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SEMI-GROUPS IN L_∞ AND LOCAL ERGODIC THEOREM

BY

R. EMILION

ABSTRACT. We show that any W^*-continuous semi-group in L_∞ is L_1-norm continuous. As an application we prove the n-dimensional local ergodic theorem in L_∞. We also note that any bounded additive process in L_∞ is absolutely continuous.

For $n = 1$ this local theorem improves those of R. Sato [14] and D. Feyel [6] and for $n \geq 1$ it generalizes M. Lin's ones which hold for positive operators [12].

1. **Introduction and Notations**. Let L_1 (resp. L_∞) be the usual space of equivalence classes of *complex* valued integrable (resp. bounded) functions on a σ-finite measure space $(X, \mathcal{F}, \mu)$.

Let $T^* = \{T_t^*, t \in \mathbb{R}_+^n\}$ be an n-parameter W^*-continuous semi-group of linear contractions on L_∞. This means that T^* verifies the following properties.

(1.1) Each T_t^* is a W^*-continuous linear contraction on L_∞

(1.2) $$T_{t+s}^* = T_t^* T_s^* \text{ for any } t, s \in \mathbb{R}_+^n$$

(1.3) $$\lim_{s \to 0} \int (T_{t+s}^* f - T_t^* f)\, g \, d\mu = 0 \text{ for any } t \in \mathbb{R}_+^n,$$
$$\text{any } f \in L_\infty \text{ and any } g \in L_1.$$

1.1 and 1.2 then imply that T^* is the adjoint semi-group of a L_1-semi-group, say $T = \{T_t, t \in \mathbb{R}_+^n\}$ and 1.3 shows that T is weakly continuous and thus strongly continuous (see [7] p. 306).

We will assume that μ is finite without loss of generality in the results of this note. Indeed it suffices to replace μ by $\bar{\mu} = u \cdot \mu$ and T_t by T_t' where $T_t' g = 1/u\, T_t(gu)$ for any $g \in L_1(\bar{\mu})$ and $u \in L_1(\mu)$, $u > 0$ μ a.e.

Let λ_n denotes the Lebesgue measure on $\mathbb{R}^n$ and for any interval I of $\mathbb{R}_+^n$ such that $\lambda_n(I) > 0$, let

(1.4) $$M_I f = \lambda_n(I)^{-1} \int_I T_t f \, dt \text{ for any } f \in L_1.$$
$$M_I^* f = (M_I)^* f \text{ for any } f \in L_\infty$$

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so that M_I^* is a linear contraction on L_∞.

For any $\alpha > 0$ we also put M_α (resp. M_α^*) $= M_{[0,\alpha]^n}$ (resp. $M^*_{[0,\alpha]^n}$).

The main result of this paper is the following local ergodic theorem which answers a question raised by M. Lin ([12], p. 301): For any $f \in L_\infty$, $M_\alpha^* f$ converges a.e. to $T_0^* f$ as $\alpha \to 0^+$.

Recall that M. Lin [11] has shown that for any fixed $f \in L_\infty$, there exists a scalar representative of $\{M_\alpha^* f, \alpha \in \mathbb{R}_+^n\}$ so that the pointwise convergence may be studied as $\alpha \to 0^+$.

In fact we first show that the W^*-continuity of T^* (1.3 above) implies a stronger property of L_1-norm continuity:

$$\lim_{s\to 0} \int |T^*_{t+s} f - T_t^* f| \, g \, d\mu = 0 \qquad \text{(theorem 3.1 below).}$$

From this result, the n-dimensional local theorem in L_∞ is proved by using the one-dimensional one for positive contractions on L_1 (U. Krengel [10]).

In the last section we will note that any bounded additive process in L_∞ is absolutely continuous (that is equal to $(M_I^* f)_I$ for some $f \in L_\infty$), just as in the case of L_p $(1 < p < \infty)$ ([3], [4]).

Local ergodic theorems in L_∞ were first proved by N. Wiener [15] and then by U. Krengel [9]. They have been recently generalized by R. Sato ([13], [14]), M. Lin [12] and D. Feyel [6] in the setting of semi-groups of operators.

