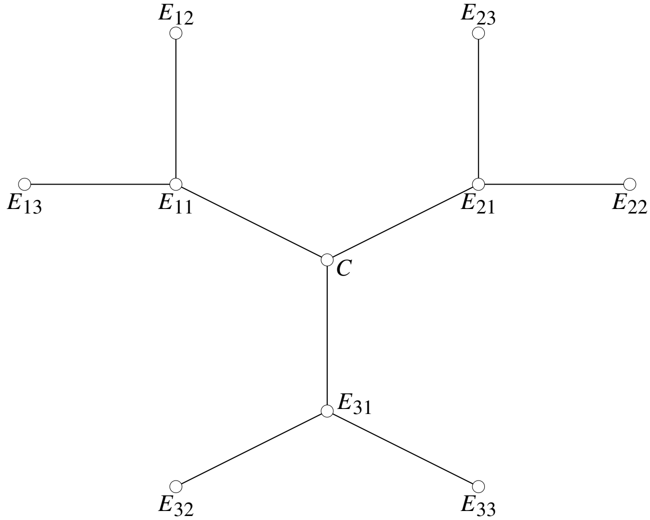

Figure 1.4 Dual graph of the resolution of singularities.

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Denote by $g\colon Y \to Z$ the contraction of six curves E_{i2}, E_{i3} ($i = 1,2,3$) in the support of N and $D = g_*C$. Then $K_Z + D$ is ample and $P = g^*(K_Z + D)$. The pair (Z, D) is a minimal model of the DLT pair (Y, C) which is also the canonical model.

In Chapter 2, we will generalize the definition of Zariski decomposition in a weak sense for pseudo-effective **R**-divisors in any dimension, which is called the "divisorial Zariski decomposition."

1.14 The 3-Dimensional Case

Let us consider the 3-dimensional case. In this situation, results in higher dimensional algebraic geometry discussed in subsequent chapters are necessary. Indeed, higher dimensional algebraic geometry starts from dimension 3. However, there are also special phenomena and results that only appear in dimension 3. We will describe them briefly as a comparison to results in dimensions up to 2. The results in this section will not be used in subsequent sections.

The MMP, including the existence of flips, the termination of flips, and the abundance conjecture which will be discussed in Chapter 2, is completely understood in dimension 3 even for the log version.

As a consequence of the minimal model theory, the following theorem holds:

Theorem 1.14.1 *Let X be a smooth projective* 3*-dimensional algebraic variety over a field of characteristic* 0*. Then there exists a projective algebraic variety X' with at most* **Q**-factorial terminal singularities *and a birational map $f\colon X \dashrightarrow X'$ surjective in codimension* 1 *such that one of the following assertions holds:*

(1) X' is a minimal model. *That is, the canonical divisor $K_{X'}$ is nef.*
(2) X' admits a Mori fiber space *structure. That is, there exists a surjective morphism $g\colon X' \to Y$ to a normal algebraic variety Y with* $\dim Y < \dim X$ *and connected geometric fibers such that $-K_X$ is g-ample and* $\rho(X/Y) = 1$.

Remark 1.14.2 (1) f is not necessarily a morphism and X' is not necessarily smooth, this is a feature in dimensions 3 and higher.
(2) X' has terminal singularities means that the pair $(X', 0)$ with divisor 0 has terminal singularities. The concept of terminal singularities was originally defined by Reid in dimension 3 ([121]). This was the starting point of higher dimensional minimal model theory. However, log terminal

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singularities for algebraic surfaces already appeared before this ([51]). In dimension 2, terminal singularities are impossible to be aware of since they are automatically smooth.

(3) Any terminal singularity can appear in some minimal model. Terminal singularities in dimension 3 are isolated singularities and are completely classified (Theorem 1.14.5). For example, for two coprime positive integers r, b with $b < r$, a cyclic quotient singularity of type $\frac{1}{r}(1, -1, b)$ is a terminal singularity (see Example 1.10.5 for the notation). The *Cartier index* of a singularity $P \in X$ is the minimal positive integer m such that mK_X is Cartier in a neighborhood of P. For example, the Cartier index of a cyclic quotient singularity of type $\frac{1}{r}(1, -1, b)$ is r. In particular, there are minimal models with arbitrarily large Cartier indices.

