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A dynamical classification for crossed products of fiberwise essentially minimal zero-dimensional dynamical systems

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Abstract. We prove that crossed products of fiberwise essentially minimal zero-dimensional dynamical systems, a class that includes systems in which all orbit closures are minimal, have isomorphic K-theory if and only if the dynamical systems are strong orbit equivalent. Under the additional assumption that the dynamical systems have no periodic points, this gives a classification theorem including isomorphism of the associated crossed product C^*-algebras as well. We additionally explore the K-theory of such crossed products and the Bratteli diagrams associated to the dynamical systems.

Key words: C^*-algebras, dynamical systems, Bratteli diagrams, K-theory
2020 Mathematics Subject Classification: 46L55 (Primary); 46L35, 46L80 (Secondary)

1. *Introduction*

In 1990, Putnam proved in [17] that the crossed product C^*-algebras associated to minimal Cantor systems are A$\mathbb{T}$-algebras of real rank zero. Using the classification results of Elliott in [7] and Dadarlat and Gong in [5], one sees that such C^*-algebras are classifiable by their K-theory. In 1995, Putnam, along with Thierry Giordano and Christian Skau, expanded this classification theorem to include dynamics; in [9], they showed that there is a condition on minimal Cantor systems, called 'strong orbit equivalence', that is equivalent to isomorphism of the K-theory associated to the dynamical systems, and therefore is equivalent to the isomorphism of the crossed product C^*-algebras. This dynamical classification was motivated by Krieger's theorem [12, 13], which says that for ergodic non-singular systems, the associated von Neumann crossed product factors are isomorphic if and only if the systems are orbit equivalent. The goal of this paper is to provide an extension of the dynamical classification theorem of Gioradno, Putnam, and Skau [9, Theorem 2.1] to include a larger class of zero-dimensional dynamical systems which we describe below.

In our previous paper [11], we determined a condition on a zero-dimensional dynamical system called 'fiberwise essentially minimal' (see Definition 2.2) that guarantees that the associated crossed product is an A$\mathbb{T}$-algebra. As its name suggests, this class is a broadening of minimal (and also essentially minimal). It additionally includes all systems whose orbit closures are minimal; more generally, it includes all systems whose points are all positively and negatively recurrent. If we assume that fiberwise essentially minimal zero-dimensional systems additionally have no periodic points, their crossed products have real rank zero and are therefore classifiable by K-theory (due to the work of Elliott in [7] and Dadarlat and Gong in [5]). This was an expansion of the work done on the minimal Cantor case in 1990 by Putnam [16, 17] in which the crossed products are simple, and work done on the essentially minimal case in 1992 by Putnam and Skau along with Herman [10] in which the crossed products are not necessarily simple. Some more non-simple results in this realm can be found in [3]. Our result from [11] includes many more non-simple crossed products.

This paper expands on the work in [11] in two major ways. The first is what we explore in §3, where we discuss some specifics about the K-theory of the crossed products. We define 'large subalgebras' of our crossed products (see Definition 3.6) which are AF-subalgebras (see Theorem 3.7) that have the same K_0 group as the crossed product (see Theorem 3.9). This mirrors the result of large subalgebras in the minimal case by Putnam [17]. We also give a simple description of the K_1 group of the crossed product in Theorem 3.10.

The second major aspect of this paper is expanding the dynamical classification of minimal Cantor systems that coincides with the K-theoretic classification, introduced by Giordano, Putnam, and Skau [9]. They introduce the notion of 'strong orbit equivalence', which we expand to the fiberwise essentially minimal case in Definition 2.2. In §4, we discuss how the circle algebra direct system that gives the A$\mathbb{T}$-algebra of the crossed product gives us a sequence of Kakutani–Rokhlin partitions, which we then use to create a Bratteli–Vershik–Kakutani model of the dynamical system, which is an ordered Bratteli diagram whose Vershik system is conjugate to the original dynamical system. Using this Bratteli diagram along with our K-theory results, we then prove Theorem 5.2, which tells us that for fiberwise essentially minimal zero-dimensional systems, K-theory isomorphism of the crossed products is equivalent to strong orbit equivalence of the dynamical systems. This, combined with the classification result of [5, 7], gives us Theorem 5.3, which tells us that if the dynamical systems have no periodic points, this is also equivalent to isomorphism of the crossed products.

The Bratteli–Vershik–Kakutani models developed in this paper suggest that more results in [9] could be generalized to the semisimple (minimal orbits) case. One cannot expect the results to hold for the entire fiberwise essentially minimal case. For example, [9, Theorem 2.2] tells us that orbit equivalence of minimal Cantor systems is equivalent to an isomorphism between the K_0 groups modulo their infinitesimal groups. Considering the shift on the one-point compactification of the integers (an essentially minimal system), this K_0 group modulo the infinitesimal group is isomorphic to $\mathbb{Z}$, which is the same as if the space was just a single point. Certainly there is no orbit equivalence between these spaces. However, in the semisimple case, these K_0 modulo infinitesimal groups are more interesting, and an application of the techniques developed in this paper to expand this result is a possibility. Considering [9, Theorem 2.2] also tells us that orbit equivalence is equivalent to a map preserving invariant probability measure, so expanding this result is of interest to ergodic theory.

2. *Preliminaries*

This section introduces terms that will be used to prove the main theorems of the paper, along with some examples and some previous relevant results.

Let X be a totally disconnected compact metrizable space and let $h : X \to X$ be a homeomorphism of X. We call (X, h) a *zero-dimensional system*. Let α be the automorphism of $C(X)$ induced by h; that is, α is defined by $\alpha(f)(x) = f(h^{-1}(x))$ for all $f \in C(X)$ and all $x \in X$. Then we denote the crossed product of $C(X)$ by α by $C^*(\mathbb{Z}, X, h)$ (or, less commonly, $C^*(\mathbb{Z}, C(X), \alpha)$). We denote the 'standard unitary' of $C^*(\mathbb{Z}, X, h)$ by u, which is a unitary element of $C^*(\mathbb{Z}, X, h)$ that satisfies $ufu^* = \alpha(f)$ for all $f \in C(X)$.

We will use the disjoint union symbol $\bigsqcup$ to denote unions of disjoint sets. We will not always say explicitly that the sets in this union are disjoint, as this will be implied by the notation. By a partition $\mathcal{P}$ of X, we mean a finite set of mutually disjoint compact open subsets of X whose union is X.

We say that a non-empty closed subset Y of X is a *minimal set* if it is h-invariant and has no non-empty h-invariant proper closed subsets. By Zorn's lemma, minimal sets exist for every zero-dimensional system. We say that a dynamical system (X, h) is *essentially minimal* if it has a unique minimal set, and additionally that (X, h) is *minimal* if the unique minimal set is X.

The following definition is introduced as [11, Definition 1.9]. Given a subset U of X, we use the notation

$$\lambda_U : U \to \mathbb{Z}_{>0} \cup \{\infty\} = \inf\{n \in \mathbb{Z}_{>0} \mid h^n(x) \in U\}.$$

This is the 'first return time map' of U under the homeomorphism h.

Definition 2.1. Let (X, h) be a zero-dimensional system and let $\mathcal{P}$ be a partition of X (see Figure 1). We define a *system of finite first return time maps subordinate to* $\mathcal{P}$ to be a tuple

$$\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, (K_t)_{t=1,\ldots,T}, (Y_{t,k})_{t=1,\ldots,T;k=1,\ldots,K_t}, (J_{t,k})_{t=1,\ldots,T;k=1,\ldots,K_t})$$

such that we have the following.

(1) We have $T \in \mathbb{Z}_{>0}$.

(2) For each $t \in \{1, \ldots, T\}$, X_t is a compact open subset of X. That $\mathcal{S}$ is subordinate to $\mathcal{P}$ means that for each $t \in \{1, \ldots, T\}$, X_t is contained in an element of $\mathcal{P}$.

(3) For each $t \in \{1, \ldots, T\}$, $K_t \in \mathbb{Z}_{>0}$.

(4) For each $t \in \{1, \ldots, T\}$ and each $k \in \{1, \ldots, K_t\}$, $Y_{t,k}$ is a compact open subset of X_t. Moreover, for each $t \in \{1, \ldots, T\}$, $\{Y_{t,1}, \ldots, Y_{t,K_t}\}$ is a partition of X_t; that is,

$$\bigsqcup_{k=1}^{K_t} Y_{t,k} = X_t.$$

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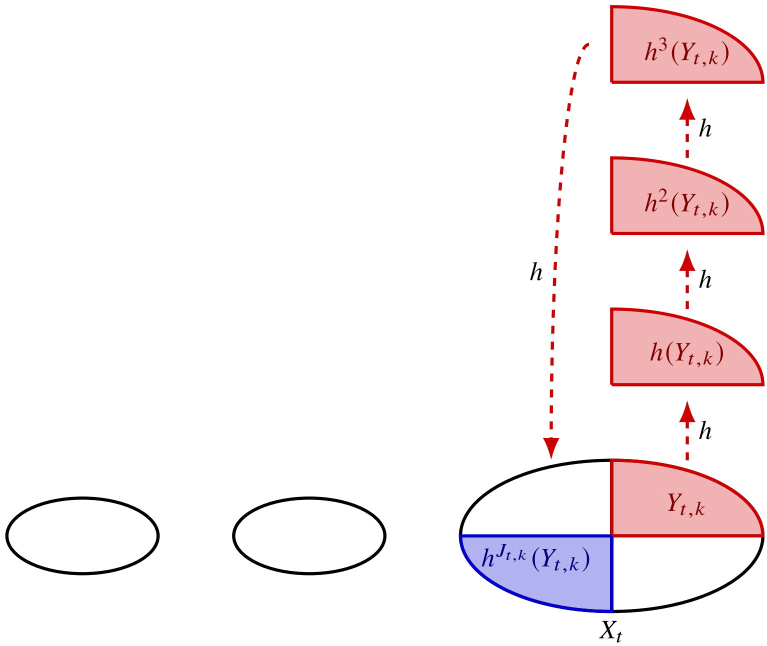

FIGURE 1. An illustration of Definition 2.1. For each base X_t, we divide the base into pieces $Y_{t,k}$ that come back to X_t only after applying $J_{t,k}$ iterations of the homeomorphism.

(5) For each $t \in \{1, \ldots, T\}$ and each $k \in \{1, \ldots, K_t\}$, $J_{t,k} \in \mathbb{Z}_{>0}$. Using Definition 2.1, we define $\{J_{t,k}\} = \lambda_{X_t}(Y_{t,k})$. Moreover, for each $t \in \{1, \ldots, T\}$, $\{h^{J_{t,1}}(Y_{t,1}), \ldots, h^{J_{t,K_t}}(Y_{t,K_t})\}$ is a partition of X_t; that is,

$$\bigsqcup_{k=1}^{K_t} h^{J_{t,k}}(Y_{t,k}) = X_t.$$

(6) The set

$$\mathcal{P}_1(\mathcal{S}) = \{h^j(Y_{t,k}) \mid t \in \{1, \ldots, T\}, k \in \{1, \ldots, K_t\}, \text{ and } j \in \{0, \ldots, J_{t,k} - 1\}\}$$

is a partition of X. Note that this combined with condition (5) also implies

$$\mathcal{P}_2(\mathcal{S}) = \{h^j(Y_{t,k}) \mid t \in \{1, \ldots, T\}, k \in \{1, \ldots, K_t\}, \text{ and } j \in \{1, \ldots, J_{t,k}\}\}$$

is a partition of X.

The following definition is introduced as [11, Definition 1.20].

Definition 2.2. Let (X, h) be a zero-dimensional system and let $Z \subset X$ be a closed subset. We say that the triple (X, h, Z) is a *fiberwise essentially minimal zero-dimensional system* if there is a quotient map $\psi : X \to Z$ such that:

(1) $\psi|_Z : Z \to Z$ is the identity map;

(2) $\psi \circ h = \psi$;

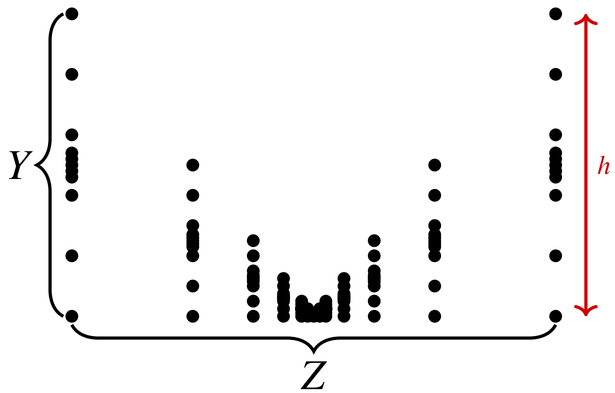

FIGURE 2. An illustration of Example 2.4(2). Each fiber over $z \in Z$ is a copy of Y, aside from the fiber at $\infty \in Z$, which is a singleton (this is pictured as the middle fiber). The picture is meant to depict how the singleton is connected to nearby fibers topologically.

(3) for each $z \in Z$, $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$ is an essentially minimal system and z is in its minimal set.

The following is [**11**, Theorem 2.1], linking Definitions 2.1 and 2.2.

THEOREM 2.3. *Let (X, h) be a zero-dimensional system. Then there exists some closed $Z \subset X$ such that (X, h, Z) is fiberwise essentially minimal if and only if for any partition $\mathcal{P}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$.*

Examples 2.4

(1) By [**11**, Proposition 1.18], zero-dimensional systems in which all points are positively and negatively recurrent are fiberwise essentially minimal. In particular, this includes systems in which orbit closures are minimal, called 'semisimple' by Furstenberg in [**8**]. Thus, examples of fiberwise essentially minimal zero-dimensional systems can be found by looking at the local structure of orbit closures.

(2) The following is [**11**, Example 1.21(c)]. Let $Z = \mathbb{Z} \cup \{\infty\}$ be the one-point compactification of the integers and let (Y, h') be an essentially minimal zero-dimensional system. Let $X = (Y \times Z)/(Y \times \{\infty\})$ and let $\pi : Y \times Z \to X$ be the quotient map. Let $\widetilde{h} = \pi(h' \times \mathrm{id}) : Y \times Z \to X$ and let $h : X \to X$ be the continuous map satisfying $h \circ \pi = \widetilde{h}$, which is obtained from the universal property of the quotient map. One checks that h is a homeomorphism. Define $\widetilde{\psi} : Y \times Z \to Z$ by $\widetilde{\psi}((y, z)) = z$ and then let $\psi : X \to Z$ be the continuous map satisfying $\psi \circ \pi = \widetilde{\psi}$, which is obtained from the universal property of the quotient map. One checks that ψ itself is a quotient map, and then one checks that (X, h) is a fiberwise essentially minimal zero-dimensional system (using Z and ψ as defined above). See this in Figure 2.

