A THEOREM IN ASYMPTOTIC NUMBER THEORY

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1. Introduction

Let $\alpha(n)$ be a multiplicative arithmetic function. H. Delange [1] has proved that if $|\alpha(n)| \leqq 1$ for all n and for a certain constant ρ, $\sum_{p \leqq x} \alpha(p) \sim \rho x/\log x$ as $x \to \infty$, where if $\rho = 1$ then $\sum_p (1-\mathscr{R}\alpha(p))/p = +\infty$, then $\sum_{n \leqq x} \alpha(n) = o(x)$ as $x \to \infty$. He applied this result to several problems such as uniform distribution (mod 1) of certain types of sequences.

The present author found that the error term in Delange's main lemma can be improved from $o(x)$ to $O(x/\log x)$. Delange suggested that we find a O-theorem analogous to his o-theorem above. This paper presents such a result.

In the text, the letter p ranges over the primes while m and n range over the positive integers.

$p \| n$ means $p|n$ but $p^2 \nmid n$.

$K, K_1, K_2, \cdots$ are positive absolute constants.

$\theta, \theta_1, \theta_2, \cdots$ are numbers with modulus $\leqq 1$ which depend on s, ω or x, where $s = \omega+1$, $0 < \omega < 1$ and $x > 1$.

$$li\, x = \int_2^x (1/\log t)dt. \tag{1}$$

$$A(x) = \sum_{n \leqq x} \alpha(n), \qquad \Phi(x) = \Phi_\alpha(x) = \sum_{p \leqq x} \alpha(p). \tag{2}$$

$$\vartheta(x) = \vartheta_\alpha(x) = \sum_{p \leqq x} \alpha(p) \log p. \tag{3}$$

L, L_1, L_2 and L_3 are *slowly varying functions* in the sense of J. Karamata [3] (also called *slowly oscillating*; see [4]); i.e., functions on $[x_0, \infty)$ for some $x_0 > 0$ which are positive-valued and continuous and satisfy

$$\lim_{x \to \infty} L(cx)/L(x) = 1 \tag{4}$$

for every $c > 0$. Such a function is characterized as asymptotically proportional to a function of the form

$$L(x) = \exp \int_1^x t^{-1}\delta(t)dt \tag{5}$$

where δ is bounded and measurable and

$$\lim_{x\to\infty} \delta(x) = 0.$$

For a slowly varying function L we define

$$L^*(x) = \int_1^x u^{-1}L(u)du, \quad L_*(x) = \exp K\int_1^x t^{-1}L(e^t)dt \tag{6}$$

where K is a suitable positive constant.

With the aid of l'Hospital's rule we have

(7) $L^*(x)$ *is slowly varying.*

$$L(x) = o[L^*(x)] \tag{8}$$

as $x \to \infty$, and for $\kappa > 0$

$$\int_1^x u^{\kappa-1}L(u)du \sim x^\kappa L(x)/\kappa \tag{9}$$

as $x \to \infty$.

If $L(x) \to 0$ then L_* is slowly varying by (5).

Note that products, powers and quotients of slowly varying functions are slowly varying. If L and M are slowly varying and M is non-decreasing with $M(x) \geqq x_0$ eventually, then $L(M(x))$ is slowly varying. $\operatorname{Log} x$ is slowly varying and so is $\exp \log^\alpha x$ if $-1 < \alpha < 1$.

2. The theorems

Consider the hypotheses

(H1) $\alpha(n)$ *is multiplicative.*

(H2) *For all* $n \geqq 1$, $|\alpha(n)| \leqq 1$.

(H3) *For a certain constant* $\rho \neq 1$ *either*

(a) $$\Phi_\alpha(x) = \sum_{p\leqq x} \alpha(p) = \rho\, li\, x + O[xL(x)/\log x]$$

or

(b) $$\Phi_\alpha(x) = \rho x/\log x + O[xL(x)/\log x],$$

where L *is a monotonic slowly varying function and*

$$L(x) \to 0$$

as $x \to \infty$.

(H4) *For a certain real* β *all values of* $\alpha(n)$ *lie in the right angle region* $\beta \leqq \arg z \leqq \beta + \pi/2$ *with vertex at the origin.*

THEOREM 1. *Under hypotheses* (H1), (H2) *and* (H3a *or* b) *we have for* $x \geqq e^2$

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$$\text{(C1)} \qquad \sum_{n \leqq x} \alpha(n) = O[xL^*(x)/\log x] + O[\rho x/\log\log x].$$

THEOREM 2. *Under hypotheses* (H1), (H2), H3a *or* b) *and* (H4), *for* $x \geqq e$

$$\text{(C2)} \qquad \sum_{n \leqq x} \alpha(n) = O(x\,(\log x)^{\rho-1} L_*(\log x)) + O(x/\log x)$$

where the K *in* $L*$ *is not greater than the O-constant in* (H3) (*see* (1.6)).