For $n = 1$ our result completes the partial ones of R. Sato [14] and D. Feyel [6]. Indeed in these papers it is assumed that the initially conservative part of the modulus of T, say C, is equal to X. Although this condition is not a restriction for L_1-theorems (because $1_{(X/C)} T_t = 0$), it is for L_∞-theorems since $1_{(X/C)} T_t^*$ need not be 0 (see [12] p. 304, Remark 1). For $n \geq 1$ our result generalises those of M. Lin which hold for positive operators [12].

On a part of the space X, we will use some nice arguments of measure change due to R. Sato ([13], [14]). These arguments were also used by M. Lin [12].

2. **Reduction of the Dimension**. In ([5], 4.2, 4.1), we have proved that there exists a constant $c_n > 0$ and a one-parameter strongly continuous semi-group of *positive* contractions in L_1, say $(U_t)_{t\geq 0}$, such that the Cesàro averages of T are dominated by those of U.

The same holds for T^*:

2.1. THEOREM: *For any $f \in L_\infty$ and any $\alpha > 0$, we have*

$$|M_\alpha^* f| \leq c_n \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} U_t^* |f| \, dt, \text{ where } \bar{\alpha} = \alpha^{2^{-k}} \text{ if } 2^{k-1} < n \leq 2^k.$$

REMARK: The representation theorem and the proof below show that we may replace $|M_\alpha^* f|$ by $\alpha^{-n} \int_0^\alpha \ldots \int_0^\alpha |T^*_{(t_1,\ldots,t_n)} f| \, dt_1 \ldots dt_n$.

PROOF: For any positive real function $g \in L_1$ one has

$$\langle |M^*_\alpha f|, g\rangle = \langle \epsilon_\alpha M^*_\alpha f, g\rangle \text{ where } \epsilon_\alpha \in L_\infty \text{ and } |\epsilon_\alpha| = 1 \text{ a.e.}$$

$$= \langle f, M_\alpha(\epsilon_\alpha g)\rangle \leq \langle |f|, c_n \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} U_t(|\epsilon_\alpha g|)\, dt\rangle$$

([5], 4.2, 4.1)

$$= \langle c_n \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} U^*_t |f|\, dt, g\rangle.$$

This clearly implies 2.1.

3. W^*-Continuity of T^* implies L_1-norm-continuity.

3.1. THEOREM: *Let $T^* = (T^*_t)_{t \in \mathbb{R}^n_+}$ be a W^*-continuous semi-group of L_∞-contractions. Then, for any $f \in L_\infty$*

$$\lim_{t_i \to 0^+} \|T^*_{(t_1, \ldots, t_n)} f - T^*_0 f\|_{L_1} = 0. \qquad i = 1, \ldots, n$$

REMARKS: • If μ is only σ-finite, we have:

$$(3.2) \qquad \lim_{t_i \to 0^+} \|(T^*_{(t_1, \ldots, t_n)} f - T^*_0 f)\, g\|_{L_1} = 0 \qquad \text{for all } f \in L_\infty \text{ and } g \in L_1.$$

• (1.4) clearly implies that $\|T^*_t M^*_I - M^*_I\|_\infty \leq \lambda_n(I)^{-1} \lambda_n((I + t)\, \Delta I)$, where Δ stands for the symmetrical difference.

Consequently, we have

$$(3.3) \qquad \lim_{t_i \to 0^+} \|T^*_t M^*_I - M^*_I\|_\infty = 0$$

and thus

$$(3.4) \qquad \lim_{t_i \to 0^+} \|M^*_{[0, (t_1, \ldots, t_n)]} M^*_I - M^*_I\|_\infty = 0.$$

• The assumption contractions may be replaced by T^* locally bounded and T^*_t positive.