The following is [**11**, Theorem 2.2]. By an A$\mathbb{T}$-algebra, we mean a C^*-algebra that is the direct limit of 'circle algebras'. By 'circle algebra', we mean an algebra isomorphic to a finite direct sum of matrices and matrices over $C(S^1)$.

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THEOREM 2.5. *Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system. Then $C^*(\mathbb{Z}, X, h)$ is an $A\mathbb{T}$-algebra.*

The following is a consequence of the proof of Theorem 2.5 that will be useful later on.

COROLLARY 2.6. *Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system, let $\mathcal{P}$ be a partition of X, let $a_1, \ldots, a_n \in C^*(\mathbb{Z}, X, h)$, and let $\varepsilon > 0$. Then there is a circle algebra $A \subset C^*(\mathbb{Z}, X, h)$ and a partition $\mathcal{P}'$ of X that is finer than $\mathcal{P}$ such that we have the following.*

(1) *The diagonal matrices of A are $C(\mathcal{P}')$.*
(2) *For each $k \in \{1, \ldots, n\}$, there is a $b_k \in A$ such that $\|a_k - b_k\| < \varepsilon$.*

On its own, Theorem 2.5 gives us a reason to study fiberwise essentially minimal zero-dimensional systems in the context of operator algebras, as $A\mathbb{T}$-algebras are C^*-algebras with very nice structure. If the following conjecture holds, then fiberwise essentially minimality plays a very important role in the structure of zero-dimensional dynamical systems.

Conjecture 2.7. Let (X, h) be a zero-dimensional system. Then $C^*(\mathbb{Z}, X, h)$ is an $A\mathbb{T}$-algebra if and only if (X, h) is fiberwise essentially minimal.

What Conjecture 2.7 would imply is that there is something unique about the structure of fiberwise essentially minimal zero-dimensional systems that gives rise to the nicest possible direct limit structure. If this is true, there is something very likely unique and inherent about the dynamical systems themselves that is worth studying. It will also likely have many consequences in the future of non-simple C^*-algebra classification, as this direct limit structure (and lack thereof) tells us a lot about the K-theory of the crossed product C^*-algebra.

Without specifying all of the details, there is ample evidence in the literature for the validity of Conjecture 2.7. One direction is clear from Theorem 2.5. For the other direction, one can prove that if $C^*(\mathbb{Z}, X, h)$ has stable rank one (which is a consequence of being $A\mathbb{T}$), then all orbit closures of (X, h) are essentially minimal. Although this is being stated without proof, [**17**, Theorem 3.1] tells us that if an orbit closure has at least two minimal sets, then the crossed product cannot have stable rank one. More results in this direction are in [**3**], which closely examines what happens to Cantor systems when one has multiple minimal sets in a single orbit closure. Since one can show that minimal orbit closures imply fiberwise essential minimality (Example 2.4(1)), it is not hard to believe that essentially minimal orbit closures also imply fiberwise essential minimality.

We now introduce the concepts important to the dynamical side of the discussion in this paper. Let (X_1, h_1) and (X_2, h_2) be dynamical systems. By an *orbit map*, we mean a homeomorphism $F : X_1 \to X_2$ such that for all $x \in X_1$, we have $F(\mathrm{orb}_{h_1}(x)) = \mathrm{orb}_{h_2}(F(x))$, where $\mathrm{orb}_{h_1}(x)$ denotes the h_1-orbit of x (and likewise for orb_{h_2}). We say that (X_1, h_1) and (X_2, h_2) are *orbit equivalent* if there exists such an F. If the orbit map satisfies $F \circ h_1 = h_2 \circ F$, we say that (X_1, h_1) and (X_2, h_2) are *conjugate*.

Definition 2.8. Let (X_1, h_1) and (X_2, h_2) be dynamical systems and let $F : X_1 \to X_2$ be an orbit map. Then there are functions $\beta, \gamma : X_1 \to \mathbb{Z}$, called *orbit cocycles*, that satisfy $(F \circ h_1^{\beta(x)})(x) = (h_2 \circ F)(x)$ and $(h_2^{\gamma(x)} \circ F)(x) = (F \circ h_1)(x)$.

The following is a generalization of [9, Definition 1.3] from the minimal case to the fiberwise essentially minimal case (to see this, simply take Z below to be a singleton). When we consider orbit maps between (X_1, h_1, Z_1) and (X_2, h_2, Z_2) for closed sets $Z_1 \subset X_1$ and $Z_2 \subset X_2$, we require that $F(Z_1) = Z_2$.

Definition 2.9. Let (X_1, h_1, Z_1) and (X_2, h_2, Z_2) be fiberwise essentially minimal zero-dimensional systems. We say that (X_1, h_1, Z_1) and (X_2, h_2, Z_2) are *strong orbit equivalent* if there is an orbit map $F : X_1 \to X_2$ such that the orbit cocycles $\beta, \gamma : X_1 \to \mathbb{Z}$ are continuous on $X_1 \setminus Z_1$.

3. *K-theory*

In this section, we discuss the K-theory of the crossed products associated to fiberwise essentially minimal zero-dimensional systems. The K-theory of operator algebras has an ordered group (defined below) of equivalence classes of projections, (K_0, K_0^+), and a group of equivalence classes of unitaries, K_1. K-theory is a classifying invariant for C^*-algebras in sufficiently nice cases (being an A$\mathbb{T}$-algebra is sufficient). For a general reference on K-theory of operator algebras, see [4]. For some references on work done in the minimal case, see [9, 17]. For a reference on work done in the essentially minimal case, see [10].

Definition 3.1. An *ordered group* is a pair (G, G^+), where G is a countable abelian group and G^+ is a subset of G, called the *positive cone*, that satisfies the following:

(1) for all $g_1, g_2 \in G^+$, we have $g_1 + g_2 \in G^+$;
(2) for all $g \in G$, there are $g_1, g_2 \in G^+$ such that $g = g_1 - g_2$;
(3) the identity of G is the only element in both G^+ and $-G^+$.

We call $e \in G^+$ an *order unit* if for all $g \in G^+$, there is some $n \in \mathbb{Z}_{>0}$ such that $ne - g \in G^+$.

Given an ordered group (G, G^+), we may write $g \geq 0$ to denote $g \in G^+$. The notation $g_1 \geq g_2$ means that $g_1 - g_2 \in G^+$. By a homomorphism of ordered groups (G_1, G_1^+) and (G_2, G_2^+), we mean a homomorphism of groups $\varphi : G_1 \to G_2$ such that $\varphi(G_1^+) \subset G_2^+$.

When we fix a particular order unit $e \in G^+$, we may write the triple (G, G^+, e) and call this an ordered group with distinguished order unit. By a homomorphism of ordered groups with distinguished order units (G_1, G_1^+, e_1) and (G_2, G_2^+, e_2), we mean a homomorphism of ordered groups $\varphi : G_1 \to G_2$ such that $\varphi(e_1) = e_2$.

We introduce notation important to the following proposition, which is [11, Proposition 2.2], and a direct consequence of [14, Theorem 2.4]. Let $\mathcal{T}$ denote the Toeplitz algebra, the universal C^*-algebra generated by a single isometry s. Let $\mathcal{K}$ denote the C^*-algebra of compact operators on a separable Hilbert space. Let A be a unital C^*-algebra and let α be an automorphism of A, and let u be the standard unitary of $C^*(\mathbb{Z}, A, \alpha)$. We denote by $\mathcal{T}(A, \alpha)$ the Toeplitz extension of A by α, which is the subalgebra of $C^*(\mathbb{Z}, A, \alpha) \otimes \mathcal{T}$

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generated by $A \otimes 1$ and $u \otimes s$. The ideal generated by $A \otimes (1 - ss^*)$ is isomorphic to $A \otimes \mathcal{K}$, and the quotient by this ideal is isomorphic to $C^*(\mathbb{Z}, A, \alpha)$.

PROPOSITION 3.2. *Let (X, h) be a zero-dimensional system. Let α be the automorphism of $C(X)$ induced by h; that is, α is defined by $\alpha(f)(x) = f(h^{-1}(x))$ for all $f \in C(X)$ and all $x \in X$. Let δ be the connecting map obtained from the exact sequence*

$$0 \longrightarrow C(X) \otimes \mathcal{K} \longrightarrow \mathcal{T}(C(X), \alpha) \longrightarrow C^*(\mathbb{Z}, A, \alpha) \longrightarrow 0,$$

where $K_0(C(X) \otimes \mathcal{K})$ is identified with $K_0(C(X))$ in the standard way. Let $i : C(X) \to C^*(\mathbb{Z}, X, h)$ be the natural inclusion. Then there is an exact sequence

$$0 \longrightarrow K_1(C^*(\mathbb{Z}, X, h)) \xrightarrow{\delta} K_0(C(X)) \xrightarrow{\mathrm{id}-\alpha_*} K_0(C(X)) \xrightarrow{i_*} K_0(C^*(\mathbb{Z}, X, h)) \longrightarrow 0.$$

Proof. Since $K_1(C(X)) = 0$, this follows immediately from [14, Theorem 2.4]. □

Note that $K_0(C(X)) \cong C(X, \mathbb{Z})$; we will use this identification throughout the paper. Let $C(X, \mathbb{Z})^+$ denote the subset of $C(X, \mathbb{Z})$ consisting of f such that $f(x) \geq 0$ for all $x \in X$. Then it is easy to check that $(C(X, \mathbb{Z}), C(X, \mathbb{Z})^+)$ is an ordered group and the function χ_X is an order unit.

The following is closely related to [10, Proposition 5.1]; although the hypotheses are broadened, the proof is essentially the same. Adopting the notation of Proposition 3.2, we denote $K_0(C(X))/\mathrm{im}(\mathrm{id} - \alpha_*)$ by $K^0(X, h)$.

PROPOSITION 3.3. *Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system and adopt the notation of Proposition 3.2. Let $\pi : C(X, \mathbb{Z}) \to K^0(X, h)$ denote the quotient map. Define $K^0(X, h)^+ = \pi(C(X, \mathbb{Z})^+)$. Then $(K^0(X, h), K^0(X, h)^+, \pi(1))$ is an ordered group with distinguished order unit.*

Proof. We check the conditions of Definition 3.1. Conditions (1) and (2) follow from surjectivity of π. For condition (3), let $g \in K^0(X, h)^+ \cap -K^0(X, h)^+$. This means that there is $f_1 \in C(X, \mathbb{Z})^+$ such that $\pi(f_1) = g$ and $f_2 \in C(X, \mathbb{Z})^+$ such that $\pi(-f_2) = g$. However, then $\pi(f_1 + f_2) = 0$ and so $f_1 + f_2 \in \mathrm{im}(\mathrm{id} - \alpha_*)$. Let $f \in C(X, \mathbb{Z})$ satisfy $f - \alpha_*(f) = f_1 + f_2$. Let $E = f^{-1}(\max_{x \in X} f(x))$. Since $f - \alpha_*(f) \geq 0$, we must have $h(E) \subset E$. Let ψ be as in Definition 2.2 and let $z \in \psi(E)$ and define $E_z = E \cap \psi^{-1}(z)$. Since $h(E_z) \subset E_z$, E_z is invariant so must intersect the minimal set. However, then by [10, Theorem 1.1], since $E_z \neq \varnothing$, we have $\bigcup_{n \in \mathbb{Z}_{\geq 0}} h^n(E_z) = \psi^{-1}(z)$, and so $E_z = \psi^{-1}(z)$. Since this holds for all $z \in \psi(E)$, we see that f is constant on $\psi^{-1}(z)$ for all $z \in Z$. Since $\psi^{-1}(z)$ is invariant for all $z \in Z$, we must have $f = \alpha_*(f)$, and so $f_1 + f_2 = 0$, and since $f_1, f_2 \geq 0$, we see $f_1 = f_2 = 0$, and finally we see $g = 0$. This proves condition (3).

Finally, the fact that $\pi(1)$ is an order unit is also clear from the surjectivity of π. □

THEOREM 3.4. *Let (X, h) be a zero-dimensional system. Then adopting the notation of Proposition 3.2, we have $(K_0(C^*(\mathbb{Z}, X, h)), K_0(C^*(\mathbb{Z}, X, h))^+, 1) \cong (K^0(X, h), K^0(X, h)^+, 1)$.*

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Proof. Since $\ker(i_*) = \mathrm{im}(\mathrm{id} - \alpha_*)$, and since $i : C(X) \to C^*(\mathbb{Z}, X, h)$ is the natural inclusion, the map i_* induces a map $\varphi : K^0(X, h) \to K_0(C^*(\mathbb{Z}, X, h))$ which is an isomorphism of groups and satisfies $K^0(X, h)^+ \subset K_0(C^*(\mathbb{Z}, X, h))^+$.

Let $p \in C^*(\mathbb{Z}, X, h)$ be a projection. By applying Corollary 2.6 with $a_1 = p$ and $\varepsilon = 1/2$, p is unitarily equivalent to χ_U for some compact open $U \subset X$. Let q be the image of χ_U under the quotient map $C(X) \to C(X)/\mathrm{im}(\mathrm{id} - \alpha)$. Then $[\varphi(q)] = [\chi_U] = [p]$. Repeating this argument for $M_n(C^*(\mathbb{Z}, X, h))$, we see that $K_0(C^*(\mathbb{Z}, X, h))^+ \subset K^0(X, h)^+$.

Finally, that $\varphi(1) = 1$ is clear, proving the theorem. □

What we have also shown in the previous proof is the following.

COROLLARY 3.5. *Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system. Let $i : C(X) \to C^*(\mathbb{Z}, X, h)$ denote the canonical inclusion. Then the induced map $i_* : K_0(C(X)) \to K_0(C^*(\mathbb{Z}, X, h))$ is surjective as a map between ordered groups.*

The following definition is from [**15**, §2], later studied in the minimal case in [**17**]. These have been referred to as 'large subalgebras' in the literature. They are called large due to Theorem 3.9, as they capture the entire K_0 of the larger crossed product. Given a locally compact Hausdorff space X, we denote the continuous functions on X that 'vanish at infinity' by $C_0(X)$. More formally, $C_0(X)$ is the C^*-algebraic closure of the continuous functions on X with compact support.

Definition 3.6. Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system. We define A_Z to be the C^*-algebra generated by $C(X)$ and $uC_0(X \setminus Z)$.

The following theorem is contained in [**15**, Theorem 2.3]. We provide a direct proof in our context for the reader, which helps give an idea of the AF structure of the large subalgebra.

