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On microfunctions at the boundary along CR manifolds

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Abstract. Let X be a complex analytic manifold, $M \subset X$ a C^2 submanifold, $\Omega \subset M$ an open set with C^2 boundary $S = \partial\Omega$. Denote by $\mu_M(\mathcal{O}_X)$ (resp. $\mu_\Omega(\mathcal{O}_X)$) the microlocalization along M (resp. Ω) of the sheaf $\mathcal{O}_X$ of holomorphic functions.

In the literature (cf. [A-G], [K-S 1,2]) one encounters two classical results concerning the vanishing of the cohomology groups $H^j\mu_M(\mathcal{O}_X)_p$ for $p \in \dot{T}^*_M X$. The most general gives the vanishing outside a range of indices j whose length is equal to $s^0(M,p)$ (with $s^{+,-,0}(M,p)$ being the number of respectively positive, negative and null eigenvalues for the 'microlocal' Levi form $L_M(p)$). The sharpest result gives the concentration in a single degree, provided that the difference $s^-(M,p') - \gamma(M,p')$ is locally constant for $p' \in T^*_M X$ near p (with $\gamma(M,p) = \dim^{\mathbf{C}}(T^*_M X \cap iT^*_M X)_z$ for z the base point of p).

The first result was restated for the complex $\mu_\Omega(\mathcal{O}_X)$ in [D'A-Z 2], in the case $\mathrm{codim}_M S = 1$. We extend it here to any codimension and moreover we also restate for $\mu_\Omega(\mathcal{O}_X)$ the second vanishing theorem.

We also point out that the principle of our proof, related to a criterion for constancy of sheaves due to [K-S 1], is a quite new one.

Key words: Solvability of the $\bar{\mathcal{O}}$-complex.

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1. Notations

Let X be a complex analytic manifold and $M \subset X$ a C^2 submanifold. One denotes by $\pi: T^*X \to X$ and $\pi: T^*_M X \to M$ the cotangent bundle to X and the conormal bundle to M in X respectively. Let $\dot{T}^*X$ be the cotangent bundle with the zero section removed, and let $\rho: M \times_X T^*X \to T^*M$ be the projection associated to the embedding $M \hookrightarrow X$.

For a subset $A \subset X$ one defines the strict normal cone of A in X by $N^X(A) := TX \setminus C(X \setminus A, A)$ where $C(\cdot,\cdot)$ denotes the normal Whitney cone (cf [K-S 1]).

Let $z_0 \in M$, $p \in (\dot{T}^*_M X)_{z_0}$. We put $T^{\mathbf{C}}_{z_0} M = T_{z_0}M \cap iT_{z_0}M$; $\lambda_M(p) = T_p T^*_M X$; $\lambda_0(p) = T_p(\pi^{-1}\pi(p))$, $\nu(p) =$the complex Euler radial field at p, and we set $\gamma(M,p) = \dim^{\mathbf{C}}(T^*_M X \cap iT^*_M X)_{z_0}$. If no confusion may arise, we will sometimes drop the indices z_0 or p in the above notations.

Let ϕ be a C^2 function in X with $\phi\mid_M \equiv 0$ and $p = (z_0; d\phi(z_0))$. In a local system of coordinates (z) at z_0 in X we define $L_\phi(z_0)$ as the Hermitian form with matrix $(\partial z_i \overline{\partial} z_j \phi)_{ij}$. Its restriction $L_M(p)$ to $T^{\mathbf{C}}_{z_0}M$ does not depend on the choice of ϕ and is called the Levi form of M at p. Let $s^{+,-,0}(M,p)$ denote the number of respectively positive, negative and null eigenvalues of $L_M(p)$.

One denotes by $\mathrm{D}^b(X)$ the derived category of the category of bounded complexes of sheaves of $\mathbf{C}$-vector spaces and by $\mathrm{D}^b(X;p)$ the localization of $\mathrm{D}^b(X)$ at $p \in T^*X$, i.e. the localization of $\mathrm{D}^b(X)$ with respect to the null system $\{F \in \mathrm{D}^b(X); p \notin \mathrm{SS}(F)\}$ (here $\mathrm{SS}(F)$ denotes the micro-support in the sense of [K-S 2], a closed conic involutive subset of T^*X).

