NOTE ON A MATRIX THEOREM OF A. BRAUER AND ITS EXTENSION

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1. Introduction. In one of his papers on limits for the characteristic roots of a Matrix, Brauer (1) has stated a theorem, which connects the roots of a given square matrix A, with those of a matrix A^* derived from A by a certain process. The proof of this theorem involves a continuity argument and in a recent paper on the construction of stochastic matrices Hazel Perfect (5) has given a proof which avoids considerations of continuity. However, her proof, involving several multiple derivatives (not with respect to the elements of the matrix), is unnecessarily heavy, and in the present note I give a proof which is simple, short and avoids both continuity and differentiation.

Two extensions of Brauer's theorem are then considered. In each matrix A^* is of the form $A^* = A + XK'$ where X is an $n \times r$ matrix (n being the order of A) whose columns are latent vectors of A, and K is an arbitrary $n \times r$ matrix. These extensions arise according as the columns of X are associated with the same latent root of A, or different roots.

2. Brauer's theorem. In what follows, symbols in bold type represent column vectors. A row vector is represented by an attached prime, which is also used to denote the transpose of a matrix. The unit matrix is denoted by I. Brauer's result may be stated as follows:

THEOREM 1. *Let the square matrix A of order n have latent roots* $\lambda_1, \ldots, \lambda_s$ *with multiplicities* $m_1, \ldots, m_s$*; let x be a latent column vector of A associated with the root* λ_1*, and let k be an arbitrary column vector. Then the matrix* $A^* = A + \mathbf{x}\mathbf{k}'$ *has latent roots* $\lambda_1 + \mathbf{k}'\mathbf{x}, \lambda_1, \ldots, \lambda_s$ *with multiplicities* $1, m_1 - 1, m_2, \ldots, m_s$.

Proof. We have

$$|\lambda I - A| = \prod_{i=1}^{s} (\lambda - \lambda_i)^{m_i}$$

and also $A\mathbf{x} = \lambda_1\mathbf{x}$. Now, since $\mathbf{x}\mathbf{k}'$ is of rank 1, we have

$$\begin{aligned} |\lambda I - A^*| &= |\lambda I - A - \mathbf{x}\mathbf{k}'| \\ &= |\lambda I - A| - \sum_{i=1}^{n} k_i |\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n|, \end{aligned}$$

where, in the last determinant, we have written $\mathbf{a}_1, \ldots, \mathbf{a}_n$ for the columns of $\lambda I - A$.

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If we write $A_{ij}(\lambda)$ for the co-factor of the (i, j)th element of $\lambda I - A$, we have

$$|\lambda I - A^*| = |\lambda I - A| - \sum_{i=1}^{n} k_i \sum_{j=1}^{n} x_j A_{ji}(\lambda).$$

Now $A_{ji}(\lambda)$ is the (i, j)th element of adj $(\lambda I - A)$. Also, from the well-known identity

$$\operatorname{adj}(\lambda I - A) \,.\, (\lambda I - A) = |\lambda I - A| I,$$

it follows at once that

$$(\lambda - \lambda_1) \operatorname{adj}(\lambda I - A) \,.\, \mathbf{x} = |\lambda I - A| \mathbf{x}.$$

This gives

$$\sum_{j=1}^{n} A_{ji}(\lambda) x_j = x_i F(\lambda) \qquad (i = 1, \ldots, n),$$

where $F(\lambda) = |\lambda I - A|/(\lambda - \lambda_1)$. Hence

$$\begin{aligned} |\lambda I - A^*| &= |\lambda I - A| - (\mathbf{k}'\mathbf{x})\, F(\lambda) \\ &= \{\lambda - \lambda_1 - (\mathbf{k}'\mathbf{x})\} (\lambda - \lambda_1)^{m_1 - 1} (\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_s)^{m_s} \end{aligned}$$

and this proves the theorem.

3. Extensions. The effect of Brauer's modification of the matrix A is to bring about a "splitting" of the spectrum of latent roots, the new root $\lambda_1 + \mathbf{k}'\mathbf{x}$ differing from λ_1 by an infinitesimal provided the elements of $\mathbf{k}$ are small quantities of the first order. It is a natural question to ask how far this splitting process may be carried, and consideration of this question leads to an extension of Brauer's theorem.

We begin with the well-known

DEFINITION. *Two square matrices A and B, of the same order n, are said to possess property P (the Frobenius property) if the characteristic roots of A and B, say $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ may be so ordered that the characteristic roots of any polynomial $f(A, B)$ are $f(\alpha_i, \beta_i)$ $(i = 1, \ldots, n)$.*

It is known (2; 4) that a necessary and sufficient condition that A and B possess property P is that the matrix $AB - BA$ belong to the radical of the algebra generated over the base field by A and B. We shall prove

THEOREM 2. *Let A be a square matrix of order n and let $B = XK'$ where* (i) *X is an $n \times r$ matrix whose column vectors are characteristic vectors (independent or otherwise) of A associated with the same latent root λ,* (ii) *K is an arbitrary $n \times r$ matrix. Then A and B have property P.*

Proof. It is sufficient to prove that $f(A, B)(AB - BA)$ is nilpotent for all polynomials $f(A, B)$. Write $N = AB - BA$. Since the column vectors of X are all characteristic vectors of A associated with the root λ, we have

$$AX = \lambda X.$$

Thus $AB = (AX)K' = \lambda B$, $N = AB - BA = B(\lambda I - A)$. It follows that $N^2 = 0$.