THEOREM 3.7. *Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system. Then A_Z is an AF-algebra.*

Proof. Let $(\mathcal{P}^{(n)})$ be a generating sequence of partitions of X. For each $n \in \mathbb{Z}_{>0}$, we inductively define systems $\mathcal{S}^{(n)} = (T^{(n)}, (X_t^{(n)}), (K_t^{(n)}), (Y_{t,k}^{(n)}), (J_{t,k}^{(n)}))$ of finite first return time maps. First, let $\mathcal{S}^{(1)} = (T^{(1)}, (X_t^{(1)}), (K_t^{(1)}), (Y_{t,k}^{(1)}), (J_{t,k}^{(1)}))$ be any system of finite first return time maps subordinate to $\mathcal{P}^{(n)}$ such that $\mathcal{P}_1(\mathcal{S}^{(1)})$ is finer than $\mathcal{P}^{(1)}$ and such that $\bigsqcup_{t=1}^{T^{(1)}} X_t^{(1)} \supset Z$ (the former is possible by [**11**, Proposition 1.13] and the latter is possible by [**11**, Lemma 4.12]). Now, let $n \in \mathbb{Z}_{>0}$ and suppose we have chosen $\mathcal{S}^{(n)} = (T^{(n)}, (X_t^{(n)}), (K_t^{(n)}), (Y_{t,k}^{(n)}), (J_{t,k}^{(n)}))$. Let $\mathcal{S}^{(n+1)} = (T^{(n+1)}, (X_t^{(n+1)}), (K_t^{(n+1)}), (Y_{t,k}^{(n+1)}), (J_{t,k}^{(n+1)}))$ be any system of finite first return time maps subordinate to $\mathcal{P}^{(n+1)}$ such that $\mathcal{P}_1(\mathcal{S}^{(n+1)})$ is finer than $\mathcal{P}^{(n+1)}$ and finer than $\mathcal{P}_1(\mathcal{S}^{(n)})$ and such that $\bigsqcup_{t=1}^{T^{(n+1)}} X_t^{(n+1)} \supset Z$.

Let $n \in \mathbb{Z}_{>0}$. Let $A^{(n)}$ be the finite dimensional C^*-subalgebra of $C^*(\mathbb{Z}, X, h)$ spanned by the matrix units $u^{i-j}\chi_{h^j(Y_{t,k}^{(n)})}$ for $t \in \{1, \ldots, T^{(n)}\}$, $k \in \{1, \ldots, K_t^{(n)}\}$, and i,

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$j \in \{0, \ldots, J_{t,k}^{(n)} - 1\}$. We see $A^{(n)} \cong \bigoplus_{t=1}^{T^{(n)}} \bigoplus_{k=1}^{K_t^{(n)}} M_{J_{t,k}^{(n)}}$. Notice that $C(\mathcal{P}_1(\mathcal{S}^{(n)})) \subset A^{(n)}$ as the diagonal matrices. Set $Z^{(n)} = \bigsqcup_{t=1}^{T^{(n+1)}} X_t^{(n)}$ and then notice that $uC(X \setminus Z^{(n)}) \subset A^{(n)}$ as the superdiagonal matrices, so $A^{(n)}$ is generated by $C(\mathcal{P}_1(\mathcal{S}^{(n)}))$ and $uC(X \setminus Z^{(n)})$.

Notice that $A^{(n)} \subset A^{(n+1)}$, so we get a directed system of finite dimensional C^*-algebras, whose limit $A^{(\infty)}$ contains $C(X)$ since $(\mathcal{P}_1(\mathcal{S}^{(n)}))$ is a generating sequence of partitions, and since $\bigcap_{n=1}^{\infty} Z^{(n)} = Z$, we have $uC(X \setminus Z^{(n)}) \to uC_0(X \setminus Z) \subset A_Z$. It now clear that $A^{(\infty)}$ is generated by $C(X)$ and $uC_0(X \setminus Z)$, and is therefore equal to A_Z. □

The following is [16, Lemma 4.2].

LEMMA 3.8. *Adopt the notation of Theorem 3.7 and its proof. Let p be a projection in $C(X) \cap A^{(n)}$ and suppose that $p = 0$ on $Z^{(n)}$. Then $\alpha(p) \in C(X) \cap A^{(n)}$ and $[\alpha(p)] = [p]$ in $K_0(A^{(n)})$.*

We finally have the following theorem, which tells us enough about the K_0 structure of the crossed product to be able to prove Theorems 5.2 and 5.3. The proof follows that of [16, Theorem 4.1].

THEOREM 3.9. *Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system. Then $K_0(A_Z) \cong K_0(C^*(\mathbb{Z}, X, h))$ as ordered groups.*

Proof. Let $i : A_Z \to C^*(\mathbb{Z}, X, h)$ denote the inclusion map, and let $i_* : K_0(A_Z) \to K_0(C^*(\mathbb{Z}, X, h))$ denote the map induced by i on K_0. Let $i_1 : C(X) \to A_Z$ denote the canonical inclusion, let $i_2 : C(X) \to C^*(\mathbb{Z}, X, h)$ denote the canonical inclusion, and let $(i_1)_*$ and $(i_2)_*$ denote the induced maps on K_0. We then clearly have $i \circ i_1 = i_2$. By Corollary 3.5, $(i_2)_* : K_0(C(X)) \to K_0(C^*(\mathbb{Z}, X, h))$ is a surjective map between ordered groups, and therefore so is i^*.

By Proposition 3.2, $\ker((i_2)_*) = \operatorname{ran}(\mathrm{id} - \alpha_*)$. Thus, since $(i_2)_* = (i_1)_* \circ i_*$, we have $(i_1)_*(\operatorname{ran}(\mathrm{id} - \alpha_*)) \subset \ker(i_*)$. Now suppose that $a \in \ker(i_*)$. Because $(i_1)_*$ is surjective, we can find $g \in C(X, \mathbb{Z})$ such that $(i_1)_*(g) = a$. Then $(i_2)_*(g) = i_*(a) = 0$, so $g \in \ker((i_2)_*) = \operatorname{ran}(\mathrm{id} - \alpha_*)$, so $a \in (i_1)_*(\operatorname{ran}(\mathrm{id} - \alpha_*))$. Altogether, we have

$$(i_1)_*(\operatorname{ran}(\mathrm{id} - \alpha_*)) = \ker(i_*). \tag{3.1}$$

Let $(\mathcal{P}^{(n)})$ be a sequence of partitions, let $(\mathcal{S}^{(n)})$ be a sequence of systems of finite first return time maps, and let $(A^{(n)})$ be a sequence of subalgebras of $C^*(\mathbb{Z}, X, h)$ as in the proof of Theorem 3.7. We now claim that $(i_1)_*(\operatorname{ran}(\mathrm{id} - \alpha_*)) = 0$. Suppose $g_1, g_2 \in C(X, \mathbb{Z})$ satisfy $g_1|_Z = g_2|_Z$. Since $(\mathcal{P}^{(n)})$ is a generating sequence of partitions and for each $n \in \mathbb{Z}_{>0}$, we have $\mathcal{P}_1(\mathcal{S}^{(n)})$ is finer than $\mathcal{P}^{(n)}$, there is some $n \in \mathbb{Z}_{>0}$ such that $g_1, g_2 \in A^{(n)}$ and such that $g_1|_{Z^{(n)}} = g_2|_{Z^{(n)}}$ (where $Z^{(n)}$ is defined in the proof of Theorem 3.7). So $g_1 - g_2$ is 0 on $Z^{(n)}$, so we can write $g_1 - g_2$ as a linear combination of projections in $C(X) \cap A^{(n)}$ each of which is zero on $Z^{(n)}$. So by Lemma 3.8, we have $[\alpha(g_1 - g_2)] = [g_1 - g_2]$ in $K_0(A^{(n)})$, and so $(i_1)_*(g_1 - \alpha(g_1)) = (i_1)_*(g_2 - \alpha(g_2))$ in $K_0(A_Z)$.

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So let $g \in C(X, \mathbb{Z})$. Define $f \in C(X, \mathbb{Z})$ by $f = g \circ \psi$. Then $f|_Z = g|_Z$, and so by the above paragraph, we have $(i_1)_*(g - \alpha(g)) = (i_1)_*(f - \alpha(f))$. However, then notice that $\alpha(f) = g \circ \psi \circ h^{-1} = g \circ \psi = f$, and so $(i_1)_*(f - \alpha(f)) = 0$. Thus, we have $(i_1)_*(\mathrm{ran}(\mathrm{id} - \alpha_*)) = 0$. Combining this with equation (3.1), we see $\ker(i_*) = 0$. Altogether, we have shown that i_* is an isomorphism of ordered groups. □

THEOREM 3.10. *Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system. Then $K_1(C^*(\mathbb{Z}, X, h)) \cong C(Z, \mathbb{Z})$.*

Proof. Adopt the notation of Proposition 3.2. Then $K_1(C^*(\mathbb{Z}, X, h)) \cong \ker(\mathrm{id} - \alpha_*)$. Identifying $K_0(C(X))$ with $C(X, \mathbb{Z})$, we may replace α_* with α.

Let $f \in \ker(\mathrm{id} - \alpha)$ and let $z \in Z$. Suppose $f|_{\psi^{-1}(z)}$ is not constant. Then there is some $x \in \psi^{-1}(z)$ such that $f(z) \neq f(x)$. Let U be a compact open subset of $\psi^{-1}(z)$ such that $f(U) = f(z)$. Since $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$ is an essentially minimal zero-dimensional system, by [10, Theorem 1.1], there is an $n \in \mathbb{Z}_{>0}$ such that $x \in h^{-n}(U)$. Let $x' = h^n(x) \in U$. Then $f(x') \neq f(x) = f(h^{-n}(x')) = \alpha^n(f)(x')$, and so $f \neq \alpha^n(f)$, and so $f \neq \alpha(f)$, which is a contradiction to $x \in \ker(\mathrm{id} - \alpha)$. Therefore, $f|_{\psi^{-1}(z)}$ is constant.

Now suppose $f \in C(X, \mathbb{Z})$ and suppose $f|_{\psi^{-1}(z)}$ is constant for each $z \in Z$. Then for each $x \in X$, we have $\alpha(f)(x) = f(h^{-1}(x)) = f(\psi(x)) = f(x)$, and so $f = \alpha(f)$, and so $f \in \ker(\mathrm{id} - \alpha)$.

Thus, we have

$$\ker(\mathrm{id} - \alpha) = \{f \in C(X, \mathbb{Z}) \mid f|_{\psi^{-1}(z)} \text{ is constant for each } z \in Z\} \cong C(Z, \mathbb{Z})$$

as desired. □

COROLLARY 3.11. *Let (X_1, h_1, Z_1) and (X_2, h_2, Z_2) be fiberwise essentially minimal zero-dimensional systems such that $C^*(\mathbb{Z}, X_1, h_1) \cong C^*(\mathbb{Z}, X_2, h_2)$. Then $Z_1 \cong Z_2$.*

A consequence of Corollary 3.11 is that given a zero-dimensional system, all choices of Z that make it fiberwise essentially minimal are homeomorphic.

4. *Bratelli diagrams*

In this section, we explore the construction of ordered Bratelli diagrams associated to fiberwise essentially minimal zero-dimensional systems. This correspondence is used to prove Theorem 5.2. For work done in the minimal case, see [2, 9]. For work done in the essentially minimal case, see [1, 10].

Definition 4.1. A *Bratteli diagram* B is a pair of sets (V, E) such that we have the following.

(1) The set V is called *the set of vertices of* B. We can write $V = \bigsqcup_{n=0}^{\infty} V_n$, where V_0 contains a single point v_0 and V_n is finite and non-empty for all $n \in \mathbb{Z}_{\geq 0}$. For each $n \in \mathbb{Z}_{\geq 0}$, we call V_n *the set of vertices of* B *at level* n.
(2) The set E is called *the set of edges of* B. We can write $E = \bigsqcup_{n=1}^{\infty} E_n$, where E_n is finite and non-empty for all $n \in \mathbb{Z}_{>0}$. For each $n \in \mathbb{Z}_{>0}$, we call E_n *the set of edges at level* n.

(3) There are maps $r, s : E \to V$ such that for $n \in \mathbb{Z}_{>0}$ and $e \in E_n$, $s(e) \in V_{n-1}$ and $r(e) \in V_n$. Moreover, $r^{-1}(v)$ is non-empty for all $v \in V$ and $s^{-1}(v)$ is non-empty for all $v \in V \setminus V_0$. The map r is called the *range map of B* and the map s is called *the source map of B*.

Notation 4.2. Let $B = (V, E)$ be a Bratteli diagram. For each $v \in V$, we denote $R(v) = r(s^{-1}(v))$, and for each $v \in V \setminus V_0$, we denote $S(v) = s(r^{-1}(v))$. If $v \in V_n$, then $R(v)$ is the set of all vertices in V_{n+1} that are connected to v by an edge, and $S(v)$ is the set of all vertices in V_{n-1} connected to v by an edge. In a reasonable sense, this gives us range and source maps for vertices.

For each $k, k' \in \mathbb{Z}_{>0}$ with $k < k'$, we denote by $P_{k,k'}$ the set of all paths from V_k to $V_{k'}$. Formally, $P_{k,k'}$ is the set of $(e_{k+1}, \ldots, e_{k'})$ such that for all $j \in \{k+1, \ldots, k'\}$, $e_j \in E_j$ and for all $j \in \{k+1, \ldots, k'-1\}$, we have $r(e_j) = s(e_{j+1})$.

Definition 4.3. Let $B = (V, E)$ and $B' = (V', E')$ be Bratteli diagrams. We say that B' is a *telescoping* of B if there is a sequence (k_n) such that for all $n \in \mathbb{Z}_{\geq 0}$, setting $k_0 = 0$, we have $k_n \in \mathbb{Z}_{\geq 0}$, $k_0 = 0$, $k_n > k_{n-1}$, $V'_n = V_{k_n}$, and $E'_n = P_{k_{n-1},k_n}$.

Remark 4.4. We create an equivalence class of Bratteli diagrams from isomorphism (bijections of vertices and edges at each level respecting range and source maps) and telescoping. If B_1 and B_2 are in the same equivalence class, we denote this by $B_1 \sim B_2$. In [6], K_0 (an ordered group) of a Bratteli diagram is defined. Although we will not go into the detail as it is not important for this paper, the result that is important is that $K_0(B_1) \cong K_0(B_2)$ if and only if $B_1 \sim B_2$. As described in [9], we also have $B_1 \sim B_2$ if and only if there is a Bratteli diagram B, called the *aggregate Bratteli diagram of B_1 and B_2*, such that telescoping B to odd levels yields a telescoping of B_1 and telescoping B to even levels yields a telescoping of B_2.

