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FURTHER COMMENTS ON THE SOLUTION OF THE $M/M/1$ QUEUE

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The elegant transient solution of the $M/M/1$ queue obtained by Parthasarathy [3] may be carried one step further to show that it is equivalent to the solution in finite terms obtained by J. W. Cohen before 1960. This is in contrast to other works quoted by Parthasarathy which express the solution by an infinite series.

Following Parthasarathy, one has immediately from his definition of $q_n(t)$ that

$$p_n(t) - \partial_{na} = \int_0^t \exp(-(\lambda+\mu)\tau)[q_{n+1}(\tau) - q_n(\tau)]\,d\tau, \qquad n \geqq 0.$$

Using his expression (6), it is easy to see that for $n \geqq 1$ (assuming for simplicity that $a \neq 0$):

$$q_{n+1}(t) - q_n(t) = \mu\beta^{n-a}A_n(t) + \mu\beta^{n-a+1}B_n(t)$$

where

$$A_n(t) = I_{n+a}(\alpha t) - 2\beta I_{m+a+1}(\alpha t) + \beta^2 I_{n+a+2}(\alpha t)$$

and

$$B_n(t) = I_{n-a-1}(\alpha t) - (\lambda^+\mu)\alpha^{-1}I_{n-a}(\alpha t) + I_{n-a+1}(\alpha t).$$

Substituting this into the above integral, one finds the integral expression already obtained by Bailey in 1954; see [1]. However, the relation for Bessel functions

$$I_{n-a-1}(x) + I_{n-a+1}(x) = 2dI_{n-a}(x)/dx$$

allows us to integrate the part involving the term $B_n(\tau)$. Hence, the final result is:

$$p_n(t) = \beta^{n-a}\exp(-(\lambda+\mu)t)I_{n-a}(\alpha t) + \mu\beta^{n-a}\int_0^t \exp(-(\lambda+\mu)\tau)A_n(\tau)\,d\tau.$$

This is the result obtained by Cohen; see formula (4.31), p. 82 in [2]. It was quoted earlier in [4] as formula (5.32a) on p. 234. Cohen used the Laplace transform method which is of course dual to the method of Parthasarathy, both depending on properties of modified Bessel functions.

It is perhaps a matter of opinion which solution is simpler—Parthasarathy's involving one integral and a finite series, or Cohen's involving three integrals.

References

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