PSEUDO-IDEMPOTENTS IN SEMIGROUPS OF FUNCTIONS

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The aim of this paper is to generalize Theorem 2.10 (i) of [2]. As stated in [2] this theorem deals with the semigroup of all selfmaps on a discrete space and provides a characterization of $\mathscr{H}$-classes which contain an idempotent. We will generalize this theorem to the case of other semigroups of functions on a discrete space, some semigroups of continuous functions on non-discrete topological spaces, and one semigroup of binary relations. The results in this paper form the main part of chapter 3 of [1]. Some results will be quoted from [1] without proof; the required proofs can easily be supplied by the reader.

Notation for composition of functions will be written in topological-analytic order: $(fg)(x) = f(g(x))$. Thus the concepts of left and right in this paper will be the mirror images of left and right in [2]. Juxtaposition will always denote ordinary composition. Definition 4 will be concerned with a semigroup multiplication which is not ordinary composition.

We will let Dom(f) denote the domain of a function f, and Im(f) will denote the image. The equivalence class of f under a Green's relation, say $\mathscr{L}$, will be called an $\mathscr{L}$-class and will be written L_f.

Definition 1. a. $\pi_f = \pi_g$ means that $\mathrm{Dom}(f) = \mathrm{Dom}(g)$ and that for arbitrary x and y in $\mathrm{Dom}(f) = \mathrm{Dom}(g)$ we have $f(x) = f(y)$ if and only if $g(x) = g(y)$.

b. A semigroup T is said to be L_π if for arbitrary f and g in T we have $L_f = L_g$ if and only if $\pi_f = \pi_g$.

c. A semigroup T is said to be R_{im} if for arbitrary f and g in T we have $R_f = R_g$ if and only if $\mathrm{Im}(f) = \mathrm{Im}(g)$.

Definition 2. Let X be a topological space.

a. $S(X)$ is the semigroup of all continuous functions from all of X into X under ordinary composition.

b. $S_1(X)$ is the subsemigroup of one-to-one functions in $S(X)$.

c. $Q(X)$ is the semigroup of all continuous functions from X into X whose domains are *open* subsets of X. Multiplication is ordinary composition.

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d. $Q_1(X)$ is the subsemigroup of one-to-one functions in $Q(X)$.

From the above definition we see that $S_1(X) = S(X) \cap Q_1(X)$, while $Q(X)$ can be thought of as the smallest semigroup containing $S(X)$ and $Q_1(X)$. We note that $S(X)$ with X discrete is an L_π and R_{im} semigroup according to Lemmas 2.5 and 2.6 of [2]. It is easy to see that for X discrete the semigroups $Q(X)$ and $Q_1(X)$ are also L_π and R_{im}; however, $S_1(X)$ is R_{im} but not L_π for infinite discrete X.

In the following three lemmas we assume that $\mathrm{Dom}(f)$ and $\mathrm{Im}(f)$ are subsets of a topological space X.

LEMMA 1. *The following conditions are equivalent*:

i) $\mathrm{Im}(ff) = \mathrm{Im}(f)$;

ii) $f^{-1}(f(x)) \cap \mathrm{Im}(f) \neq \varnothing$ *for each* $x \in \mathrm{Dom}(f)$.

PROOF. i $\to$ ii. For each $x \in \mathrm{Dom}(f)$ we have $f(x) = ff(z)$ for some z. Since

$$f(z) \in f^{-1}(ff(z)) = f^{-1}(f(x))$$

we see that $f(z) \in f^{-1}(f(x)) \cap \mathrm{Im}(f) \neq \varnothing$.

ii $\to$ i. Clearly $\mathrm{Im}(ff) \subset \mathrm{Im}(f)$. Let $y \in \mathrm{Im}(f)$ be arbitrary; $y = f(x)$. Let

$$z \in f^{-1}(f(x)) \cap \mathrm{Im}(f).$$

Then $z = f(t)$ for some t, and hence $y = f(x) = f(z) = ff(t) \in \mathrm{Im}(ff)$.

LEMMA 2. *The following conditions are equivalent*:

i) $\mathrm{Im}(ff) = \mathrm{Im}(f)$ *and* $f|_{\mathrm{Im}(f)}$ *is one-to-one*;

ii) $f^{-1}(f(x)) \cap \mathrm{Im}(f)$ *is a single point for each* $x \in \mathrm{Dom}(f)$.