Definition 4.5. An *ordered Bratteli diagram B* is a Bratteli diagram (V, E) together with a partial order $\leq$ on E such that $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. We write $B = (V, E, \leq)$.

Let $B = (V, E, \leq)$ be an ordered Bratteli diagram. We define $E_{\min}$ ($E_{\max}$) to be the set of all edges that are minimal (maximal, respectively) with respect to $\leq$. We define $V_{\min}$ ($V_{\max}$) to be the set of all $v \in V$ such that there is an e in $E_{\min}$ ($E_{\max}$, respectively) with $s(e) = v$.

If $B = (V, E, \leq)$, then any telescoping B' of B has an order induced by B. In general, we can put an order on $P_{i,j}$ by $(e_{i+1}, \ldots, e_j) \leq (e'_{i+1}, \ldots, e'_j)$ if $e_k \leq e'_k$ for the smallest $k \in \{i+1, \ldots, j\}$ such that $e_k \notin E_{\max}$.

Definition 4.6. Let $B = (V, E, \leq)$ be an ordered Bratteli diagram. We define a *partial Vershik transformation* $\widetilde{h}_B : (X_B \setminus X_{B,\max}) \cup X_{B,\min} \to (X_B \setminus X_{B,\min}) \cup X_{B,\max}$ in the following way. If $x \in X_{B,\max} \cap X_{B,\min}$, we define $\widetilde{h}_B(x) = x$. If $x = (x_1, x_2, \ldots) \in X_B \setminus X_{B,\max}$, then there is some smallest $k \in \mathbb{Z}_{>0}$ such that $x_k \notin E_{\max}$. Let y_k denote the successor of x_k in E and let $(y_1, y_2, \ldots, y_{k-1})$ be the unique path from

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v_0 to $s(y_k)$ such that $y_j \in E_{\min}$ for all $j \in \{1, \ldots, k-1\}$. We define $\widetilde{h}_B(p) = (y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots)$.

Definition 4.7. Let $B = (V, E, \leq)$ be an ordered Bratteli diagram. We define the *infinite path space* X_B to be the set of all sequences $x = (x_1, x_2, \ldots)$ where $x_n \in E_n$ and $r(x_n) = s(x_{n+1})$ for all $n \in \mathbb{Z}_{>0}$ together with the topology generated by sets of the form $U(e_1, \ldots, e_k)$, which is the set of all $x = (x_1, x_2, \ldots)$ with $x_j = e_j$ for all $j \in \{1, \ldots, k\}$.

Let $B = (V, E, \leq)$ be an ordered Bratteli diagram. It is easy to see that the infinite path space is a zero-dimensional space. We define $X_{B,\min}$ ($X_{B,\max}$) to be the set of all $x = (x_1, x_2, \ldots) \in X_B$ such that x_j is in $E_{\min}$ ($E_{\max}$, respectively) for all $j \in \mathbb{Z}_{>0}$.

The following terminology appears in [2, Definition 2.18], although we restate it to give more clarity as to when the definition applies.

Definition 4.8. Let $B = (V, E, \leq)$ be an ordered Bratteli diagram and let $\widetilde{h}_B$ be its partial Vershik transformation. We say that the ordering on B is *perfect* if for every $e \in X_{B,\min}$, $\overline{\mathrm{orb}_{\widetilde{h}_B(e)}} \cap X_{B,\max}$ contains a single element, and if for every $e \in X_{B,\max}$, $\overline{\mathrm{orb}_{\widetilde{h}_B(e)}} \cap X_{B,\min}$ contains a single element. In this case, we define the *Vershik transformation* of X_B, denoted by h_B, to be the extension of $\widetilde{h}_B$ which, for each $e \in X_{B,\min}$, sends the unique element of $\overline{\mathrm{orb}_{\widetilde{h}_B(e)}} \cap X_{B,\max}$ to e.

Thus, given an ordered Bratteli diagram $B = (V, E, \leq)$ with a perfect ordering, the system (X_B, h_B) is a zero-dimensional system. There is a standard way of using systems of finite first return time maps to associate minimal (and essentially minimal) Cantor systems to 'minimal' (and 'essentially minimal') ordered Bratteli diagrams and *vice versa*; see [9, §3] (and [10, §2]). We give a couple of brief examples of the Bratteli diagram to dynamical system direction in Example 4.9. The procedure in the other direction is part of the proof of Proposition 4.11.

Examples 4.9. We provide a couple of examples illustrating the definitions above.

(1) Consider the ordered Bratteli diagram B in Figure 3. Assume the pattern shown in the diagram continues forever. As we can see, there is a linear order on the set of edges that share a range vertex. For example, there are three edges going into v, and those three edges are ordered 1 to 3. There is exactly one minimal infinite path and one maximal infinite path; these are the same path, shown in red. It is not too hard to see that the ordering on B is perfect. The dynamical system (X_B, h_B) is conjugate to the shift on the one-point compactification of the integers, where the path in red on the right corresponds to the point at ∞ and the path on the left (consisting of all edges labeled '2') corresponds to $0 \in \mathbb{Z}$. This is not the only possible diagram that yields such a Vershik system, but one that reflects a nice choice of a sequence of systems of finite first return time maps.

(2) We now show an perfectly ordered Bratteli diagram B such that (X_B, h_B) is conjugate to the dynamical system in Example 2.4(2), where we take Y to be $\mathbb{Z} \cup \{\infty\}$ and h' to be the shift. See Figure 4. The straight line down the middle is the

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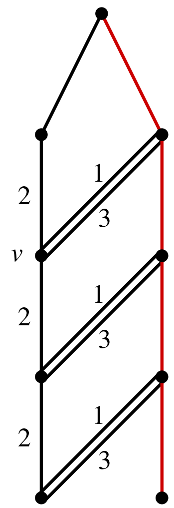

FIGURE 3. An illustration of Example 4.9(1). This is a Bratteli diagram associated to the shift on the one-point compactification of the integers.

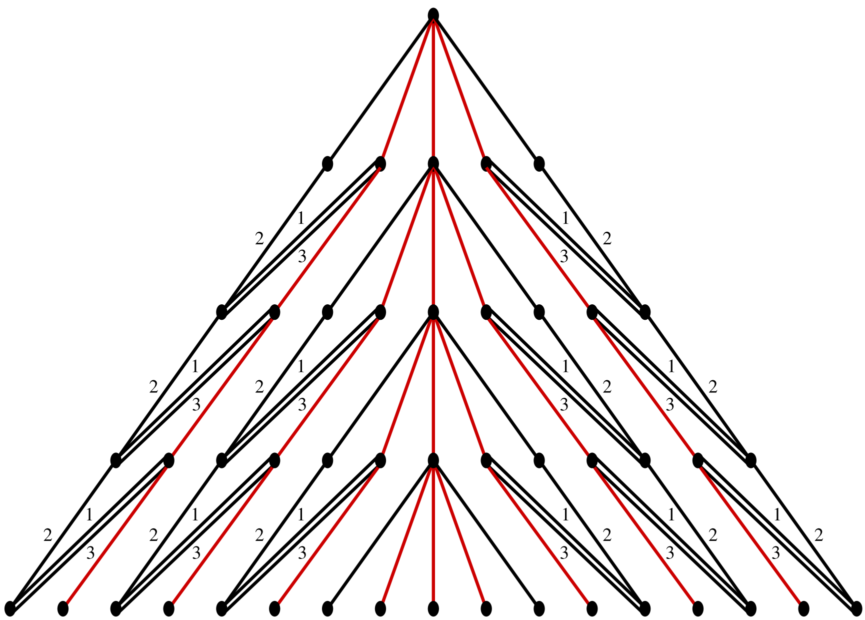

FIGURE 4. An illustration of Example 4.9(2). This is a Bratteli diagram associated to the fiberwise essentially minimal zero-dimensional system in Example 4.9(2).

infinite path corresponding to the crushed point in Figure 2. We can see a bunch of subtrees branching off that look like Figure 3; these correspond to the fibers, which are conjugate to the shift on the one-point compactification of the integers.

There are many minimal and maximal paths, and we color these with red. However, since the orbit closures are the fibers, each orbit closure has exactly one minimal and maximal path; these paths are the same, like in Figure 3. Therefore, the order is perfect, and the Vershik system does in fact turn out to be conjugate to the system in Example 2.4(2).

The purpose of the following lemma is used to build a sequence of systems of finite first return time maps with desirable properties in the proof of Proposition 4.11.

LEMMA 4.10. *Let (X, h) be a fiberwise essentially minimal zero-dimensional system, let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of X, and let $\mathcal{S} = (T, (X_t), (K_t), (Y_{t,k}), (J_{t,k}))$ be a system of finite first return time maps subordinate to $\mathcal{P}$ such that for each $t \in \{1, \ldots, T\}$, we have $\psi(X_t) \subset X_t$. Then there is a system $\mathcal{S}^0 = (T^0, (X_t^0), (K_t^0), (Y_{t,k}^0), (J_{t,k}^0))$ of finite first return time maps subordinate to $\mathcal{P}$ and a system $\mathcal{S}' = (T', (X_t'), (K_t'), (Y_{t,k}'), (J_{t,k}'))$ of finite first return time maps subordinate to $\mathcal{P}'$ such that:*

(1) *the partition $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}'$ and $\mathcal{P}_1(\mathcal{S}^0)$;*
(2) *the partition $\mathcal{P}_1(\mathcal{S}^0)$ is finer than $\mathcal{P}_1(\mathcal{S})$;*
(3) *we have $T^0 = T$ and for each $t \in \{1, \ldots, T\}$, we have $X_t^0 = X_t$;*
(4) *for each $t' \in \{1, \ldots, T'\}$, there is a $t \in \{1, \ldots, T^0\}$ and $k \in \{1, \ldots, K_t^0\}$ such that $X'_{t'} \subset Y^0_{t,k}$;*
(5) *for each $t' \in \{1, \ldots, T'\}$, there is a $t \in \{1, \ldots, T^0\}$ and $k \in \{1, \ldots, K_t^0\}$ such that $X'_{t'} \subset h^{J^0_{t,k}}(Y^0_{t,k})$;*
(6) *for each $t \in \{1, \ldots, T^0\}$ and each $k \in \{1, \ldots, K^0\}$, there is a $t' \in \{1, \ldots, T'\}$ such that $Y^0_{t,k} \subset \bigcup_{j \in \mathbb{Z}} h^j(X'_{t'})$.*

Proof. We first construct $\mathcal{S}'$ and then use it to modify $\mathcal{S}$ to obtain $\mathcal{S}^0$. By applying [**11**, Lemma 4.13], we may assume that $\mathcal{S}$ satisfies its conclusions; in particular, for all $t \in \{1, \ldots, T\}$, we have $\psi(X_t) \subset Y_{t,1}$, and the partitions $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S})$ are finer than $\mathcal{P}$. Let $\mathcal{P}''$ be a partition finer than $\mathcal{P}'$, $\mathcal{P}_1(\mathcal{S})$, and $\mathcal{P}_2(\mathcal{S})$. Then apply [**11**, Lemma 3.2] to obtain a system $\mathcal{S}' = (T', (X_t'), (K_t'), (Y_{t,k}'), (J_{t,k}'))$ of finite first return time maps subordinate to $\mathcal{P}''$ such that for all $t' \in \{1, \ldots, T'\}$, there is a $t \in \{1, \ldots, T\}$ such that $X'_{t'} \subset X_t$. Since $\mathcal{P}''$ is finer than $\mathcal{P}'$, $\mathcal{S}'$ is also subordinate to $\mathcal{P}'$. By applying [**11**, Proposition 1.13], we may assume that $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}''$.

Let $t' \in \{1, \ldots, T'\}$ and let $t \in \{1, \ldots, T\}$ be such that $X'_{t'} \subset X_t$. Since $\mathcal{P}''$ is finer than $\mathcal{P}_1(\mathcal{S})$, there is some $k \in \{1, \ldots, K_t\}$ such that

$$X'_{t'} \subset Y_{t,k}. \tag{4.1}$$

Since $\mathcal{P}''$ is finer than $\mathcal{P}_2(\mathcal{S})$, there is some $l \in \{1, \ldots, K_t\}$ such that

$$X'_{t'} \subset h^{J_{t,l}}(Y_{t,l}). \tag{4.2}$$

Define $T^0 = T$ and for each $t \in \{1, \ldots, T\}$, define $X_t^0 = X_t$ (note that after finishing this construction, this verifies conclusion (c) of the lemma). Let $t \in \{1, \ldots, T\}$ and let $\{s(1), \ldots, s(N_t)\}$ be the set of all $t' \in \{1, \ldots, T'\}$ such that $X_{t'} \subset X_t$. For each $n \in \{1, \ldots, N_t\}$, let $\{a(n, 1), \ldots, a(n, C_n)\}$ be the set of all $k \in \{1, \ldots, K_t\}$ such that $Y_{t,k} \cap \bigcup_{j\in\mathbb{Z}} h^j(X'_{s(n)}) \neq \varnothing$. Define $K_t^0 = \sum_{n=1}^{N_t} C_n$ and define $C_0 = 0$. Let $k \in \{1, \ldots, K_t^0\}$ and let $n \in \{1, \ldots, N_t\}$, and $c \in \{1, \ldots, C_n\}$ be such that $k = C_{n-1} + c$. Then define

$$Y_{t,k}^0 = Y_{t,a(n,c)} \cap \bigcup_{j\in\mathbb{Z}} h^j(X'_{s(n)}) \tag{4.3}$$

and define $J_{t,k}^0 = J_{t,a(n,c)}$. It is routine to verify that $\mathcal{S}^0 = (T^0, (X_t^0), (K_t^0), (Y_{t,k}^0), (J_{t,k}^0))$ is a system of finite first return time maps subordinate to $\mathcal{P}$. By applying [11, Proposition 1.9], we may assume that $\mathcal{P}_1(\mathcal{S}^0)$ is finer than $\mathcal{P}$ (note that this proves conclusion (2) of the lemma).