Remark 1.1. We recall that a complex F which verifies $\mathrm{SS}(F) \subset T^*_MX$ in a neighborhood of $p \in T^*_MX$ is microlocally isomorphic (i.e. isomorphic in $\mathrm{D}^b(X;p)$) to a constant sheaf on M. This criterion, stated in [K-S 1] for a C^2 manifold M, extends easily to C^1 manifolds (cf [D'A-Z 1]).

Let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions on X and $\mathbf{C}_A$, ($A \subset X$ locally closed), the sheaf which is zero in $X \setminus A$ and the constant sheaf with fiber $\mathbf{C}$ in A. We shall consider the complex $\mu_A(\mathcal{O}_X) := \mu\mathrm{hom}(\mathbf{C}_A, \mathcal{O}_X)$ of microfunctions along A (where $\mu\mathrm{hom}(\cdot,\cdot)$ is the bifunctor of [K-S 1]). Special interest lies in the complexes $\mu_M(\mathcal{O}_X)$ and $\mu_\Omega(\mathcal{O}_X)$ for Ω being an open subset of the manifold M (cf [S]).

2. Statement of the results

Let X be a complex analytic manifold of dimension n, $M \subset X$ a C^2 submanifold of codimension l, $\Omega \subset M$ an open set with C^2 boundary $S = \partial\Omega$, and set $r = \mathrm{codim}_M S$ (we assume Ω locally on one side of S for $r = 1$). Let $z_0 \in M$, $p \in (\dot{T}^*_MX)_{z_0}$. Define

$$d_M(p) = \mathrm{codim}_X M + s^-(M,p) - \gamma(M,p),$$
$$c_M(p) = n - s^+(M,p) + \gamma(M,p).$$

Let us recall the following classical results concerning the cohomology of $\mu_M(\mathcal{O}_X)$.

THEOREM A. ([A-G], [K-S 1]) *Assume* $\dim^{\mathbf{R}}(\nu(p) \cap \lambda_M(p)) = 1$. *Then*

$$H^j\mu_M(\mathcal{O}_X)_p = 0 \quad \text{for } j \notin [d_M(p), c_M(p)].$$

THEOREM B. ([H], [K-S 1]) *Assume* $\dim^{\mathbf{R}}(\nu(p) \cap \lambda_M(p)) = 1$ *and* $s^-(M,p') - \gamma(M,p') \equiv \mathrm{const}$ *for* $p' \in T^*_MX$ *close to* p. *Then*

$$H^j\mu_M(\mathcal{O}_X)_p = 0 \quad \text{for } j \neq d_M(p).$$

Dealing with $\mu_\Omega(\mathcal{O}_X)$ (and choosing now $p \in S \times_M \dot{T}^*_M X$), one knows that

THEOREM C. ([D'A-Z 2]) *Assume codim* $_M S = 1$ *and* $\dim^{\mathbf{R}}(\nu(p) \cap \lambda_M(p)) = 1$. *Then*

$$H^j \mu_\Omega(\mathcal{O}_X)_p = 0 \quad \text{for } j \notin [d_M(p), c_M(p)].$$

The aim of the present note is, on the one hand, to extend Theorem C to the case of any codimension for S in M, and, on the other hand, to state the analogue of Theorem B for the complex $\mu_\Omega(\mathcal{O}_X)$. We point out that the method of our proof, based on the criterion of [K-S 1, Proposition 6.2.2] (with its C^1-variant of [D'A-Z 1]), is a quite new one.

Our results, valid for any $r = \text{codim}_M S$, go as follows.

THEOREM 2.1. *Assume*

$$\dim^{\mathbf{R}}(\nu(p) \cap \lambda_S(p)) = 1. \tag{2.1}$$

Then

$$H^j \mu_\Omega(\mathcal{O}_X)_p = 0 \quad \text{for} \quad j \notin [d_M(p), c_M(p) + r - 1]. \tag{2.2}$$

When M is a real analytic manifold of dimension n and X a complexification of M, then Theorem 2.1 states the concentration in degree n for $\mu_\Omega(\mathcal{O}_X)_p$. This should be proved as well by the aid of Proposition 3.1 of [S]. In fact, since Ω has C^2-boundary, then $M \setminus \Omega$ is C^ω-convex (i.e. convex in suitable real analytic coordinates at z_o).