(4) The existence of flips in dimension 3 was proved by Mori via an almost complete classification of small contractions ([102]). As will be discussed in Chapter 3, the existence of flips in arbitrary dimension is proved in a completely different way by induction on dimensions, where the generalization to the log version is essential.

(5) The termination of flips in dimension 3 was proved by Shokurov ([127]). The termination of log flips in dimension 3 was proved in [65]. The termination of flips remains open in arbitrary dimension.

The abundance theorem holds in dimension 3 ([63, 95–97]):

Theorem 1.14.3 *Let X be a 3-dimensional minimal model. That is, X is a projective algebraic variety with terminal singularities and K_X is nef. Then there exists a positive integer m such that the pluricanonical system $|mK_X|$ is free. Associated with this, there exists a surjective morphism $f : X \to Y$ to a normal projective algebraic variety with connected geometric fibers such that $K_X \sim_{\mathbf{Q}} f^*H$ for an ample $\mathbf{Q}$-divisor H on Y. By definition, $\dim Y = \kappa(X)$. In particular, the canonical ring is finitely generated.*

Remark 1.14.4 (1) The log version of the abundance conjecture in dimension 3 was also proved ([78]).

(2) As will be shown in Chapter 3, the finite generation of canonical rings is much weaker that the abundance theorem.

Terminal singularities in dimension 3 are completely classified as complex analytic singularities ([101, 121, 124]):

Theorem 1.14.5 *Let X be a 3-dimensional algebraic variety defined over the complex number field with terminal singularities and take $P \in X$ to be a singular point. Then (X, P) is an isolated singularity. Take r to be the Cartier*

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index, then there exists an analytic neighborhood of P isomorphic to the neighborhood of the image of the origin of one of the following singularities:

(1) A cyclic quotient singularity of type $\frac{1}{r}(a,-a,1)$. Here r, a are coprime positive integers (see Example 1.10.5 for the notation).

(2) General type*: The quotient space of the hypersurface in $\mathbf{C}^4$ defined by the equation $xy+f(z^r,w)=0$ at the origin by the cyclic group $\mathbf{Z}/(r)$. In other words, the prime divisor in a 4-dimensional quotient singularity defined by*

$$\left\{(x,y,z,w)\in\frac{1}{r}(a,-a,1,0)\middle|xy+f(z^r,w)=0\right\}.$$

Here r, a are coprime positive integers and f has no constant term or w term.

The following (3), (4) are also prime divisors in 4-dimensional quotient singularities.

(3) Special type*:*

$$\left\{(x,y,z,w)\in\frac{1}{2}(1,0,1,1)\,\middle|\,x^2+y^2+f(z,w)=0\right\},\quad f\in\mathfrak{m}^4, r=2;$$

$$\left\{(x,y,z,w)\in\frac{1}{2}(1,0,1,1)\,\middle|\,x^2+f(y,z,w)=0\right\},$$

$$f\in\mathfrak{m}^3\setminus\mathfrak{m}^4,\ f_3\neq y^3, r=2;$$

$$\left\{(x,y,z,w)\in\frac{1}{3}(0,1,2,2)\,\middle|\,x^2+f(y,z,w)=0\right\},$$

$$f\in\mathfrak{m}^3,\ f_3=y^3+z^3+w^3, y^3+zw^2,\ or\ y^3+z^3, r=3;$$

$$\left\{(x,y,z,w)\in\frac{1}{2}(1,0,1,1)\,\middle|\,x^2+y^3+yf(z,w)+g(z,w)=0\right\},$$

$$f\in\mathfrak{m}^4, g\in\mathfrak{m}^4\setminus\mathfrak{m}^5, r=2.$$

Here $\mathfrak{m}$ is the maximal ideal of the origin.

(4) Exceptional type*:*

$$\left\{(x,y,z,w)\in\frac{1}{4}(1,3,1,2)\,\middle|\,x^2+y^2+f(z^2,w)=0\right\},\quad r=4.$$

Here f has no constant term or w term.

The exceptional type is different since f is not invariant under the group action.

Example 1.14.6 A terminal singularity appearing as the target of a divisorial contraction from a smooth 3-dimensional algebraic variety is either smooth or among one of the following cases:

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(1) A cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$.
(2) The hypersurface defined by the equation $xy + zw = 0$ in $\mathbf{C}^4$.
(3) The hypersurface defined by the equation $xy + z^2 + w^3 = 0$ in $\mathbf{C}^4$.