PROOF OF 3.1: First notice that $T_{t+0} = T_t T_0$ implies $|T_t| \leq |T_t|\, |T_0|$ and thus we have

$$(3.5) \qquad \text{Strong} - \lim_{t_i \to 0^+} |T_{(t_1, \ldots, t_n)}| = |T_0|$$

(see [8], p. 374).

We prove 3.1 by induction on n.

For $n = 1$, let $S = (S_t)_{t \geq 0}$ be the modulus semi-group of T [8] so that $|T_t f| \leq S_t |f|$ if $f \in L_1$.

The argument of the proof of 2.1 then shows that we also have

$$(3.6) \qquad |T^*_t f| \leq S^*_t |f| \text{ if } f \in L_\infty.$$

On the conservative part of S we use some nice arguments of measure change due to R. Sato ([13], [14]). See also M. Lin [12].

Let $h = \int_0^\infty e^{-t} S_t 1 \, \mathrm{d}t$, $C = \{h > 0\}$ and $D = X \setminus C$. Let $\nu = \mathrm{h}.\mu$.
It is easy to see that

$$S_t h \leq e^t h \tag{3.7}$$

$$1_D S_t = 0 \tag{3.8}$$

and thus

$$S_t^*(1_D f) = 0 \text{ if } f \in L_\infty. \tag{3.9}$$

3.6 and 3.9 then imply

$$T_t^*(1_D f) = 0 \text{ for any } f \in L_\infty. \tag{3.10}$$

Following R. Sato [14], we consider the semi-group $(R_t)_{t \geq 0}$ in $L_1(C, \nu)$ defined by $R_t g = e^{-t} T_t(gh)/h$ for any $g \in L_1(C, \nu)$.

Then, for any $f \in L_\infty(C, \nu) = L_\infty(C, \mu) = L_\infty(C)$, we easily see that

$$R_t^* f = e^{-t} T_t^* f \text{ a.e. on } C \tag{3.11}$$

and that

$$\int |R_t^* f| \, \mathrm{d}\nu \leq \int |f| \, \mathrm{d}\nu \text{ because of 3.11, 3.6 and 3.7.} \tag{3.12}$$

3.12 then implies that R_t^* can be extended to a contraction on $L_1(C, \nu)$. Further, the W^*-continuity of T^* and 3.11 show that the map $\mathbb{R}_+ \to L_1(C, \nu)$ $t \to R_t^* g$ is continuous in the weak topology of $L_1(C, \nu)$ for any $g \in L_\infty(C, \nu)$ and thus for any $g \in L_1(C, \nu)$.

Hence, by ([7], p. 306), this map is also continuous in the norm topology at every point $s > 0$.

To see the continuity at 0, consider the set $H = \{g \in L_1(C, \nu) \mid \text{norm}-\lim_{t \to 0^+} R_t^* g = R_0^* g\}$.

Since the R_t^* are contractions, H is closed in the norm topology and since H is a vector space H is also weakly closed.

Let $g \in L_1(C, \nu)$. Since $R_s g \in H$ for any $s > 0$, $w-\lim_{s \to 0^+} R_s^* g = R_0^* g \in H$, that is $g \in H$ and $H = L_1(C, \nu)$.

In particular for any $f \in L_\infty(C)$ we have $\lim_{t \to 0^+} \int |e^{-t} T_t^* f - T_0^* f| h \, \mathrm{d}\mu = 0$ and by 3.10 we obtain

$$\lim_{t \to 0^+} \int |T_t^* f - T_0^* f| \, h \, \mathrm{d}\mu = 0 \text{ if } f \in L_\infty. \tag{3.13}$$

Next let $\epsilon > 0$.

3.14. There exists a number $\delta > 0$ such that $\int_B |T_0| \, 1 \, \mathrm{d}\mu < \epsilon$ whenever $B \in \mathcal{F}$ and $\mu(B) < \delta$. Furthermore $|T_0| \, 1 = S_0 \, 1 \in L_1(C, \mu)$ implies

3.15. There exists a number $\rho > 0$ and a measurable set $A \subset C = \{h > 0\}$ such that $1_A |T_0| \, 1 \leq \rho.\mathrm{h}$ a.e. and $\mu(C \setminus A) < \delta$.