Next, $NB = 0$ and, for the polynomial $f(A, B)$, we have

$$f(A, B)B = f(\lambda I, B)B, \quad f(A, B)N = f(\lambda I, B)N.$$

Thus $[f(A, B)N]^2 = [f(\lambda I, B)N]^2 = 0$.

We now use Theorem 2 to obtain an extension of Brauer's theorem. Let X have rank $s(s \leqslant r)$. Then B has at most s non-zero characteristic roots. Since A and B have property P and $AB = \lambda B$ the non-zero roots of B all associate with the root λ of A. Now suppose the roots of B are $\mu_1, \ldots, \mu_\sigma$ $(\sigma \leqslant s, \mu_i \neq 0)$ and a zero root of multiplicity $n - \sigma$; and let the roots of A be $\lambda, \lambda_1, \ldots, \lambda_t$ of multiplicities $\tau, m_1, \ldots, m_t$ respectively. Since the μ_i all associate with λ we have $\sigma \leqslant \tau$. Theorem 2 now leads to

THEOREM 3. *If A and B are the matrices occurring in Theorem 2 and $f(A, B)$ is any polynomial in A and B, then $f(\lambda, \mu_i)$ $(i = 1, \ldots, \sigma)$ are roots of $f(A, B)$ and the remaining roots are $f(\lambda, 0)$, $f(\lambda_1, 0), \ldots, f(\lambda_t, 0)$ with multiplicities $\tau - \sigma, m_1, \ldots, m_t$ respectively.*

On putting $f(A, B) = A + B$ and noting that the non-zero roots of XK' and $K'X$ are the same we obtain

THEOREM 4. *If $A^* = A + XK'$, where X and K are as defined in Theorem 2, and if the non-zero roots of $K'X$ are $\mu_1, \ldots, \mu_\sigma$ $(\sigma \leqslant s \leqslant r)$ then the roots of A^* are $\mu_i + \lambda (i = 1, \ldots, \sigma)$ and λ, $\lambda_1, \ldots, \lambda_t$ with multiplicities $\tau - \sigma$, $m_1, \ldots, m_t$ respectively.*

Another extension of Brauer's theorem is obtained by taking the columns of X to be latent vectors of A associated with distinct latent roots $\lambda_1, \ldots, \lambda_r$ of A. By a proper choice of basis, that is, by a suitable similarity transformation, A and X may be simultaneously reduced to the forms

$$A = \begin{pmatrix} \Lambda & A_1 \\ 0 & A_2 \end{pmatrix}, \quad X = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$ and I_r is the unit matrix of order r.

If $K = (K_1, K_2)$ where K_1 is of order r, we have $K'X = K_1$ and

$$A + XK' = \begin{pmatrix} \Lambda + K_1 & A_1 + K_2 \\ 0 & A_2 \end{pmatrix}.$$

From this there follows:

THEOREM 5. *If $A^* = A + XK'$, where the column vectors of the $n \times r$ matrix X are latent vectors of A associated with distinct latent roots $\lambda_1, \ldots, \lambda_r$ of A, then the numbers $\epsilon_1, \ldots, \epsilon_r$ are latent roots of A^*, where $\epsilon_1, \ldots, \epsilon_r$ are the roots of $\Lambda + K'X$; also, every root of A^* other then $\epsilon_1, \ldots, \epsilon_r$ is a root of A, with the same multiplicity.*

4. Matrices having the same characteristic equation. In Theorem 5 let K be such that

$$K'X = PTP^{-1},$$

where P is a permutation matrix and T is a matrix whose elements t_{ij} satisfy $t_{ij} = 0$ for $i \leqslant j$ or $t_{ij} = 0$ for $i \geqslant j$, that is, T is lower or upper nilpotent triangular. Then

$$\Lambda + K'X = P(P^{-1}\Lambda P + T)P^{-1}.$$

Now $P^{-1}\Lambda P$ is a diagonal matrix and the diagonal of T contains only zeros. Thus Λ and $\Lambda + K'X$ have the same characteristic equation, and hence, by Theorem 5, A and $A + XK'$ have this property also.

It is perhaps worth pointing out that this result follows from the following theorem proved recently **(3)**.

THEOREM. *Let A and B be matrices of orders n and r, such that there exists an $n \times r$ matrix X, of rank r, for which $AX = XB$. If K is any $n \times r$ matrix, then the pair of matrices A, XK' has property P if and only if the pair B, $K'X$ has this property.*

In the case at hand we have $AX = X\Lambda$, that is $B = \Lambda$. Now if $K'X = PTP^{-1}$ then Λ and $K'X$ have property P, since $P^{-1}\Lambda P$ and T have this property. Hence by this theorem A and XK' have this property. Since XK' is nilpotent it follows that A and $A + XK'$ have the same characteristic equation.

In conclusion, I wish to thank the referee for several suggestions leading to improved proofs of the various theorems.

REFERENCES

1. A. Brauer, *Limits for the characteristic roots of a matrix IV: Applications to stochastic matrices*, Duke Math. J., *19* (1952), 75.

2. M. P. Drazin, J. W. Dungey, and K. W. Gruenberg, *Some theorems on commutative matrices*, J. Lond. Math. Soc., *26* (1951), 221.

3. L. S. Goddard and H. Schneider, *Pairs of matrices with a non-zero commutator*, Proc. Cambridge Phil. Soc., *51* (1955).

4. N. H. McCoy, *On the characteristic roots of matrix polynomials*, Bull. Amer. Math. Soc., *42* (1936), 592.

5. H. Perfect, *Methods of constructing certain stochastic matrices*, Duke Math. J., *20* (1953), 395.

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