In cases (2) and (3), K_X is Cartier.

More complicated terminal singularities appear when taking divisorial contractions from singular 3-dimensional algebraic varieties. Conversely, for the equation of each singularity above, we can construct a divisorial contraction $f\colon Y \to X$ explicitly by a *weighted blowing up* of X (see [128, Appendix]).

Let X be a 3-dimensional minimal projective algebraic variety. When $\kappa(X) = 3$, we want to have a formula for plurigenera. Being of general type for X is equivalent to that the self-intersection of the canonical divisor on a minimal model is positive $(K_X^3) > 0$ (Theorem 1.5.12). However, as K_X is not necessarily Cartier, (K_X^3) is in general only a rational number.

By the finite generation of canonical rings, we can define the *canonical model* $Y = \operatorname{Proj} R(X, K_X)$. There exists a birational morphism $g\colon X \to Y$ such that $K_X = g^*K_Y$ which is the same as in dimension 2. Here this equality is in the following sense: For an integer m, mK_X is Cartier if and only if mK_Y is Cartier, moreover, in this case the equality $mK_X = g^*(mK_Y)$ holds. In particular, $|mK_X|$ is free if and only if $|mK_Y|$ is free.

In order to state *Reid's plurigenus formula* in [124], we introduce the concept of baskets of singularities. Take $\{P_1, \ldots, P_t\}$ to be the set of singular points of X. Each singular point (X, P_i) is associated with a set of couples of integers $\{\frac{1}{r_{ij}}(1, -1, b_{ij})\}$ which is called the *basket*. Here r_{ij}, b_{ij} are coprime positive integers with $b_{ij} < r_{ij}$. For example, when (X, P_i) is a cyclic quotient singularity of type $\frac{1}{r}(1, -1, b)$, its basket just consists of one couple $\{\frac{1}{r}(1, -1, b)\}$, which coincides with the type of the quotient singularity. In general, a 3-dimensional terminal singularity can be locally deformed into several cyclic quotient singularities, in which case its basket is the collection of types of those cyclic quotient singularities. The Cartier index r_i of (X, P_i) coincides with the least common multiple of r_{ij} in its basket. By considering baskets, terminal singularities can be replaced by a set of virtual cyclic quotient singularities.

Reid's plurigenus formula for $m \geq 2$ is the following:

$$\dim H^0(X, mK_X) = \frac{1}{12}m(m-1)(2m-1)(K_X^3) + (1-2m)\chi(\mathcal{O}_X) + \sum_{i,j}\left(\frac{r_{ij}^2 - 1}{12r_{ij}}(m - \bar{m}) + \sum_{k=0}^{\bar{m}-1} \frac{\overline{b_{ij}k}\cdot(r_{ij} - \overline{b_{ij}k})}{2r_{ij}}\right).$$

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Here $\overline{m}$ denotes the residue of m modulo r_{ij} ([124]). This formula is a sum of a polynomial in m and a periodic correction term with respect to m (see [144]). The correction term runs over the baskets of all singularities. As plurigenera are birational invariants, the left-hand side is the same as the starting smooth model, but the right-hand side can be only computed on a minimal model with singularities. In other words, when computing plurigenera on a smooth model, the singularities of its minimal model appear, which is a surprising phenomenon.

Also we have the following formula ([59]):

$$\chi(\mathcal{O}_X) = -\frac{1}{24}(K_X \cdot c_2(X)) + \sum_{i,j} \frac{r_{ij}^2 - 1}{24 r_{ij}}.$$

Here, since X has only isolated singularities, the intersection number $(K_X \cdot c_2(X))$ can be defined properly.

Remark 1.14.7 In this book, we will show the finite generation of canonical rings. However, it is impossible to find a bound of the degrees of generators depending only on the dimension. This can already be observed in dimension 3.

Let P be a singular point on a minimal model X. If m is not divisible by the Cartier index r of P, then P is a basepoint of $|mK_X|$. Hence for arbitrary large m, we can construct examples such that $|mK_X|$ is not free.

For example, if $\dim X = 3$ and P is a cyclic quotient singularity of type $\frac{1}{r}(a, -a, 1)$, then the canonical ring cannot be generated by elements of degree less than r. This is a completely different phenomenon from that in dimensions up to 2, because singularities appear in minimal models in dimensions 3 and higher.