Then, for any $f \in L_\infty(X)$ we obtain

$$\int |T_t^* f - T_0^* f| \, d\mu = \int |T_0^*(T_t^* f - T_0^* f)| \, d\mu$$

$$\leq \int |T_0|^* (|T_t^* f - T_0^* f|) \, d\mu = \int (|T_0| 1) \, |T_t^* f - T_0^* f| \, d\mu$$

$$\leq \rho \int h |T_t^* f - T_0^* f| \, d\mu + 2 \, \|f\|_\infty \int_{C \setminus A} |T_0| 1 \, d\mu \; (3.15)$$

$$\leq \rho \int h |T_t^* f - T_0^* f| \, d\mu + 2 \, \|f\|_\infty \, \epsilon \; (3.15 \text{ and } 3.14).$$

The property 3.13 then implies that $\limsup_{t \to 0^+} \|T_t^* f - T_0^* f\|_{L_1} \leq 2 \, \|f\|_\infty \, \epsilon$ for any $\epsilon > 0$, that is $\lim_{t \to 0^+} \|T_t^* f - T_0^* f\|_{L_1} = 0$.

In particular we also have

(3.16) $\qquad \lim_{t \to 0^+} \|(T_t^* f - T_0^* f) g\|_{L_1} = 0$ for any $f \in L_\infty$ and $g \in L_1$.

So, we have proved 3.1 if $n = 1$.

Now, suppose that 3.1 holds for an integer n and let us prove that it also holds for any $(n + 1)$-parameter w^*-continuous semi-group T^*.

First note that

$$T^*_{(t_1, \ldots, t_n, t_{n+1})} - T_0^* = T^*_{(t_1, \ldots, t_n, 0)} - T_0^* + T^*_{(t_1, \ldots, t_n, 0)} \, (T^*_{(0, \ldots, 0, t_{n+1})} - T_0^*).$$

Hence for any $f \in L_\infty$ we obtain

$$\|T^*_{(t_1, \ldots, t_n, t_{n+1})} f - T_0^* f\|_{L_1} \leq \|T^*_{(t_1, \ldots, t_n, 0)} f - T_0^* f\|_{L_1}$$

$$+ \int (|T_{(t_1, \ldots, t_n, 0)}| 1) \, |T^*_{(0, \ldots, 0, t_{n+1})} f - T_0^* f| \, d\mu$$

$$\leq \|T^*_{(t_1, \ldots, t_n, 0)} f - T_0^* f\|_{L_1} + \int (|T_0| 1) \, |T^*_{(0, \ldots, 0, t_{n+1})} f - T_0^* f| \, d\mu$$

$$+ 2 \|f\|_\infty \, \| |T_{(t_1, \ldots, t_n, 0)}| \, 1 - |T_0| 1 \|_{L_1}$$

As $t_i \to 0^+$ independently for $i = 1, \ldots, n, n + 1$, the first term of the last member tends to 0 because of the induction hypothesis, the second one also tends to 0 because of 3.16 and the last term tends to 0 because of 3.5.

The proof of the theorem is completed.

3.17. Remark: If we apply the above arguments to the semi-group of positive operators $(U_t)_{t \geq 0}$ obtained in section 2 and if $h = \int_0^\infty e^{-t} \, U_t 1 \, dt$, $C = \{h > 0\}$, $D = X \setminus C$, $\nu = h.\mu$, we get $U_t^* f = e^t R_t^* f$ a.e. on C for any $f \in L_\infty(C)$ (3.11), (where $(R_t^*)_{t \geq 0}$ is a strongly continuous semi-group of positive contractions in $L_1(C, \nu)$) and $U_t^*(1_D f) = 0$ for any $f \in L_\infty$.