PROOF. i) $\to$ ii). By the previous lemma we know that $f^{-1}(f(x)) \cap \mathrm{Im}(f) \neq \varnothing$. Let y and z be in $f^{-1}(f(x)) \cap \mathrm{Im}(f)$. Then $f(y) = f(x)$ and $y = f(s)$ for some s; also $f(z) = f(x)$ and $z = f(t)$ for some t. Therefore $f(y) = f(z)$, and thus $ff(s) = ff(t)$. Since f is one-to-one on $\mathrm{Im}(f)$ we have $f(s) = f(t)$, that is, $y = z$.

ii) $\to$ i). By the previous lemma we know that $\mathrm{Im}(ff) = \mathrm{Im}(f)$. Suppose that y and z are in $\mathrm{Im}(f)$ with $f(y) = f(z)$. Then y and z are in $f^{-1}(f(y)) \cap \mathrm{Im}(f)$, and thus $y = z$.

We note that in condition i) of the preceding lemma we do not assume that $\mathrm{Im}(f) \subset \mathrm{Dom}(f)$. If in fact we have $\mathrm{Im}(f) \subset \mathrm{Dom}(f)$, then condition i) says precisely that f is a permutation on $\mathrm{Im}(f)$. Condition ii) of the lemma enables us to define a function

$$g(x) \equiv f^{-1}(f(x) \cap \mathrm{Im}(f) \text{ on } \mathrm{Dom}(f).$$

This function is examined in the following lemma.

LEMMA 3. *Suppose* $\mathrm{Im}(f) \subset \mathrm{Dom}(f)$ *where* f *is a function which satisfies*

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the equivalent conditions in the preceding lemma. Define $g(x) \equiv f^{-1}(f(x)) \cap \mathrm{Im}(f)$ *for each* $x \in \mathrm{Dom}(f)$. *Then* $\pi_f = \pi_g$; $\mathrm{Im}(f) = \mathrm{Im}(g)$; *and* $gg = g$.

PROOF. It is clear that $\mathrm{Dom}(f) = \mathrm{Dom}(g)$. Suppose $f(x) = f(y)$. Then $g(x) = g(y)$ by inspection. On the other hand, suppose $f(x) \neq f(y)$. Then

$$f^{-1}(f(x)) \cap f^{-1}(f(y)) = \varnothing$$

and therefore $g(x) \neq g(y)$. Hence $\pi_f = \pi_g$. Now let $y \in \mathrm{Im}(f)$. Then

$$g(y) = f^{-1}(f(y)) \cap \mathrm{Im}(f) = y.$$

Therefore $g|_{\mathrm{Im}(f)} = i|_{\mathrm{Im}(f)}$. Thus $\mathrm{Im}(f) \subset \mathrm{Im}(g)$, but by definition we have $\mathrm{Im}(g) \subset \mathrm{Im}(f)$. Consequently $\mathrm{Im}(f) = \mathrm{Im}(g)$ and $g|_{\mathrm{Im}(g)} = i|_{\mathrm{Im}(g)}$, that is, $gg = g$.

In view of Lemmas 2 and 3 we now define pseudo-idempotency in $S(X)$, $S_1(X)$, $Q(X)$, and $Q_1(X)$.

DEFINITION 3. Let T be a semigroup of the form $S(X)$, $S_1(X)$, $Q(X)$, or $Q_1(X)$ for some topological space X. We say that a function $f \in T$ is *pseudo-idempotent* if $\mathrm{Im}(ff) = \mathrm{Im}(f) \subset \mathrm{Dom}(f)$ and $f|_{\mathrm{Im}(f)}$ is one-to-one, that is, f is a permutation on $\mathrm{Im}(f)$.

We remark that in $S_1(X)$ the only idempotent is the identity function, and the pseudo-idempotents are precisely the onto functions. In $S(X)$ and $S_1(X)$ the condition $\mathrm{Im}(f) \subset \mathrm{Dom}(f)$ is superfluous. In $Q_1(X)$ it is easy to see that $\mathrm{Im}(f) = \mathrm{Dom}(f)$ for a pseudo-idempotent f. On the the basis of Definition 3 we can state the central theorem of this paper.

THEOREM 1. *Let* H *be an* $\mathscr{H}$*-class in* $S(X)$, $S_1(X)$, $Q(X)$, *or* $Q_1(X)$ *with* X *discrete. The following conditions are equivalent*:

1) H *contains a pseudo-idempotent*;
2) H *contains an idempotent* (*unique*);
3) H *consists of pseudo-idempotents.*

PROOF. 1) → 2). This follows from Lemma 3 together with the remarks preceding Lemma 1 and following Definition 3. Uniqueness of the idempotent follows from Lemma 2.15 of [2].

2) → 3). Let f be the idempotent in H, and let $g \in H$ be arbitrary. Then $\pi_f = \pi_g$ and $\mathrm{Im}(f) = \mathrm{Im}(g)$, and it is easy to check therefore that the conditions in Lemma 2 hold for g and that $\mathrm{Im}(g) \subset \mathrm{Dom}(g)$.

3) → 1). Trivial.