We now verify the conclusions of the lemma. Conclusions (1), (2), and (3) have already been verified. Conclusion (4) follows from equation (4.1) and from the fact that $\mathcal{P}_1(\mathcal{S}^0)$ is finer than $\mathcal{P}_1(\mathcal{S})$. Conclusion (5) follows from equation (4.1) and the fact that that $\mathcal{P}_2(\mathcal{S}^0)$ is finer than $\mathcal{P}_2(\mathcal{S})$. Conclusion (6) is shown by equation (4.3). This proves the lemma. □

The following proposition is the key to adapting the proof of [9, Theorem 2.1] to extend from the minimal case to our case (Theorem 5.2). Using Lemma 4.10, we construct an ordered Bratteli diagram (called a 'Bratteli–Vershik–Kakutani model' in the literature) using a special sequence of partitions of X (this is a special type of what is referred to in the literature as a sequence of 'Kakutani–Rokhlin' partitions). In the minimal case, you can construct an ordered Bratteli diagram $B = (V, E, \leq)$ whose Vershik system is conjugate to a minimal system such that for any vertex v in V_n, there are multiple edges from every vertex in V_{n-1} to v (see [9, §3]).

PROPOSITION 4.11. *Let (X, h, Z) be a fiberwise essentially minimal zero-dimensional system. There is an ordered Bratteli diagram $B = (V, E, \leq)$ with a perfect ordering such that:*

(1) *the system $(X_B, h_B, X_{B,\min})$ is conjugate to (X, h, Z);*
(2) *for each v in $V_{\min}$ (or $V_{\max}$), there is a v' in $V_{\min}$ ($V_{\max}$, respectively) and an edge e in $E_{\min}$ ($E_{\max}$, respectively) such that $s(e) = v$ and $r(e) = v'$;*
(3) *for each v in $V_{\min}$ (or $V_{\max}$) and each e in $E_{\min}$ ($E_{\max}$, respectively) with $r(e) \in R(v)$ satisfies $s(e) = v$;*
(4) *for each v in $V_{\min}$ (or $V_{\max}$) and each $m \in \mathbb{Z}_{>0}$, $R^m(v) = (R^m \circ S^m \circ R^m)(v)$.*

Proof. Let Z and ψ correspond to (X, h) as in Definition 2.2, and let $(\mathcal{P}^{(n)})$ be a generating sequence of partitions of X. We will construct an ordered Bratteli diagram $B = (V, E, \leq)$ such that (X_B, h_B) is conjugate to (X, h) via a map $F : X \to X_B$ that satisfies $F(Z) = X_{B,\min}$.

First, we construct a sequence $(\mathcal{S}^{(n)})$ of finite first return time maps subordinate to $(\mathcal{P}^{(n)})$. First, let $\mathcal{S}^{(1)\prime}$ be any system of finite first return time maps subordinate to $\mathcal{P}^{(1)}$ such that $\mathcal{P}_1(\mathcal{S}^{(1)\prime})$ is finer than $\mathcal{P}^{(1)}$ (such a system exists by [11, Proposition 1.13]). We construct the other systems inductively. For each $n \in \mathbb{Z}_{>0}$, we apply Lemma 4.10 with $\mathcal{S}^{(n)\prime}$ in place of $\mathcal{S}$, $\mathcal{P}^{(n)}$ in place of $\mathcal{P}$, $\mathcal{P}^{(n+1)}$ in place of $\mathcal{P}'$, and get $\mathcal{S}^{(n)}$ (that is, $\mathcal{S}^0$ in the lemma) and $\mathcal{S}^{(n+1)\prime}$ (that is, $\mathcal{S}'$ in the lemma) satisfying the conclusions of the lemma. Thus, to construct the sequence of systems, we only need to define $\mathcal{S}^{(0)}$ by $T^{(0)} = 1$, $X_1^{(0)} = X$, $K_1^{(0)} = 1$, $Y_{1,1}^{(0)} = 1$, and $J_{1,1}^{(0)} = 1$.

Now we begin to define B. For each $n \in \mathbb{Z}_{\geq 0}$, define

$$V_n = \{(n, t, k) \mid t \in \{1, \ldots, T^{(n)}\} \text{ and } k \in \{1, \ldots, K_t^{(n)}\}\}.$$

The set of edges from $(n, t, k) \in V_n$ to $(n+1, t', k') \in V_{n+1}$ is the set of all $(n+1, t', k', j)$ such that $h^j(Y_{t',k'}^{(n+1)}) \subset Y_{t,k}^{(n)}$. Note that this is well defined; by assumption, $h^j(Y_{t',k'}^{(n+1)})$ is a subset of an element of $\mathcal{P}_1(\mathcal{S}^{(n)})$, so we do not need to include t and k in the tuple defining this edge. We define an order on the edges $r^{-1}((n, t, k))$ by $(n, t, k, j_1) \leq (n, t, k, j_2)$ if $j_1 < j_2$.

We now construct the orbit map $F : X \to X_B$. Let $x \in X$. Then for each $n \in \mathbb{Z}_{>0}$, there is precisely one $t \in \{1, \ldots, T^{(n)}\}$, one $k \in \{1, \ldots, K_t^{(n)}\}$, and one $j \in \{0, \ldots, J_{t,k}^{(n)}\}$ such that $x \in h^j(Y_{t,k}^{(n)})$. If $x \in h^j(Y_{t,k}^{(n)}) \cap h^{j'}(Y_{t',k'}^{(n+1)})$, then $j' \geq j$, since otherwise we would have $h^{j-j'}(Y_{t,k}^{(n)}) \subset X_t^{(n+1)}$ (this follows from Lemma 4.10(3) and (4)), which is not possible since by definition, $h^i(Y_{t,k}^{(n)}) \cap X_t^{(n+1)} = \varnothing$ for $i \in \{1, \ldots, J_{t,k}^{(n)} - 1\}$. This, combined with the fact that $\mathcal{P}_1(\mathcal{S}^{(n+1)})$ is finer than $\mathcal{P}_1(\mathcal{S}^{(n)})$, tells us that $h^{j'-j}(Y_{t',k'}^{(n+1)}) \subset Y_{t,k}^{(n)}$, and therefore there is an edge from (n, t, k) to $(n+1, t', k')$; namely, $(n+1, t', k', j'-j)$. Thus, this gives us an infinite path in X_B associated to x. We define F by sending x to this infinite path.

We now show that F is injective. Suppose $x, x' \in X$ and $F(x) = F(x') = (e_1, e_2, \ldots)$ where we write $e_n = (n, t_n, k_n, j_n)$ for $n \in \mathbb{Z}_{>0}$. By definition, it is clear that for each $n \in \mathbb{Z}_{>0}$, there are $i_n, i_n' \in \{0, \ldots, J_{t_n,k_n}^{(n)} - 1\}$ such that $x \in h^{i_n}(Y_{t_n,k_n}^{(n)})$ and $x' \in h^{i_n'}(Y_{t_n,k_n}^{(n)})$. First, notice that $i_1 = j_1$ and $i_1' = j_1$ by definition of F. Then, by definition of F, we have $j_2 = i_2 - j_1$ and $j_2 = i_2' - j_2$; in particular, $i_2 = i_2'$. Proceeding like this, we see that $i_n = i_n'$ for all $n \in \mathbb{Z}_{>0}$. Since $(\mathcal{P}^{(n)})$ is a generating sequence of partitions, so is $(\mathcal{P}_1(\mathcal{S}^{(n)}))$, and therefore $\bigcap_{n=0}^{\infty} h^{i_n}(Y_{t_n,k_n}^{(n)})$ contains at most one element of X. Thus, $x = x'$, and therefore F is injective.

Next, we show that F is surjective. Let $e = (e_1, e_2, \ldots) \in X_B$ and write $e_n = (n, t_n, k_n, j_n)$ for $n \in \mathbb{Z}_{>0}$. We construct a sequence (i_n) with $i_n \in \{0, \ldots, J_{t_n,k_n}^{(n)} - 1\}$ for all $n \in \mathbb{Z}_{>0}$ such that $\bigcap_{n=0}^{\infty} h^{i_n}(Y_{t_n,k_n}^{(n)})$ is non-empty and contains the element of X that F maps to e. First, let $i_1 = j_1$. Then, for all $n \in \mathbb{Z}_{>1}$, let $i_n = j_n + i_{n-1}$ (note that this can be rewritten as $i_n = \sum_{k=1}^{n} j_k$). The claim now follows from the definition of B, and F is therefore surjective.

So far, we have shown that F is bijective. We now show that F is a homeomorphism. Let $U(e_1, \ldots, e_N)$ be an element of the basis of the topology of X_B. For each $n \in \{1, \ldots, N\}$, write $e_n = (n, t_n, k_n, j_n)$. For each $n \in \{1, \ldots, N\}$, write $j_n' = \sum_{k=1}^{n} j_k$. We claim that if $x \in h^{j_N'}(Y_{t_N,k_N}^{(N)})$, then $F(x) \in U(e_1, \ldots, e_N)$. So let $x \in h^{j_N'}(Y_{t_N,k_N}^{(N)})$. First notice that by definition of e_N,

$$h^{j_N}(Y_{t_N,k_N}^{(N)}) \subset Y_{t_{N-1},k_{N-1}}^{(N-1)}.$$

Thus,

$$h^{j'_N}(Y^{(N)}_{t_N,k_N}) \subset h^{j'_{N-1}}(Y^{(N-1)}_{t_{N-1},k_{N-1}}),$$

since $j_N + j'_{N-1} = j'_N$. Similarly, we have

$$h^{j_{N-1}}(Y^{(N-1)}_{t_{N-1},k_{N-1}}) \subset Y^{(N-2)}_{t_{N-2},k_{N-2}}.$$

Thus, for every $n \in \{1, \ldots, N\}$, we have

$$x \in h^{j'_n}(Y^{(N)}_{t_n,k_n}),$$

and so since $j'_n - j'_{n-1} = j_n$, the nth edge of $F(x)$ is indeed e_n, and $F(x) \in U(e_1, \ldots, e_N)$ as desired. Next, we claim that if $x \in X$ satisfies $F(x) \in U(e_1, \ldots, e_N)$, then $x \in h^{j'_N}(Y^{(N)}_{t_N,k_N})$. So let $x \in X$ satisfy $F(x) \in U(e_1, \ldots, e_N)$. Then for each $n \in \{1, \ldots, N\}$, $x \in h^{i_n}(J^{(n)}_{t_n,k_n})$ for some $i_n \in \{0, \ldots, J^{(n)}_{t_n,k_n} - 1\}$. It is clear that $i_1 = j_1$. Then, notice that i_2 is such that $j_2 = i_2 - j_1$, and so $i_2 = j_1 + j_2 = j'_2$. Repeating this process inductively, we see that $i_N = j'_N$, and so $x \in h^{j'_N}(Y^{(N)}_{t_N,k_N})$ as desired. Altogether, this shows that F is a homeomorphism.

If $x \in Z$, then there are sequences of integers (t_n) and (k_n) such that $x \in \bigcap_{n=0}^{\infty} Y^{(n)}_{t_n,k_n}$. By definition of the order on B, this means that $F(x) \in X_{B,\min}$. Conversely, suppose $x \in X$ satisfies $F(x) \in X_{B,\min}$. Write $F(x) = (e_1, e_2, \ldots)$ and for $n \in \mathbb{Z}_{>0}$, write $e_n = (n, t_n, k_n, i_n)$. Since e_n is minimal, i_n is the minimal element of $\{0, \ldots, J^{(n)}_{t_n,k_n} - 1\}$ such that $h^{i_n}(Y^{(n)}_{t_n,k_n}) \subset X^{(n-1)}_{t_{n-1}}$. However, since $X^{(n)}_{t_n} \subset X^{(n-1)}_{t_{n-1}}$, we have $i_n = 0$. Hence, $x \in \bigcap_{n=0}^{\infty} Y^{(n)}_{t_n,k_n}$. Thus, $F(Z) = X_{B,\min}$. Also notice that if $x \in h^{-1}(Z)$, there are sequences of integers (t_n) and (k_n) such that $x \in \bigcap_{n=0}^{\infty} h^{J^{(n)}_{t_n,k_n}-1}(Y^{(n)}_{t_n,k_n})$. By definition of the order on B, this means that $F(x) \in X_{B,\max}$. Similarly, the converse holds, and so $F(h^{-1}(Z)) = X_{B,\max}$.

We now show that $(F \circ h)|_{X\setminus h^{-1}(Z)} = (\widetilde{h}_B \circ F)|_{X\setminus h^{-1}(Z)}$. Let $x \in X \setminus h^{-1}(Z)$. For each $n \in \mathbb{Z}_{>0}$, let $t_n \in \{1, \ldots, T^{(n)}\}$, let $k_n \in \{1, \ldots, K^{(n)}_{t_n}\}$, and let $j_n \in \{0, \ldots, K^{(n)}_{t_n} - 1\}$ satisfy $x \in h^{j_n}(Y^{(n)}_{t_n,k_n})$. Since $x \notin h^{-1}(Z)$, there is some smallest $N \in \mathbb{Z}_{>0}$ such that $j_N \neq J^{(N)}_{t_N,k_N} - 1$. We have

$$h^{j_N+1}(Y^{(N)}_{t_N,k_N}) \subset J^{(N-1)}_{t_{N-1},k_{N-1}}(Y^{(N-1)}_{t_{N-1},k_{N-1}})$$

and so by Lemma 4.10(4), there is a $k'_{N-1} \in \{1, \ldots, K^{(N-1)}_{t_{N-1}}\}$ such that

$$h^{j_N+1}(Y^{(N)}_{t_N,k_N}) \subset Y^{(N-1)}_{t_{N-1},k'_{N-1}}. \tag{4.4}$$

Inductively, we can find for each $n \in \{1, \ldots, N-2\}$ a $k'_n \in \{1, \ldots, K^{(n)}_{t_n}\}$ such that $Y^{(n+1)}_{t'_{n+1},k'_{n+1}} \subset Y^{(n)}_{t'_n,k'_n}$. Since $j_N + 1 < J^{(N)}_{t_N,k_N}$ and since for each $n \in \mathbb{Z}_{>N}$ we have $X^{(n)}_{t_n} \subset X^{(N)}_{t_N}$, it follows that $j_n + 1 < J^{(N)}_{t_N,k_N}$ as well. So let $k'_n = k_n$ and let $j'_n = 0$ for $n \in \{1, \ldots, N-1\}$, and let $j'_n = j_n + 1$ for $n \in \mathbb{Z}_{\geq N}$. Then for each $n \in \mathbb{Z}_{>0}$, $h(x) \in h^{j'_n}(Y^{(n)}_{t_n,k'_n})$.