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4. **The local ergodic theorem in L_∞.**

4.1. THEOREM: *Let T^* be as in section 1 then for any $f \in L_\infty$ $\lim_{\alpha\to 0^+} M^*_\alpha f = T^*_0 f$ a.e. and in L_1-norm.*

4.2. COROLLARY: Let $(\lambda V_\lambda)_{\lambda>0}$ be a resolvent in L_∞ such that W^*-$\lim_{\lambda\to\infty} \lambda V_\lambda f$ exists for any $f \in L_\infty$, then $\lim_{\lambda\to\infty} \lambda V_\lambda f$ exists a.e. (λV_λ is assumed to be a W^*-continuous contraction).

PROOF OF THEOREM 4.1: Let $f \in L_\infty$ and let $\hat{f} = \limsup_{\alpha\to 0} |M^*_\alpha f - T^*_0 f|$. By 3.4 we have $\lim_{\alpha\to 0^+} M^*_\alpha(M^*_\beta f) = M^*_\beta f = T^*_0(M^*_\beta f)$ a.e. for any $\beta > 0$. Thus, since $M^*_\alpha f = M^*_\alpha(T^*_0 f)$, we have

$$\hat{f} \leq \limsup_{\alpha\to 0^+} |M^*_\alpha(T^*_0 f - M^*_\beta f)| + |M^*_\beta f - T^*_0 f|$$

for any $\beta > 0$.

2.1 then yields $\hat{f} \leq c_n \limsup_{\alpha\to 0^+} \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} U^*_t(|T^*_0 f - M^*_\beta f|)\, dt + |M^*_\beta f - T^*_0 f|$

Denoting f_β the first term of the last member, we get

$$f_\beta = c_n \limsup_{\alpha\to 0^+} \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} U^*_0 U^*_t(T^*_0 f - M^*_\beta f|)\, dt$$

$$\leq U^*_0 f_\beta, \text{ since } f_\beta \in L_\infty \text{ and } U^*_0 \text{ is positive.}$$

Hence

$$\int f_\beta \, d\mu \leq \int (U_0\, 1) f_\beta \, d\mu$$

$$= c_n \int (U_0\, 1) \left(\limsup_{\alpha\to 0^+} \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} U^*_t(1_C|T^*_0 f - M^*_\beta f|)\, dt\right) d\mu$$

(see 3.17)

$$= c_n \int (U_0\, 1) \left(\limsup_{\alpha\to 0^+} \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} e^t R^*_t(1_C|T^*_0 f - M^*_\beta f|)\, dt\right) d\mu$$

(we may apply 3.17 because $U_0\, 1 \in L_1(C, \mu)$)

$$= c_n \int (U_0\, 1) \left(\limsup_{\alpha\to 0^+} \bar{\alpha}^{-1} \int_0^{\bar{\alpha}} R^*_t(1_C|T^*_0 f - M^*_\beta f|)\, dt\right) d\mu$$

$$= c_n \int (U_0\, 1) R^*_{t_0}(1_C|T^*_0 f - M^*_\beta f|)\, d\mu$$

(3.17 and U. Krengel's local theorem applied in $L_1(C, \nu)$ [10])

$$= c_n \int (U_0\, 1) U^*_0(1_C|T^*_0 f - M^*_\beta f|)\, d\mu$$

(see 3.17)

$$= c_n \int (U_0^2\, 1)\, (1_C |T_0^* f - M_\beta^* f|)\, d\mu$$

$$= c_n \int (U_0\, 1)\, |T_0^* f - M_\beta^* f|\, d\mu$$

(because $U_0\, 1 \in L_1(C)$ and $U_0^2 = U_0$).

Finally we have

$$\int \hat{f}\, d\mu \leq c_n \int (U_0\, 1)\, |T_0^* f - M_\beta^* f|\, d\mu + \int |M_\beta^* f - T_0^* f|\, d\mu$$

for any $\beta > 0$.

Letting $\beta \to 0$ and applying 3.2 and 3.1 we see that $\hat{f} = 0$ that is $\lim_{\alpha \to 0} M_\alpha^* f = T_0^* f$ a.e.

The L_1-norm-convergence is the property 3.1.

The proof is completed.