We will now establish the results of Theorem 1 for some semigroups of the form $S(X)$ and $S_1(X)$ where X is a non-discrete topological space. Similar results can be obtained for $Q(X)$ and $Q_1(X)$. In order to establish Theorem 1 with X non-discrete we only have to verify the implication $1 \to 2$. To do this we must show that the idempotent

$$g(x) \equiv f^{-1}(f(x)) \cap \mathrm{Im}(f)$$

is continuous and belongs to the $\mathscr{H}$-class of f.

Suppose, for instance, that X is a compact Hausdorff space for which $S(X)$ is an L_π and R_{im} semigroup. These assumptions hold, for example, for X finite discrete or X equal to $\{0\} \cup \{1/n\}_{n=1,2,\dots}$ with the usual metric topology (see [1], Propositions 2.16 and 2.17). Then for each pseudo-idempotent $f \in S(X)$ the corresponding idempotent g is continuous because X is compact Hausdorff (see [1], Lemma 3.8), and $g \in H_f$ by Lemma 3 above. Therefore Theorem 1 holds for $S(X)$ in this case.

Let I be the closed unit interval, and consider $S(I)$. By Lemma 3.8 of [1] we know that the idempotent g is continuous. Since $S(I)$ is an L_π semigroup (see [1], Proposition 2.23) we know that $g \in L_f$ by Lemma 3 above. $S(I)$ is not an R_{im} semigroup, but it is shown in Corollary 3 of Theorem 3.2 of [1] that $g \in R_f$. Hence Theorem 1 holds for $S(I)$. We can also show that Theorem 1 holds for $S(R)$ where R is the real line (see [1], Corollary 4 of Theorem 3.2).

For semigroups of the form $S_1(X)$ the situation depends on whether any onto functions $f \in S_1(X)$ fail to be invertible in $S_1(X)$. As we remarked after Definition 3, the semigroup $S_1(X)$ contains only one idempotent, the identity function i; and therefore Theorem 1 is concerned with H_i. Clearly H_i in $S_1(X)$ for any X consists of the continuously invertible onto functions. The pseudo-idempotents are the onto functions. We conclude that Theorem 1 holds for $S_1(X)$ if and only if each onto function in $S_1(X)$ is continuously invertible. It is then clear that Theorem 1 holds, for instance, for $S_1(X)$ with X equal to an interval of the real line or equal to the space $p = \{0\} \cup \{1/n\}$ with the metric topology. For $PN = \{0\} \cup \{1/n\} \cup \{n\}$ we can see that Theorem 1 is false for both $S_1(PN)$ and $S(PN)$.

Finally we will establish the results of Theorem 1 for some semigroups which are not included in Definition 2.

Definition 4. a. Let Y be a subspace of X. Then $S(X,Y)$ is the subsemigroup of functions $f \in S(X)$ such that $f(Y) \subset Y$.

b. Let X and Y be arbitrary topological spaces. Let p be a continuous function which maps all of Y into X. Then $S(X, p, Y)$ is the semigroup of all continuous functions which map all of X into Y under the multiplication $f \circ g \equiv fpg$.

c. Let X be an arbitrary set. For a binary relation T on X we let $T(x) = \{y \mid xTy\}$. Then $B_1(X)$ denotes the semigroup of all binary relations T on X such that $x \neq y$ implies $T(x) \cap T(y) = \varnothing$.

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The semigroups $S(X,Y)$ were called *restrictive* semigroups by Magill in [3], and the semigroups $S(X, p, Y)$ were discussed by Magill in [4].

First we will establish Theorem 1 for $S(X, Y)$ with X discrete. The main task is to relate the definition of pseudo-idempotency in $S(X, Y)$ to the subspace Y. To this end we observe that if $f \in S(X, Y)$ is idempotent, then each of the functions $f|_{\mathrm{Im}(f)}$, $f|_{Y \cap \mathrm{Im}(f)}$, and $f|_{f(Y)}$ is the identity on its domain. We define a function $f \in S(X, Y)$ to be pseudo-idempotent if each of the functions $f|_{\mathrm{Im}(f)}$, $f|_{Y \cap \mathrm{Im}(f)}$, and $f|_{f(Y)}$ is a permutation. Given a pseudo-idempotent, we then define the corresponding idempotent exactly as we did in Lemma 3. In order to show that the idempotent g is in the $\mathscr{H}$-class of the pseudo-idempotent f we first have to characterize $\mathscr{H}$-classes in $S(X, Y)$ with X discrete. We define the symbol $f Y g$ to mean that for each $x \in X$ we have $f(x) \in Y$ if and only if $g(x) \in Y$. Then we can show that $L_f = L_g$ if and only if $\pi_f = \pi_g$ and $f Y g$ (see [1], Proposition 2.40), and $R_f = R_g$ if and only if $\mathrm{Im}(f) = \mathrm{Im}(g)$ and $f(Y) = g(Y)$ (see [1], Proposition 2.41). With the resulting characterization of $\mathscr{H}$-classes it is easy to show that Theorem 1 is valid for $S(X,Y)$.