THEOREM 2.2. *Assume* (2.1) *and moreover*

$$\begin{cases} s^-(M, p') - \gamma(M, p') \text{ is constant for } p' \in \overline{\Omega} \times_M T^*_M X \text{ near } p, \\ s^-(S, p') - \gamma(S, p') \text{ is constant} \\ \quad \text{for } p' \in T^*_S X \cap \rho^{-1}(N^M(\Omega)^{\circ a}) \text{ near } p, \\ s^-(M, p) - \gamma(M, p) = s^-(S, p) - \gamma(S, p). \end{cases} \tag{2.3}$$

Then

$$H^j \mu_\Omega(\mathcal{O}_X)_p = 0 \quad \text{for} \quad j \notin [d_M(p), d_M(p) + r - 1].$$

Remark 2.3. We notice that the sets appearing in (2.3) are very natural in this context; one has in fact

$$\begin{cases} T^*_M X \cap \text{SS}(\mathbf{C}_\Omega) = \overline{\Omega} \times_M T^*_M X, \\ T^*_S X \cap \text{SS}(\mathbf{C}_\Omega) = T^*_S X \cap \rho^{-1}(N^M(\Omega)^{\circ a}). \end{cases}$$

3. Proofs of the results

Proof of Theorem 2.1. We set $\Omega^- = M \setminus \overline{\Omega}$ and use the distinguished triangle

$$\mu_S(\mathcal{O}_X) \to \mu_M(\mathcal{O}_X) \to \mu_\Omega(\mathcal{O}_X) \oplus \mu_{\Omega^-}(\mathcal{O}_X) \xrightarrow{+1}. \tag{3.4}$$

We remark that by its own definition: $L_S(p) = L_M(p)|_{T^{\mathbf{C}}_{z_o}S}$, $(p \in S \times_M \dot{T}^*_M X)$. This gives:

$$\begin{aligned} s^{+,-}(S,p) \leqslant s^{+,-}(M,p) &\leqslant s^{+,-}(S,p) + (\dim T^{\mathbf{C}}_{z_o}M - \dim T^{\mathbf{C}}_{z_o}S) \\ &= s^{+,-}(S,p) + (r + \gamma(M,p) - \gamma(S,p)). \end{aligned}$$

Thus if the integers $c_M(p), d_M(p)$ and $c_S(p), d_S(p)$ are defined as in Section 2, we have at once

$$\begin{aligned} c_M(p) &\leqslant c_S(p) \leqslant c_M(p) + r, \\ d_M(p) &\leqslant d_S(p) \leqslant d_M(p) + r. \end{aligned} \tag{3.5}$$

The vanishing of (2.2) for $j > c_M(p) + r - 1$ then follows by applying Theorem A to M and S.

The vanishing of (2.2) for $j < d_M(p)$ is immediate for $d_S(p) > d_M(p)$ due to Theorem A and (3.1).

When $d_S(p) = d_M(p)$ it remains to be proven that

$$H^{d_M(p)}\mu_S(\mathcal{O}_X)_p \to H^{d_M(p)}\mu_M(\mathcal{O}_X)_p \quad \text{is injective.} \tag{3.6}$$

To this end we perform a contact transformation χ near p which interchanges (setting $q = \chi(p)$)

$$\begin{cases} T^*_M X \to T^*_{\widetilde{M}} X & \text{codim}\, \widetilde{M} = 1, s^-(\widetilde{M}, q) = 0, \\ T^*_S X \to T^*_{\widetilde{S}} X & \text{codim}\, \widetilde{S} = 1, \end{cases} \tag{3.7}$$

(cf. [D'A-Z 3]). Let $\widetilde{M}^+$ and $\widetilde{S}^+$ be the closed half spaces with boundary $\widetilde{M}$ and $\widetilde{S}$ and inner conormal q. We have

PROPOSITION 3.1. *Let* $d_S = d_M$. *Then in the above situation*

$$\begin{cases} s^-(\widetilde{S}, q) = 0, \\ \widetilde{S}^+ \subset \widetilde{M}^+. \end{cases} \tag{3.8}$$