PROOF OF 4.2: λV_λ is the adjoint of a L_1-resolvent, say λW_λ, such that strong$-\lim_{\lambda \to \infty} \lambda W_\lambda = T_0$ exists.

Hence $W_\lambda = \int_0^\infty e^{-\lambda s}\, T_s\, ds$ where $(T_s)_{s \geq 0}$ is a strongly continuous semi-group of L_1-contractions.

Since $V_\lambda = \int_0^\infty e^{-\lambda s}\, T_s^* ds$, 4.2 is a consequence of 4.1 as Cesàro convergence implies Abel convergence.

Note that the condition of w^*-convergence at infinity is necessary for the pointwise convergence to hold for any $f \in L_\infty$.

4. **Additive processes in L_∞.**

In this last section we note that any bounded additive process in L_∞ is absolutely continuous.

I_n denotes the class of all intervals of $\mathbb{R}_+^n$.

DEFINITION: (M. A. Akcoglu − U. Krengel − A. Del Junco [2], [1]) A set function $F : I_n \to L_\infty$ will be called a bounded additive process with respect to $T = (T_t)_{t \in \mathbb{R}_+^n}$ if it satisfies the following conditions:

- $\sup \{\|F(I)/\lambda_n(I)\|_\infty \,|\, I \in I_n,\ \lambda_n(I) > 0\} = K(F) < +\infty$
- $T_u^* F(I) = F(I + u)$ for all $u \in \mathbb{R}_+^n$ and $I \in I_n$
- If $I_1, \ldots, I_k \in I_n$ are pairwise disjoint and if $I = \bigcup_{i=1}^{k} I_k \in I_n$ then $F(I) = \sum_{i=1}^{k} F(I_i)$.

In the following T_t^* need not be a contraction and the proof below also holds in $L_p (1 < p \leq \infty)$ [3], [4] or in any space for which bounded sets are W^*-compact.

4.1: THEOREM: *For any bounded additive process F, there exists a function $f \in L_\infty$ such that $F(I) = \int_I T_t^* f\, dt$ for any $I \in I_n$.*

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PROOF: First note that $T_0^*F(I) = F(I)$. Then the w^*-continuity of T^* at 0 and the arguments of M. Akcoglu–A. Del Junco in ([1] lemma 3.2) yield

4.2. Given $g \in L_1$, $I \in I_n$ and any $\epsilon > 0$ there is a $u \in (\mathbb{R}_+ - \{0\})^n$ such that if

$$A \in I_n^u = \{I \in I_n | I \subset [0, u], \lambda_n(I) > 0\}$$

then

$$|<F(I) - \int_I T_t^*(F(A)/\lambda_n(A))\, dt, g>| < \epsilon$$

Next, for any $x > 0$ put $F_x = F([0, x[^n)$.

The boundedness condition implies that there is a sequence $x_i \to 0^+$ such that $f = w^* - \lim_{i\to\infty} x_i^{-n} F_{x_i}$ exists.

Let $I \in I_n$, $g \in L_1$, $\epsilon > 0$ be given. Let u be as in 4.2. Let i be such that $[0, x_i]^n \subset I_n^u$ and $|\langle x_i^{-n} F_{x_i} - f, \int_I T_t\, g\, dt\rangle| < \epsilon$.

Then

$$|\langle F(I) - \int_I T_t^* f\, dt, g\rangle| \leq |\langle F(I) - \int_I T_t^*(x_i^{-n} F_{x_i})\, dt, g\rangle|$$

$$+ |\langle \int_I T_t^*(x_i^{-n} F_{x_i}) - f)\, dt, g\rangle| < \epsilon + |\langle x_i^{-n} F_{x_i} - f, \int_I T_t\, g\, dt\rangle| < 2.\epsilon.$$

ϵ and g being arbitrary we obtain $F(I) = \int_I T_t^* f\, dt$ for any $I \in I_n$.

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UNIVERSITÉ PARIS VI
LABORATOIRE DE PROBABILITÉS
4, PLACE JUSSIEU
75230 PARIS CEDEX 05