We will now consider $S(X, p, Y)$ for the case of X and Y discrete. Clearly a function $f \in S(X, p, Y)$ is idempotent if and only if $fp|_{\mathrm{Im}(f)} = i|_{\mathrm{Im}(f)}$. We then say that $f \in S(X, p, Y)$ is pseudo-idempotent if $fp|_{\mathrm{Im}(f)}$ is a permutation. Given a pseudo-idempotent f we construct the corresponding idempotent g by the formula

$$g(x) \equiv (fp)^{-1}(f(x)) \cap \mathrm{Im}(f).$$

With the help of the following lemma, which is Lemma 3.26 of [1], it can be shown that $g \in H_f$ and that Theorem 1 is valid for $S(X, p, Y)$.

LEMMA 4. *Let X and Y be discrete and let f and g be pseudo-idempotents in $S(X, p, Y)$. If $L_{pf} = L_{pg}$ in $S(X)$, then $L_f = L_g$ in $S(X, p, Y)$. If $R_{fp} = R_{fp}$ in $S(Y)$, $R_f = R_g$ in $S(X, p, Y)$.*

We will conclude by considering the semigroup of relations $B_1(X)$. For T and V in $B_1(X)$ we can show that $L_T = L_V$ if and only if $\mathrm{Dom}(T) = \mathrm{Dom}(V)$, and $R_T = R_V$ if and only if each $T(x)$ equals $V(y)$ for some y and each $V(u)$ equals $T(w)$ for some w (see [1], Propositions 5.1 and 5.2).

The definition of pseudo-idempotency in $B_1(X)$ will be based on the following lemma.

LEMMA 5. *A relation $T \in B_1(X)$ is idempotent if and only if $T(x) \cap \mathrm{Dom}(T) = \{x\}$ for each $x \in \mathrm{Dom}(T)$.*

By means of this lemma we have found an identity function, namely, $T(x) \cap \mathrm{Dom}(T)$ on $\mathrm{Dom}(T)$. We then define a relation $T \in B_1(X)$ to be pseudo-idempotent if $T(x) \cap \mathrm{Dom}(T)$ is a singleton for each $x \in \mathrm{Dom}(T)$ and the resulting function $f(x) \equiv T(x) \cap \mathrm{Dom}(T)$ is a permutation on $\mathrm{Dom}(T)$. For $T \in B_1(X)$ let $T^{-1}(y) = \{x \mid xTy\}$, which is either a singleton or the empty set.

LEMMA 6. *Let* $T \in B_1(X)$ *be pseudo-idempotent. Define* V *on* $\mathrm{Dom}(T)$ *by the formula* $V(x) \equiv T(T^{-1}(x))$. *Then* $V \in B_1(X)$; V *is idempotent, and* $H_T = H_V$.

PROOF. To see that $V \in B_1(X)$ we suppose that $V(x) \cap V(y) \neq \varnothing$. Then

$$T^{-1}(x) \cap T^{-1}(y) \neq \varnothing.$$

Since T^{-1} is a permutation on $\mathrm{Dom}(T)$, it follows that $x = y$, which completes the first part of the proof.

To see that V is idempotent we consider the expression $T^{-1}TT^{-1}(x)$. Since T is in $B_1(X)$ we have

$$T^{-1}TT^{-1}(x) = T^{-1}(x).$$

Therefore $VV(x) = T(T^{-1}TT^{-1}(x)) = TT^{-1}(x) = V(x)$.

We will now show that $H_T = H_V$. By the definition of V we have $\mathrm{Dom}(T) = \mathrm{Dom}(V)$, and thus $L_T = L_V$. Since T^{-1} is a permutation on $\mathrm{Dom}(T)$, we see from the definition of V that each $V(x)$ equals $T(y)$ for some y, and each $T(z)$ equals $V(w)$ for some w. Therefore $R_T = R_V$, and the proof is done.

From this lemma it now follows easily that Theorem 1 is true for $B_1(X)$.

References

[1] F. Cezus, *Green's Relations in Semigroups of Functions* (Ph.D. thesis submitted May, 1972 at the Australian National University).

[2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups* (Math. Surveys, No. 7, Amer. Math. Soc., 1961 and 1967).

[3] K. D. Magill, Jr, 'Subsemigroups of $S(X)$', *Math. Japonicae* 11 (1966), 109–115.

[4] K. D. Magill, Jr, 'Semigroup structures for families of functions. I', *J. Austral. Math. Soc.* 7 (1967), 81–94.

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