

Corrigendum: towards an understanding of ramified extensions of structured ring spectra

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The original version of this paper unfortunately contained a mistake in a calculation that resulted in erroneous statements in Lemma 5.1 and Theorem 5.2. We are grateful to Eva Höning who discovered the mistake.

LEMMA 0.1 (replaces Lemma 5.1). *There is an isomorphism of graded commutative augmented ku_* -algebras*

$$(ku \wedge_{ko} ku)_* \cong ku_*[s]/(s^2 - su)$$

with $|s| = 2$, where the ku_* -algebra structure on $(ku \wedge_{ko} ku)_*$ is from the left and the augmentation is given by the multiplication $m : ku \wedge_{ko} ku \rightarrow ku$ and by $s \mapsto 0$.

Proof. Smashing Wood’s cofiber sequence $ko \xrightarrow{\iota} ku \xrightarrow{j} \Sigma^2 ko$ with ku (from the left) over ko gives a split exact sequence (with unit isomorphism $ku \cong ku \wedge_{ko} ko$ suppressed)

$$0 \longrightarrow \pi_* ku \xrightarrow{(1 \wedge \iota)_*} \pi_*(ku \wedge_{ko} ku) \xrightarrow{(1 \wedge j)_*} \pi_* \Sigma^2 ku \longrightarrow 0.$$

Let $u \in \pi_2 ku$ be the generator with $j_*(u) = \Sigma^2 2 \in \pi_2 \Sigma^2 ko \cong \mathbb{Z}$. Let u_l and u_r be the images of u in $\pi_2(ku \wedge_{ko} ku)$ induced by the left and right inclusion of ku in $ku \wedge_{ko} ku$. If s is the

unique element in $\pi_2(ku \wedge_{ko} ku)$ with $(1 \wedge j)_*s = -\Sigma^2 1$ and $m_*s = 0$, then $(1 \wedge j)_*(u_r + 2s) = 1 \cdot j_*u - 2 = 0$ and $m_*(u_r + 2s) = u$. Since also $(1 \wedge j)_*u_l = 0$ and $m_*u_l = u$ we must have $u_r + 2s = u_l$.

In $ku_* \otimes_{ko_*} ku_*$, and hence also in $\pi_4(ku \wedge_{ko} ku)$, we have that $2u_r^2 - 2u_l^2 = 0$. As $\pi_*(ku \wedge_{ko} ku)$ is torsion free, we get $u_r^2 - u_l^2 = 0$ and therefore

$$u_r^2 - u_l^2 = (u_l - 2s)^2 - u_l^2 = 4s^2 - 4su_l = 0.$$

Again, since there is no torsion, this yields $s^2 - su_l = 0$.

THEOREM 0.2 (replaces Theorem 5.2). *The Tor spectral sequence*

$$E_{*,*}^2 = \text{Tor}_{*,*}^{(ku \wedge_{ko} ku)_*}(ku_*, ku_*) \implies \text{THH}_*^{ko}(ku)$$

collapses at the E^2 -page and $\text{THH}_*^{ko}(ku)$ is a square zero extension of ku_* :

$$\text{THH}_*^{ko}(ku) \cong ku_* \rtimes (ku_*/u)\{y_0, y_1, \dots\}$$

with $|y_j| = (1 + |u|)(2j + 1) = 3(2j + 1)$.

Proof. Lemma 0.1 implies that the E^2 -term of the Tor spectral sequence is

$$E_{*,*}^2 = \text{Tor}_{*,*}^{(ku \wedge_{ko} ku)_*}(ku_*, ku_*) = \text{Tor}_{*,*}^{ku_*[s]/(s^2-su)}(ku_*, ku_*).$$

We have a periodic free resolution of ku_* as a module over $ku_*[s]/(s^2 - su)$

$$\dots \xrightarrow{s} \Sigma^4 ku_*[s]/(s^2 - su) \xrightarrow{s-u} \Sigma^2 ku_*[s]/(s^2 - su) \xrightarrow{s} ku_*[s]/(s^2 - su).$$

Tensoring this down to ku_* yields

$$\dots \xrightarrow{0} \Sigma^4 ku_* \xrightarrow{-u} \Sigma^2 ku_* \xrightarrow{0} ku_*.$$

As ku_* splits off $\text{THH}_*^{ko}(ku)$, the zero column has to survive and cannot be hit by differentials and hence all differentials are trivial.

For the E^∞ -term we therefore get $E_{0,*}^\infty \cong ku_*$, $E_{2j,*}^\infty = 0$ for $j > 0$, and $E_{2j+1,*}^\infty \cong (ku_*/u)\{y_j\}$ for y_j in bidegree $(2j + 1, 4j + 2)$ if $j > 0$. So even total degrees occur only in $E_{0,*}^\infty$ and odd total degrees occur only in at most one bidegree and we do not need to worry about additive extensions. As y_j corresponds to $\Sigma^{4j+2} 1 \in \Sigma^{4j+2} ku_*$, the action of $u^i \in ku_*$ on y_j is $u^i y_j$ and this is trivial for $i \geq 1$ in $(ku_*/u)\{y_j\}$ so $u^i y_j$ is zero in $E_{2j+1,4j+2}^\infty$. But $E_{2j+1,4j+2}^\infty$ has all the elements of total degree $6j + 3$ in the entire E^∞ -term, so in fact the element in $\text{THH}_*^{ko}(ku)$ that y_j represents is killed by multiplication by u^i for any $i \geq 1$. Thus we have no nontrivial products of the u^i , $i \geq 1$, and the odd dimensional elements.

Since the elements of $\text{THH}_*^{ko}(ku)$ represented by the y_i are all in odd degrees, if there were nonzero products among them they would have to be elements in $E_{0,*}^\infty \cong ku_*$. But elements in ku_* are not killed by multiplying by u , whereas the elements represented by the y_j are. So there can be no such nontrivial products.