Now write $F(x) = (e_1, e_2, \ldots)$ and $\widetilde{h}_B(F(x)) = (e'_1, e'_2, \ldots)$ and for each $n \in \mathbb{Z}_{>0}$, write $e_n = (n, s_n, l_n, i_n)$ and $e'_n = (n, s'_n, l'_n, i'_n)$. By definition of F, for all $n \in \mathbb{Z}_{>0}$, we have $s_n = t_n$, $l_n = k_n$, and $i_n = j_n - j_{n-1}$ where $j_0 = 0$. We also see that N is the smallest element of $\mathbb{Z}_{>0}$ such that $e_N \notin E_{\max}$, so e'_N is the successor of e_N, $(e'_1, \ldots, e'_{N-1})$ is the minimal path such that $r(e'_{N-1}) = s(e'_N)$, and $e'_n = e_n$ for all $n \in \mathbb{Z}_{>N}$. In particular, we see that $s'_n = t_n$ for all $n \in \mathbb{Z}_{>0}$, $l'_n = k_n$ for all $n \in \mathbb{Z}_{\geq N}$, $i'_n = 0$ for $n \in \{1, \ldots, N-1\}$, and $i'_n = i_n = j_n - j_{n-1}$ for all $n \in \mathbb{Z}_{>N}$. Observe that i'_N is the smallest integer greater than i_N such that $h^{i'_N}(Y^{(N)}_{t_N,k_N}) \subset Y^{(N-1)}_{t_{N-1},l'_{N-1}}$. Thus, this combined with

$$h^{i_N}(Y^{(N)}_{t_N,k_N}) \subset Y^{(N-1)}_{t_{N-1},k_{N-1}} \tag{4.5}$$

tells us

$$\begin{aligned} i'_N &= i_N + J^{(N-1)}_{t_{N-1},k_{N-1}} \\ &= j_N - j_{N-1} + J^{(N-1)}_{t_{N-1},k_{N-1}} \\ &= j_N - (J^{(N-1)}_{t_{N-1},k_{N-1}} - 1) + J^{(N-1)}_{t_{N-1},k_{N-1}} \\ &= j_N + 1 \\ &= j'_N. \end{aligned}$$

From equations (4.4) and (4.5), we see $l'_{N-1} = k'_{N-1}$. Similarly, for $n \in \{2, \ldots, N-1\}$, since $i'_n = j'_n = 0$, we have $l'_{n-1} = k'_{n-1}$. Write $F(h(x)) = (e''_1, e''_2, \ldots)$ and for each $n \in \mathbb{Z}_{>0}$, write $e''_n = (n, s''_n, l''_n, i''_n)$. For each $n \in \mathbb{Z}_{>0}$, we have $s''_n = t'_n = t_n$ and $l''_n = k'_n$. For $n \in \{1, \ldots, N-1\}$, we have $i''_n = i'_n = 0$. We also have $i''_N = j'_N - j'_{N-1} = j'_N - 0 = i'_N$. For $n \in \mathbb{Z}_{>N}$, we have

$$\begin{aligned} i''_n = j'_n - j'_{n-1} &= j_n + 1 - (j_{n-1} - 1) \\ &= j_n - j_{n-1} \\ &= i_n \\ &= i'_n. \end{aligned}$$

Altogether, we see $F(h(x)) = \widetilde{h}_B(F(x))$, and so

$$(F \circ h)|_{X \setminus h^{-1}(Z)} = (\widetilde{h}_B \circ F)|_{X \setminus h^{-1}(Z)}. \tag{4.6}$$

We now show that the order on B is perfect. For each $x \in X$, z is in the minimal set of the essentially minimal zero-dimensional system $(\psi^{-1}(Z), h|_{\psi^{-1}(Z)})$, $\overline{\operatorname{orb}}(x)$ contains exactly one element of Z and exactly one element of $h^{-1}(Z)$. Now, equation (4.6) combined with $F(Z) = X_{B,\min}$ and $F(h^{-1}(Z)) = X_{B,\max}$ tells us that the ordering on B is perfect. It is now clear that $F \circ h = h_B \circ F$. This proves conclusion (1) of the proposition.

Before we prove the rest, we first prove two claims that will be used a few times in the proof.

Claim $(*)$: let $n \in \mathbb{Z}_{>0}$, $t_n \in \{1, \ldots, T^{(n)}\}$, $k_n \in \{1, \ldots, K^{(n)}_{t_n}\}$, and $t_{n+1} \in \{1, \ldots, T^{(n+1)}\}$. If $X^{(n+1)}_{t_{n+1}} \subset h^{J^{(n)}_{t_n,k_n}}(Y^{(n)}_{t_n,k_n})$, then for any $k_{n+1} \in \{1, \ldots, K^{(n+1)}_{t_{n+1}}\}$, there is an

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$i_{n+1} \in \{1, \ldots, J^{(n+1)}_{t_{n+1},k_{n+1}}\}$ such that $e = (n+1, t_{n+1}, k_{n+1}, i_{n+1})$ is a maximal edge with $s(e) = (n, t_n, k_n)$.

We now prove claim $(*)$. Let $k_{n+1} \in \{1, \ldots, K^{(n+1)}_{t_{n+1}}\}$. Then

$$h^{J^{(n+1)}_{t_{n+1},k_{n+1}}}(Y_{t_{n+1},k_{n+1}}) \subset X_{t^{(n+1)}_{n+1}} \subset h^{J^{(n)}_{t_n,k_n}}(Y^{(n)}_{t_n,k_n}),$$

so we have

$$h^{J^{(n+1)}_{t_{n+1},k_{n+1}} - J^{(n)}_{t_n,k_n}}(Y_{t_{n+1},k_{n+1}}) \subset Y^{(n)}_{t_n,k_n}.$$

Set $i_{n+1} = J^{(n+1)}_{t_{n+1},k_{n+1}} - J^{(n)}_{t_n,k_n}$. Let $j \in \{1, \ldots, J^{(n)}_{t_n,k_n} - 1\}$. Then

$$h^{i_{n+1}+j}(Y^{(n+1)}_{t_{n+1},k_{n+1}}) \subset h^j(Y^{(n)}_{t_n,k_n}),$$

and since $h^j(Y^{(n)}_{t_n,k_n}) \cap (\bigsqcup_{t=1}^{T^{(n)}} X^{(n)}_t) = \varnothing$, we indeed see that $e = (n+1, t_{n+1}, k_{n+1}, i_{n+1}) \in E_{\max}$. This proves claim $(*)$.

Claim $(**)$: if there is a $k_{n+1} \in \{1, \ldots, K^{(n+1)}_{t_{n+1}}\}$ and a $i \in \{1, \ldots, J^{(n+1)}_{t_{n+1},k_{n+1}}\}$ such that $e = (n+1, t_{n+1}, k_{n+1}, i_{n+1})$ is a maximal edge with $s(e) = (n, t_n, k_n)$, then $X^{(n+1)}_{t_{n+1}} \subset h^{J^{(n)}_{t_n,k_n}}(Y^{(n)}_{t_n,k_n})$.

We now prove claim $(**)$. Since $s(e) = (n, t_n, k_n)$, we have

$$h^{i_{n+1}}(Y^{(n+1)}_{t_{n+1},k_{n+1}}) \subset Y^{(n)}_{t_n,k_n}.$$

Since e is maximal, there is no $j \in \{i_{n+1}+1, \ldots, J^{(n+1)}_{t_{n+1},k_{n+1}} - 1\}$ with $h^j(Y^{(n+1)}_{t_{n+1},k_{n+1}}) \subset X^{(n)}_{t_n,k_n}$. However, notice that

$$h^{i_{n+1}+J^{(n)}_{t_n,k_n}}(Y^{(n+1)}_{t_{n+1},k_{n+1}}) \subset h^{J^{(n)}_{t_n,k_n}}(Y^{(n)}_{t_n,k_n}) \subset X^{(n)}_{t_n,k_n}.$$

Thus, we must have $i_{n+1} + J^{(n)}_{t_n,k_n} = J^{(n+1)}_{t_{n+1},k_{n+1}}$. Thus, $X^{(n+1)}_{t_{n+1}} \cap h^{J^{(n)}_{t_n,k_n}}(Y^{(n)}_{t_n,k_n}) \neq \varnothing$, and so by Lemma 4.10(5), we actually have $X^{(n+1)}_{t_{n+1}} \subset h^{J^{(n)}_{t_n,k_n}}(Y^{(n)}_{t_n,k_n})$. This proves claim $(**)$.

We now prove conclusion (2) of the proposition. Let $v \in V_{\min}$. Write $v = (n, t_n, k_n)$. Since $v \in V_{\min}$, there is some $v'' \in V_{n+1}$ and some $e' \in E_{\min}$ with $s(e') = v$ and $r(e') = v''$. Write $v'' = (n+1, t_{n+1}, k_{n+1})$ and then $e' = (n+1, t_{n+1}, k_{n+1}, i_{n+1})$. By Lemma 4.10(4), since $e' \in E_{\min}$, we have $i_{n+1} = 0$, so $Y^{(n+1)}_{t_{n+1},k_{n+1}} \subset Y^{(n)}_{t_n,k_n}$. Again by Lemma 4.10(4), we have

$$X^{(n+1)}_{t_{n+1}} \subset Y^{(n)}_{t_n,k_n}. \tag{4.7}$$

Now, let $t_{n+2} \in \{1, \ldots, T^{(n+2)}\}$ satisfy

$$X^{(n+2)}_{t_{n+2}} \cap \left(\bigcup_{j\in\mathbb{Z}} h^j(X^{(n+1)}_{t_{n+1}}) \right) \neq \varnothing.$$

By Lemma 4.10(4), there is actually a $k'_{n+1} \in \{1, \ldots, K^{(n+1)}_{t_{n+1}}\}$ such that

$$X^{(n+2)}_{t_{n+2}} \subset Y^{(n+1)}_{t_{n+1},k'_{n+1}}.$$

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So for any $k_{n+2} \in \{1, \ldots, K^{(n+2)}_{t_{n+2}}\}$, we have a minimal edge $e'' = (n+2, t_{n+2}, k_{n+2}, 0)$ with $s(e'') = v' = (n+1, t_{n+1}, k'_{n+1})$. Thus, $v' \in V_{\min}$. By equation (4.7), $Y^{(n+1)}_{t_{n+1},k'_{n+1}} \subset Y^{(n)}_{t_n,k_n}$, and so there is a minimal edge $e = (n+1, t_{n+1}, k'_{n+1}, 0)$ with $s(e) = v$ and $r(e) = v'$.

Now let $v \in V_{\max}$ and write $v = (n, t_n, k_n)$. Since $v \in V_{\max}$, there is some $v'' \in V_{n+1}$ and some $e' \in E_{\max}$ with $s(e') = V$ and $r(e') = v''$. Write $v'' = (n+1, t_{n+1}, k_{n+1})$ and then $e' = (n+1, t_{n+1}, k_{n+1}, i_{n+1})$. By claim $(**)$, we have

$$X^{(n+1)}_{t_{n+1}} \subset h^{J^{(n)}_{t_n,k_n}}(Y^{(n)}_{t_n,k_n}).$$

Now, let $t_{n+2} \in \{1, \ldots, T^{(n+2)}\}$ satisfy

$$X^{(n+2)}_{t_{n+2}} \cap \bigg(\bigcup_{j\in\mathbb{Z}} h^j(X^{(n+1)}_{t_{n+1}})\bigg) \neq \varnothing.$$

By Lemma 4.10(5), there is actually a $k'_{n+1} \in \{1, \ldots, K^{(n+1)}_{t_{n+1}}\}$ such that

$$X^{(n+2)}_{t_{n+2}} \subset h^{J^{(n+1)}_{t_{n+1},k'_{n+1}}}(Y^{(n+1)}_{t_{n+1},k'_{n+1}}).$$

By claim $(*)$, $v' = (n+1, t_{n+1}, k'_{n+1}) \in V_{\max}$. By claim $(*)$, there is $e \in E_{\max}$ with $s(e) = v$ and $r(e) = v'$. This completes the proof of conclusion (2).

We now prove conclusion (3) of the proposition. Let $v \in V_{\min}$ and $e \in E_{\min}$ satisfy $r(e) \in R(v)$. Write $v = (n, t_n, k_n)$ and $e = (n+1, t_{n+1}, k_{n+1}, i_{n+1})$. Since $r(e) \in R(v)$, there is an edge $e' = (n+1, t_{n+1}, k_{n+1}, i'_{n+1})$ with $s(e') = v$, which tells us there is $j_{n+1} \in \{0, \ldots, J^{(n+1)}_{t_{n+1},k_{n+1}} - 1\}$ such that $h^{j_{n+1}}(Y_{t_{n+1},k_{n+1}}) \subset Y^{(n)}_{t_n,k_n}$. By Lemma 4.10(6), $Y^{(n)}_{t_n,k_n} \subset \bigcup_{j\in\mathbb{Z}} h^j(X^{(n+1)}_{t_{n+1}})$. Since $v \in V_{\min}$, there is some edge $e'' = (n+1, t_{n+1}, k''_{n+1}, i''_{n+1}) \in E_{\min}$ with $s(e) = v$. By Lemma 4.10(4), since $e'' \in E_{\min}$, we have $i''_{n+1} = 0$, so $Y^{(n+1)}_{t_{n+1},k''_{n+1}} \subset Y^{(n)}_{t_n,k_n}$. Again by Lemma 4.10(4), we actually have

$$X^{(n+1)}_{t_{n+1}} \subset Y^{(n)}_{t_n,k_n}. \tag{4.8}$$

Now, since $e \in E_{\min}$, by Lemma 4.10(4), we have $i_{n+1} = 0$. But by (4.8), we must have $Y^{(n+1)}_{t_{n+1},k_{n+1}} \subset Y^{(n)}_{t_n,k_n}$, which means that $s(e) = v$.

Let $v \in V_{\max}$ and $e \in E_{\max}$ satisfy $r(e) \in R(v)$. Write $v = (n, t_n, k_n)$ and $e = (n+1, t_{n+1}, k_{n+1}, i_{n+1})$. Since $r(e) \in R(v)$, there is an edge $e' = (n+1, t_{n+1}, k_{n+1}, i'_{n+1})$ with $s(e') = v$, which tells us there is $j_{n+1} \in \{0, \ldots, J^{(n+1)}_{t_{n+1},k_{n+1}} - 1\}$ such that $h^{j_{n+1}}(Y^{(n+1)}_{t_{n+1},k_{n+1}}) \subset Y^{(n)}_{t_n,k_n}$. By Lemma 4.10(6), $Y^{(n)}_{t_n,k_n} \subset \bigcup_{j\in\mathbb{Z}} h^j(X^{(n+1)}_{t_{n+1}})$. Since $v \in V_{\max}$, there is some edge $e'' = (n+1, t_{n+1}, k''_{n+1}, i''_{n+1}) \in E_{\max}$ with $s(e) = v$. By claim $(**)$, we have

$$X^{(n+1)}_{t_{n+1}} \subset h^{J^{(n)}_{t_n,k_n}}(Y^{(n)}_{t_n,k_n}).$$

By claim $(*)$, $s(e) = v$. This completes the proof of conclusion (3).