Proof. Quantizing χ by a kernel $K \in \mathrm{Ob}(\mathrm{D}^b(X \times X))$ we get by [K-S 1, Proposition 11.2.8]

$$\begin{cases} \phi_K(\mathbf{C}_M) \cong \mathbf{C}_{\widetilde{M}+}[d_M(p)-1] \quad \text{in } \mathrm{D}^b(X;q), \\ \phi_K(\mathbf{C}_S) \cong \mathbf{C}_{\widetilde{S}+}[d_S(p)-s^-(\widetilde{S},q)-1] \quad \text{in } \mathrm{D}^b(X;q). \end{cases}$$

Moreover the natural morphism $\mathbf{C}_M \to \mathbf{C}_S$ is transformed via ϕ_K to a non null morphism $\mathbf{C}_{\widetilde{M}+}[d_M(p)-1] \to \mathbf{C}_{\widetilde{S}+}[d_S(p)-s^-(\widetilde{S},q)-1]$. Thus

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{D}^b(X;q)}(\mathbf{C}_{\widetilde{M}+}[d_M(p)-1], \mathbf{C}_{\widetilde{S}+}[d_S(p)-s^-(\widetilde{S},q)-1]) \\ &\quad = H^0(\mathrm{R}\Gamma_{\widetilde{M}+}(\mathbf{C}_{\widetilde{S}+})_y[d_S(p)-d_M(p)-s^-(\widetilde{S},q)]) \\ &\quad \neq 0, \end{aligned}$$

where $y = \pi(q)$. Since we are assuming $d_M(p) = d_S(p)$, (3.5) follows. □

End of the proof of Theorem 2.1. From the proof of Proposition 3.1 it follows that ϕ_K transforms the morphism (3.3) in

$$\mathcal{H}^1_{\widetilde{S}+}(\mathcal{O}_X)_y \to \mathcal{H}^1_{\widetilde{M}+}(\mathcal{O}_X)_y, \tag{3.9}$$

where $y = \pi(q)$, which is clearly injective. The proof of Theorem 2.1 is now complete. □

Proof of Theorem 2.2. From now on we will drop p in our notations, due to the constancy assumptions (2.3).

If $r > 1$ one has $\overline{\Omega} = M$ and $N^M(\Omega)^{\circ a} = T^*M$. Thus, by (2.3), we enter the hypotheses of Theorem B for both M and S. The claim follows in this case from (3.1), (3.3) and from the inequalities (3.2).

We may then assume $r = 1$. The problem in this case is that (2.3) holds only along $\mathrm{SS}(\mathbf{C}_\Omega)$.

Let χ: $T^*X \to T^*X$ be a contact transformation from a neighborhood of p to a neighborhood of $q = \chi(p)$, such that

$$\begin{cases} T^*_M X \to T^*_{\widetilde{M}} X & \operatorname{codim} \widetilde{M} = 1, \\ T^*_S X \to T^*_{\widetilde{S}} X & \operatorname{codim} \widetilde{S} = 1, s^-(\widetilde{S}, q') \equiv 0. \end{cases}$$

Notice that, for $y = \pi(q)$, $T_y\widetilde{M} = T_y\widetilde{S}$. Quantizing χ by a kernel K, we thus have that either $\phi_K(\mathbf{C}_{\overline{\Omega}})$ or $\phi_K(\mathbf{C}_\Omega)$ is a simple sheaf along the conormal bundle to a C^1 submanifold $Y \subset X$. Since $d_M = d_S - 1$, then $s^-(\widetilde{M}, q) = 0$, $\widetilde{M}^+ \subset \widetilde{S}^+$ and $\phi_K(\mathbf{C}_\Omega) = \mathbf{C}_Y[d_M - 1]$. Denoting by W the open domain with boundary Y and exterior conormal q, we have by Lemma 3.3 of [Z] that W is pseudoconvex at y, and one concludes since

$$\chi_*\mu_\Omega(\mathcal{O}_X)_q[-d_M] \cong \mathcal{H}^1_{X\setminus W}(\mathcal{O}_X)_y, \qquad \square$$

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