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Now we prove conclusion (4) of this lemma. Let $v \in V_{\min}$, let $m \in \mathbb{Z}_{>0}$, and write $v = (n, t_n, k_n)$. It is clear that $v \in (S^m \circ R^m)(v)$, and so

$$R^m(v) \subset (R^m \circ S^m \circ R^m)(v).$$

Now, let $w = (n+m, t_{n+m}, k_{n+m}) \in R^m(v)$. By Lemma 4.10(4) (applied m times), we have

$$\bigcup_{j\in\mathbb{Z}} h^j(X^{(n+m)}_{t_{n+m}}) \subset \bigcup_{j\in\mathbb{Z}} h^j(X^{(n)}_{t_n}).$$

So every element of $(S^m \circ R^m)(v)$ has the form (n, t_n, k'_n) for some $k'_n \in \{1, \ldots, K^{(n)}_{t_n}\}$. Let $w' = (n+m, t'_{n+m}, k'_{n+m}) \in (R^m \circ S^m \circ R^m)(v)$, meaning there is an path $(e_1, \ldots, e_m)$ with $s(e_1) = (n, t_n, k'_n)$ for some $k'_n \in \{1, \ldots, K^{(n)}_{t_n}\}$ and $r(e_m) = w'$. Write $e_m = (n+m, t'_{n+m}, k'_{n+m}, i'_{n+m})$. This means that $h^{i'_{n+m}}(Y^{(n+m)}_{t'_{n+m},k'_{n+m}}) \subset Y^{(n)}_{t_n,k'_n}$. However, then by Lemma 4.10(4) (applied m times), we must have $Y^{(n+m)}_{t'_{n+m},k'_{n+m}} \subset Y^{(n)}_{t_n,k_n}$. Therefore, $w' \in R^m(v)$ (by a minimal path), as desired. An identical argument using Lemma 4.10(5) in place of conclusion (4) shows that this equation also holds when $v \in V_{\max}$. This proves conclusion (4) of this proposition and therefore finishes the proof of the proposition. □

Remark 4.12. We describe and illustrate what it means for a Bratteli diagram to satisfy the conclusions of Proposition 4.11. As you read the proof of Theorem 5.2, keep these descriptions and illustrations in mind as Proposition 4.11 gets cited many times. We adopt the notation of the proposition. For simplicity, we discuss the minimal vertex/edge case as the maximal case is analogous.

(1) Conclusion (1), which says that that the system $(X_B, h_B, X_{B,\min})$ is conjugate to (X, h, Z), just means that the Bratteli diagram's dynamics agrees with the dynamics of the original system. This tells us that we have created a 'Bratteli–Vershik–Kakutani' model of the system.

(2) What it means for $v \in V_{\min}$ is that there is some minimal edge e' with $s(e') = v$. What conclusion (2) guarantees is that at least one of those minimal edges, e, comes from another minimal vertex, v'. See Figure 5 for an illustration of a situation that violates this. This property tells us that given any $v \in V_{\min}$, there is an infinite minimal path containing v.

(3) What conclusion (3) tells us is that for every minimal vertex v, everything in its range $R(v)$ has the property that the source of its minimal edge is v. See Figure 6 for an illustration of a situation that violates this. One of the consequences of this is that no two minimal vertices share anything in their range.

(4) For $m = 1$, what conclusion (4) tells us is that the range of a vertex is either contained in the range of a minimal vertex or disjoint from it. See Figure 7 for an illustration of a situation that violates this. For larger values of m, this tells us that telescopings of the Bratteli diagram still have this property (see Proposition 4.13).

We show in the following proposition that the class of Bratteli diagrams satisfying the conclusions of Proposition 4.11 is closed under telescoping. This is important, as we will require some amount of telescoping of diagrams.

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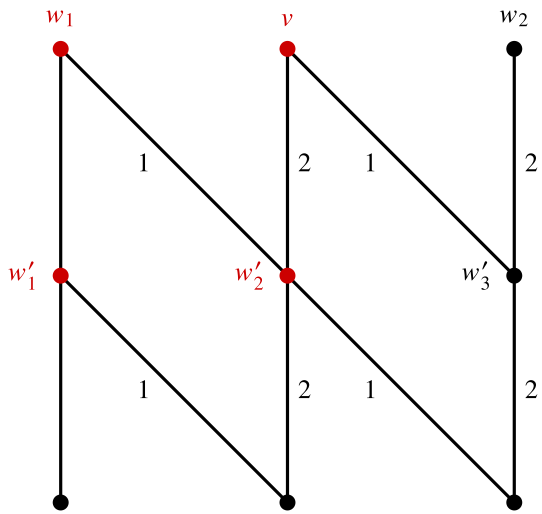

FIGURE 5. A partial ordered Bratteli diagram illustrating an example of a situation that violates conclusion (2) of Proposition 4.11. In this picture, $v \in V_{\min}$ since there is a minimal edge with a range of w'_3 and a source of v. However, there is no minimal edge with a source of v and a range that is a minimal vertex (the possibilities are w'_1 and w'_2).

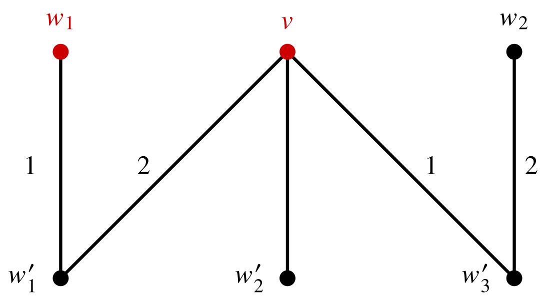

FIGURE 6. A partial ordered Bratteli diagram illustrating an example of a situation that violates conclusion (3) of Proposition 4.11. In this picture, $v \in V_{\min}$ since there is a minimal edge with a range of w'_3 and a source of v (also one with a range of w'_2). However, even though w'_1 is in $R(v)$, the minimal edge with a range of w'_1 has a source of w_1, not v.

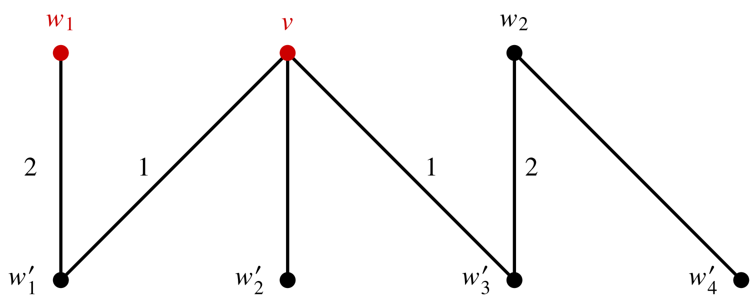

FIGURE 7. A partial ordered Bratteli diagram illustrating an example of a situation that violates conclusion (4) of Proposition 4.11. In this picture, $v \in V_{\min}$ since there is a minimal edge with a range of w'_1 and a source of v (also one with a range of w'_2 and one with a range of w'_3). However, w'_4 is not in $R(v)$, but since w_2 is in $(S \circ R)(v)$, w'_4 is in $(R \circ S \circ R)(v)$.

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PROPOSITION 4.13. *Let (X, h) be a fiberwise essentially minimal zero-dimensional system and let $B = (V, E, \leq)$ be an ordered Bratteli diagram that satisfies the conclusions of Proposition 4.11. If $B' = (V', E', \leq')$ is a telescoping of B, then B' also satisfies the conclusions of Proposition 4.11.*

Proof. Let (k_n) be the telescoping sequence corresponding to B', so that $V'_n = V_{k_n}$ and $E'_n = P_{k_n+1,k_{n+1}}$ for all $n \in \mathbb{Z}_{>0}$ (where $k_0 = 0$). This identification induces a map $\varphi : X_B \to X_{B'}$ by sending $e = (e_1, e_2, \ldots)$ to $\varphi(e) = ((e_1, \ldots, e_{k_1}), (e_{k_1+1}, \ldots, e_{k_2}), \ldots)$. By definition of the induced order on a telescoped Bratteli diagram, φ is a conjugation, and therefore B' satisfies conclusion (1) of Proposition 4.11.

We now show that conclusion (2) holds. Let $v \in V'_{n,\min}$. What we are looking for is a $v' \in V'_{n+1,\min}$ and $e \in E'_{n+1,\min}$ such that $s(e) = v$ and $r(e) = v'$. We can regard all of this as happening in B instead of B', so that we have $v \in V_{k_n,\min}$, and we are looking for $v' \in V_{k_{n+1},\min}$ and a minimal path $(e_{k_n+1}, \ldots, e_{k_{n+1}}) \in P_{k_n+1,k_{n+1}}$. By Proposition 4.11(2), there is a $e_{k_n+1} \in E_{k_n+1,\min}$ and a $v_{k_n+1} \in V_{k_n+1,\min}$ with $s(e) = v$ and $r(e) = v_{k_n+1}$. Proceeding inductively, we construct the desired result, with $v' = v_{k_{n+1}}$. The same argument works for $V'_{\max}$ in place of $V'_{\min}$. This proves that conclusion (2) holds.

Next, we prove that conclusion (3) holds. Let $v \in V'_{n,\min}$ and let $e \in E'_{n+1,\min}$ satisfy $r(e) \in R(v)$. We want to show that $s(e) = v$. We once again regard all of this as happening in B instead of B'. What this means is that we have $v \in V'_{k_n,\min}$ and a minimal path $(e_{k_n+1}, \ldots, e_{k_{n+1}}) \in P_{k_n+1,k_{n+1}}$ such that there is some path $(e'_{k_n+1}, \ldots, e'_{k_{n+1}}) \in P_{k_n+1,k_{n+1}}$ such that $s(e'_{k_n+1}) = v$ and $r(e'_{k_{n+1}}) = r(e_{k_{n+1}})$, and we want to show that $s(e_{k_n+1}) = v$. Suppose not. Then by Proposition 4.11(3), we have $r(e_{k_n+1}) \notin R(v)$; in particular, we have $r(e_{k_n+1}) \neq r(e'_{k_n+1})$. Proceeding like this, we eventually see $r(e_{k_{n+1}}) \neq r(e'_{k_{n+1}})$, which is a contradiction. Thus, $s(e_{k_n+1}) = v$. The proof for $V'_{\max}$ in place of $V'_{\min}$ is analogous. This proves that conclusion (3) holds.

That conclusion (4) holds is immediate; replace m with $k_{n+m} - k_n$. This completes the proof of the proposition. $\square$

5. *The dynamical classification theorem*

We now prove our main theorems, Theorems 5.2 and 5.3.

LEMMA 5.1. *Let $B = (V, E, \leq)$ be an ordered Bratteli diagram with a perfect ordering. Let $e, f \in X_B$ and suppose e and f pass through the same vertex v at level k. Then there is some $N \in \mathbb{Z}$ such that $h_B^N(e) = f$.*

Proof. Write $e = (e_1, e_2, \ldots)$ and write $f = (f_1, f_2, \ldots)$. Let n_1 be the largest element of $\{1, \ldots, k\}$ such that $e_{n_1} \neq f_{n_1}$.

Suppose $e_{n_1} < f_{n_1}$. Then there is an $N_1 \in \mathbb{Z}_{>0}$ such that

$$h_B^{N_1}(e) = (e'_1, \ldots, e'_{n_1-1}, f_{n_1}, e_{n_1+1}, e_{n_1+2}, \ldots),$$

where $(e'_1, \ldots, e'_{n_1-1})$ is the minimal path from $v_0 \in V_0$ to $s(f_{n_1})$. Now, $h_B^M(e)$ and f pass through the same vertex at level $n_1 - 1$. So let n_2 be the largest element of $\{1, \ldots, n_1 - 1\}$ such that $e'_{n_2} \neq f_{n_2}$. Clearly, $e'_{n_2} < f_{n_2}$. So repeating the above process, we find an integer

$N_2 \in \mathbb{Z}_{>0}$ such that $h_B^{N_1+N_2}(e) = (e''_1, \ldots, e''_{n_2-1}, f_{n_2}, \ldots, f_{n_1}, e_{n_1+1}, \ldots)$. Repeating this process inductively, we arrive at an integer N such that $h_B^N(e) = f$. $\square$

THEOREM 5.2. *Let (X_1, h_1, Z_1) and (X_2, h_2, Z_2) be fiberwise essentially minimal zero-dimensional systems. Then (X_1, h_1, Z_1) and (X_2, h_2, Z_2) are strong orbit equivalent if and only if*

$$\begin{aligned}(K_0(C^*(\mathbb{Z}, X_1, h_1)), &K_0(C^*(\mathbb{Z}, X_1, h_1))^+, 1)\\ &\cong (K_0(C^*(\mathbb{Z}, X_2, h_2)), K_0(C^*(\mathbb{Z}, X_2, h_2))^+, 1)\end{aligned}$$

and

$$K_1(C^*(\mathbb{Z}, X_1, h_1)) \cong K_1(C^*(\mathbb{Z}, X_2, h_2)).$$

Proof. ($\Leftarrow$). Let B_1 and B_2 be the Bratteli diagrams satisfying the conclusions of Proposition 4.11 for (X_1, h_1, Z_1) and (X_2, h_2, Z_2), respectively. By Theorem 3.4, we have $K^0(X_1, h_1) \cong K^0(X_2, h_2)$. By Proposition 4.11(1), we have $K^0(X_{B_1}, h_{B_1}) \cong K^0(X_{B_2}, h_{B_2})$. By a slight but trivial extension of [**10**, Theorem 5.4], we have $K_0(B_1) \cong K_0(B_2)$. By [**6**], we have $B_1 \sim B_2$ (in the equivalence class of Bratteli diagrams generated by telescoping and isomorphism), which tells us that there is a (non-ordered) Bratteli diagram B such that telescoping B to odd levels yields a telescoping of B_1 and telescoping B to even levels yields a telescoping of B_2. By replacing B_1 and B_2 with their telescopings (which can be done without changing the above due to Proposition 4.13), we may assume that telescoping B to odd levels yields B_1 and telescoping B to even levels yields B_2. Let B' be the telescoping of B by the sequence $(3n-2)$, so that telescoping B to odd levels yields a telescoping of B_1 by $(3n-2)$ and telescoping B to even levels yields a telescoping of B_2 by $(3n-1)$. Note that by Proposition 4.13, these telescopings of B_1 and B_2 also satisfy the conclusions of Proposition 4.11.

We denote by $V'_{\min}$ ($V'_{\max}$) the minimal (respectively maximal) vertices as inherited by B_1 and B_2. We claim that B' has the following property:

(*) let $v \in V'_{\min}$. There is precisely one $v' \in V'_{\min}$ with $v' \in S(v)$.

We now prove property (*). Let $v \in V'_{\min,n}$ and, without loss of generality, suppose that n is odd. The case $n = 1$ is trivial so suppose $n > 1$. View v as a vertex in $V_{\min,3n-2}$, so that the statement we are trying to prove is that $S^3(v)$ contains precisely one element. Let $w \in S(v)$ and let (e_1, e_2) be a minimal path with $r(e_2) = w$. Since $s(e_1) \in V_{\min}$, we have $S^3(v) \cap V_{\min} \neq \varnothing$. Now, suppose $S^3(v) \cap V_{\min}$ contains two elements, v_1 and v_2. By Proposition 4.11(3), we have $R^2(v_1) \cap R^2(v_2) = \varnothing$. By Proposition 4.11(2), $R^2(v_1) \cap V_{\min} \neq \varnothing$ and $R^2(v_2) \cap V_{\min} \neq \varnothing$. By Proposition 4.11(3) again,

$$R^2(R^2(v_1) \cap V_{\min}) \cap R^2(R^2(v_2) \cap V_{\min}) = \varnothing. \tag{5.1}$$

Since $v \in R^3(v_1)$ and $v \in R^3(v_2)$, we have

$$R^4(v_1) \cap R^4(v_2) \neq \varnothing. \tag{5.2}$$

However, now notice that by Proposition 4.11(4), we have $R^4(v_1) = R^2(R^2(v_1) \cap V_{\min})$ and $R^4(v_2) = R^2(R^2(v_2) \cap V_{\min})$. Thus, we see that equations (5.1) and (5.2) yield a contradiction. Therefore, $S^3(v) \cap V_{\min}$ contains precisely one element. The proof for $v \in V'_{\max}$ is analogous. This proves property $(*)$.

For convenience, replace $B^{(1)}$ with its telescoping by $(3n-2)$ and replace $B^{(2)}$ with its telescoping by $(3n-1)$. Now, let $e = (e_1, e_2, \ldots) \in X_{B^{(1)},\min}$ and let $(v_1, v_2, \ldots)$ be its associated vertices. Let $n \in \mathbb{Z}_{>1}$. We view v_n as a vertex in V'_{2n-1}. By property $(*)$, there is a unique vertex $v'_{n-1} \in V'_{\min,2n-2}$ such that $v'_{n-1} \in S(v_n)$. We claim that $v_{n-1} \in S(v'_{n-1})$. If not, then by property $(*)$, there is some $w \in S(v'_{n-1})$ such that $w \neq v_{n-1}$. However, then $R^2(w) \cap R^2(v_{n-1}) \neq \varnothing$, which is a contradiction again by Proposition 4.11(3). Thus, for each $n \in \mathbb{Z}_{>1}$, v'_{n-1} is connected by a path to v'_n, and there is an $e'_n \in E^{(2)}_{\min,n}$ with $s(e'_n) = v'_{n-1}$ and $r(e'_n) = v'_n$ by Proposition 4.11(3). Thus, this gives us $e' = (e'_1, e'_2, \ldots) \in X_{B^{(2)},\min}$ such that for each $n \in \mathbb{Z}_{>0}$, we have $s(e'_n) = v'_{n-1}$ and $r(e'_n) = v'_n$.

Let $f = (f_1, f_2, \ldots) \in X_{B'}$ be any path with $r(f_n) = s(f_{n+1}) = v_{2n-1}$ for all odd $n \in \mathbb{Z}_{>0}$ and $r(f_n) = s(f_{n+1}) = v'_{2n}$ for all even $n \in \mathbb{Z}_{>0}$. Since this respects the vertices and the range and source maps, we are free to define $F_1(e) = f$ and $F_2(e') = f$.

We now show that the above pairing is a bijection between $X^{(1)}_{B,\min}$ and $X^{(2)}_{B,\min}$. Let $e' = (e'_1, e'_2, \ldots) \in X_{B^{(2)},\min}$ and (v'_n) be as above. By property $(*)$, for each $n \in \mathbb{Z}_{>0}$, there is precisely one $w_n \in V'_{\min}$ with $w_n \in S(v'_n)$. Since $v_n \in S(v'_n)$, we must have $w_n = v_n$. Thus, the bijection is established, and so $F_1(X_{B^{(1)},\min}) = F_2(X_{B^{(2)},\min})$. We now repeat the process for maximal vertices, choosing edges which are not minimal if the number of edges between the vertices is more than one. This extends F_1 and F_2 so that $F_1(X_{B^{(1)},\max}) = F_2(X_{B^{(2)},\max})$.

We now extend F_1 and F_2 by any bijection between $E^{(1)}_n$ and E'_{2n-1}, and any bijection between $E^{(2)}_n$ and E_{2n} that respect the range and source maps. In this way, we get homeomorphisms $F_1 : X_{B^{(1)}} \to X_{B'}$ and $F_2 : X_{B^{(2)}} \to X_{B'}$. Define $F = F_2^{-1} \circ F_1 : X_{B^{(1)}} \to X_{B^{(2)}}$.

Let $\beta, \gamma : X_{B^{(1)}} \to \mathbb{Z}$ be the orbit cocycles of F. We will show that β and γ are continuous on $X_{B^{(1)}} \setminus X_{B^{(1)},\max}$. So let $e = (e_1, e_2, \ldots) \in X_{B^{(1)}} \setminus X_{B^{(1)},\max}$. Let k be the smallest element of $\mathbb{Z}_{>0}$ such that $e_k \notin E^{(1)}_{\max}$. Then e and $h_{B^{(1)}}(e)$ are confinal from level k. This means that $F_1(e)$ and $F_1(h_{B^{(1)}}(e))$ are cofinal from level $2k-1$, and so $F(e)$ and $F(h_{B^{(1)}}(e))$ are cofinal from level k. In particular, $F(e)$ and $F(h_{B^{(1)}}(e))$ pass through the same vertex v at level k. By Lemma 5.1, there is an integer N such that

$$h^N_{B^{(1)}}(e) = F(h_{B^{(1)}}(e)).$$

Let $f \in U(e_1, \ldots, e_{k+1})$. Then y and $h_{B^{(1)}}(f)$ are confinal from level k. This means that $F_1(f)$ and $F_1(h_{B^{(1)}}(f))$ are cofinal from level $2k-1$, and so $F(y)$ and $F(h_{B^{(1)}}(f))$ are cofinal from level k. Since $e, f \in U(e_1, \ldots, e_{k+1})$, $h_{B^{(1)}}(x)$ and $h_{B^{(1)}}(f)$ have the same initial segment from level 0 to level $k+1$, and so $F(h_{B^{(1)}}(e))$ and $F(h_{B^{(1)}}(f))$ have the same initial segment from level 0 to level k. Similarly, $F(e)$ and $F(f)$ have the same initial segment from level 0 to level k. Thus, the integer N from above satisfies

$$h^N_{B^{(1)}}(f) = F(h_{B^{(1)}}(f)).$$

Since $f \in U(e_1, \ldots, e_{k+1})$ was arbitrary, this shows that β is continuous at e.

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The argument for γ is analogous to β. Thus, $(X_{B^{(1)}}, h_{B^{(1)}}, X_{B^{(1)},\max})$ and $(X_{B^{(2)}}, h_{B^{(2)}}, X_{B^{(2)},\max})$ are strong orbit equivalent. By replacing F with $h_{B^{(1)}}^{-1} \circ F$, we see that $(X_{B^{(1)}}, h_{B^{(1)}}, X_{B^{(1)},\min})$ and $(X_{B^{(2)}}, h_{B^{(2)}}, X_{B^{(2)},\min})$ are strong orbit equivalent. Therefore, by Proposition 4.11(1), $(X_1, h_1.Z_1)$ and (X_2, h_2, Z_2) are strong orbit equivalent.

($\Rightarrow$). Assume (X_1, h_1, Z_1) and (X_2, h_2, Z_2) are strong orbit equivalent. By the definition of strong orbit equivalence, we have $Z_1 \cong Z_2$ and so by Theorem 3.10, we have $K_1(C^*(\mathbb{Z}, X_1, h_1)) \cong K_1(C^*(X_2, h_2, Z_2))$.

Let $\beta, \gamma : X_1 \to \mathbb{Z}$ be the associated orbit cocyles defined by $F \circ h_1 = h_2^{\beta} \circ F$ and $F \circ h_1^{\gamma} = h_2 \circ F$. Recall that strong orbit equivalence tells us that β and γ are continuous on $X_1 \setminus Z_1$ and $F(Z_1) = Z_2$.

Let $\widetilde{h}_2 = F^{-1} \circ h_2 \circ F : X_1 \to X_1$. Then $\widetilde{h}_2$ is conjugate to h_2, has the same orbits as h_1, and

$$h_1 = \widetilde{h}_2^{\beta}, \widetilde{h}_2 = h_1^{\gamma}.$$

We also see that $(X_1, \widetilde{h}_2, Z_1)$ is a fiberwise essentially minimal zero-dimensional system that is conjugate to (X_2, h_2, Z_2), so we work with the former for the remainder of the proof.

Let α_1 be the automorphism of $C(X_1)$ induced by h_1 and let α_2 be the automorphism of $C(X_1)$ induced by $\widetilde{h}_2$. We now show that $\operatorname{ran}(\mathrm{id} - (\alpha_1)_*) \subset \operatorname{ran}(\mathrm{id} - (\alpha_2)_*)$. It is enough to show that for any compact open set $E \subset X_1$, we have $(\mathrm{id} - (\alpha_1)_*)(\chi_E) \in \operatorname{ran}(\mathrm{id} - (\alpha_2)_*)$.

Given $f \in C(X_1, \mathbb{Z})$, we denote the image of f in $K_0(C^*(\mathbb{Z}, X_1, h_i))$ by $[f]_1$ and denote the image of f in $K_0(C^*(\mathbb{Z}, X_1, \widetilde{h}_2))$ by $[f]_2$. Since h_1 and $\widetilde{h}_2$ have the same orbits, there is a probability measure μ that is both h_1- and $\widetilde{h}_2$-invariant. Note that $C^*(\mathbb{Z}, X_1, h_1)$ is the C^*-subalgebra of $L(L^2(X_1, \mu))$ generated by $C(X)$ and the unitary $u_1 : g \to g \circ h_1^{-1}$. Also note that $C^*(\mathbb{Z}, X_1, \widetilde{h}_2)$ is the C^*-subalgebra of $L(L^2(X_1, \mu))$ generated by $C(X)$ and the unitary $u_2 : g \to g \circ \widetilde{h}_2^{-1}$. For the remainder of the proof, we identify these crossed products with these corresponding subalgebras of $L(L^2(X_1, \mu))$.

We claim that map $\varphi : K_0(C^*(\mathbb{Z}, X_1, h_1)) \to K_0(C^*(\mathbb{Z}, X_1, \widetilde{h}_2))$ defined by $\varphi([\chi_U]_1) = [\chi_U]_2$ is an isomorphism of ordered groups. Since the positive cone and distinguished order units agree via this map, the only thing to check is that $\operatorname{im}(\mathrm{id} - (\alpha_1)_*) = \operatorname{im}(\mathrm{id} - (\alpha_2)_*)$.

Let E be a compact open subset of X_1 such that $E \cap Z_1 = \varnothing$. Then since β is continuous on E, $\operatorname{ran}(\beta|_E) = \{k_1, \ldots, k_N\}$. Then

$$u_1 \chi_E = \sum_{n=1}^{N} u_2^{k_n} \chi_{E \cap \beta^{-1}(k_n)}.$$

Thus, $u_1 \chi_E \in C^*(\mathbb{Z}, X_1, \widetilde{h}_2)$, and, letting $A_{Z_1}^{(1)}$ correspond to $C^*(\mathbb{Z}, X_1, h_1)$ as in Definition 3.6, we have $A_{Z_1}^{(1)} \subset C^*(\mathbb{Z}, X_1, \widetilde{h}_2)$. Since $E \cap Z_1 = \varnothing$, there is a unitary $v \in A_{Z_1}^{(1)}$ such that $v \chi_E v^* = \chi_{h_1(E)}$. Since we also have $v \in C^*(\mathbb{Z}, X_1, \widetilde{h}_2)$, we have $[\chi_E]_2 = [\chi_{h_1(E)}]_2$, so $\chi_E - \chi_{h_1(E)} \in \operatorname{im}(\mathrm{id} - (\alpha_2)_*)$. Theorem 3.9 (and the proof) tells us that $K_0(A_{Z_1})^{(1)} \cong K_0(\mathbb{Z}, X_1, h_1)$ by the injection map. Thus, $\operatorname{im}(\mathrm{id} - (\alpha_1)_*) \subset$

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$\mathrm{im}(\mathrm{id} - (\alpha_2)_*)$. By repeating the above process with γ instead of β, we can similarly show that $\mathrm{im}(\mathrm{id} - (\alpha_2)_*) \subset \mathrm{im}(\mathrm{id} - (\alpha_1)_*)$. This completes the proof. □

THEOREM 5.3. *Let (X_1, h_1, Z_1) and (X_2, h_2, Z_2) be fiberwise essentially minimal zero-dimensional systems with no periodic points. The following are equivalent:*

(1) $C^*(\mathbb{Z}, X_1, h_1) \cong C^*(\mathbb{Z}, X_2, h_2)$;

(2) $(K_0(C^*(\mathbb{Z}, X_1, h_1)), K_0(C^*(\mathbb{Z}, X_1, h_1))^+, 1) \cong (K_0(C^*(\mathbb{Z}, X_2, h_2)), K_0(C^*(\mathbb{Z}, X_2, h_2))^+, 1)$ *and* $K_1(C^*(\mathbb{Z}, X_1, h_1)) \cong K_1(C^*(\mathbb{Z}, X_2, h_2))$;

(3) (X_1, h_1, Z_1) *and* (X_2, h_2, Z_2) *are strong orbit equivalent.*

Proof. (1) $\Longleftrightarrow$ (2). By [**11**, Theorems 2.2 and 2.3], $C^*(\mathbb{Z}, X_1, h_1)$ and $C^*(\mathbb{Z}, X_2, h_2)$ are A$\mathbb{T}$-algebras of real rank zero, so this result follows from [**5**].

(2) $\Longleftrightarrow$ (3). This is implied by Theorem 5.2. □

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