3. The Lemmas

Lemma 1 is our improvement of Delange's main lemma.

LEMMA 1. *If* $\alpha(n)$ *is bounded and multiplicative then as* $x \to \infty$

$$(1) \qquad \sum_{n \leqq x} \alpha(n) = (1/\log x) \sum_{n \leqq x} \alpha(n) \vartheta_\alpha(x/n) + O(x/\log x).$$

ϑ_α is defined in (1.3).

PROOF OF LEMMA 1. Using both hypotheses we have

$$(2) \qquad \begin{aligned} \sum_{m \leqq x} \alpha(m)\vartheta_\alpha(x/m) &= \sum_{n \leqq x} \sum_{p|n} \alpha(n/p)\alpha(p) \log p \\ &= \sum_{n \leqq x} \alpha(n) \sum_{p||n} \log p + O\Big(\sum_{n \leqq x} \sum_{p^2|n} \log p\Big) \\ &= \sum_{n \leqq x} \alpha(n) \log n - \sum_{n \leqq x} \alpha(n) \log [n/q(n)] + O(x), \end{aligned}$$

where $q(n) = \prod_{p||n} p$. In particular, if $\alpha(n) = 1$ identically

$$\vartheta_\alpha(x) = \vartheta(x) = \sum_{p \leqq x} \log p$$

and (2) says

$$\sum_{n \leqq x} \vartheta(x/n) = \sum_{n \leqq x} \log n - \sum_{n \leqq x} \log [n/q(n)] + O(x).$$

Hence, since

$$\sum_{n \leqq x} \log n = \sum_{n \leqq x} \psi(x/n),$$

where

$$\psi(x) = \sum_{n \leqq x} \Lambda(n) = \sum_{p^k \leqq x} \log p,$$

we have

$$\begin{aligned} \Big| \sum_{n \leqq x} \alpha(n) \log [n/q(n)] \Big| &\leqq K_1 \sum_{n \leqq x} \log [n/q(n)] \\ &= K_1 \sum_{n \leqq x} \log n - K_1 \sum_{n \leqq x} \vartheta(x/n) + O(x) \\ &= K_1 \sum_{n \leqq x} [\psi(x/n) - \vartheta(x/n)] + O(x) = O(x). \end{aligned}$$

Substituting in (2) we have

$$\sum_{n \leqq x} \alpha(n) \vartheta_\alpha(x/n) = \sum_{n \leqq x} \alpha(n) \log n + O(x). \tag{3}$$

Now if $A(x) = \sum_{n \leqq x} \alpha(n)$ then

$$\begin{aligned} A(x) \log x - \sum_{n \leqq x} \alpha(n) \log n &= \sum_{n \leqq x} \alpha(n) \log (x/n) \\ &= \sum_{n \leqq x} \alpha(n) \int_n^x t^{-1} dt = \int_2^x t^{-1} A(t) dt = O(x) \end{aligned} \tag{4}$$

since $\alpha(n)$ is bounded. Hence by (3)

$$\sum_{n \leqq x} \alpha(n) \vartheta_\alpha(x/n) = A(x) \log x + O(x),$$

which is equivalent to (1).

LEMMA 2. (a) *Hypothesis* (H3a) *implies that for* $x \geqq 2$

$$\vartheta_\alpha(x) = \rho x + O(xL(x)).$$

(b) *Hypothesis* (H3b) *implies for* $x \geqq 2$

$$\vartheta_\alpha(x) = \rho x + O[xL(x)] + O[x/\log x].$$

PROOF. Let $\vartheta = \vartheta_\alpha$, $\Phi = \Phi_\alpha$. Then as in (4) we have

$$\vartheta(x) = \Phi(x) \log x - \int_2^x t^{-1} \Phi(t) dt. \tag{5}$$

Hypothesis (H3a) gives

$$\int_2^x t^{-1} \Phi(t) dt = \rho \int_2^x t^{-1}\, li\, t\, dt + O\left[\int_2^x (L(t)/\log t) dt\right].$$

Substitute $li\, t = \int_2^t \log^{-1} u du$ and invert the order of integration and apply (1.9) with $\kappa = 1$ to the error term to yield

$$\int_2^x t^{-1} \Phi(t) dt = \rho\, li\, x \log x - \rho x + O[xL(x)/\log x]. \tag{6}$$

(6) combined with (5) and (H3a) yields lemma 2(a).

Under hypothesis (H3b)

$$\begin{aligned} \int_2^x t^{-1} \Phi(t) dt &= \rho\, li\, x + O\left[\int_2^x (L(t)/\log t) dt\right] \\ &= O[x/\log x] + O[xL(x)/\log x]. \end{aligned} \tag{7}$$

Combine (7), (5) and (H3b) to yield lemma 2(b).

LEMMA 3. *Suppose* L_1 *is slowly varying,* $\alpha(n)$ *is bounded and multiplicative and as* $x \to \infty$

$$\sum_{n \leq x} \alpha(n)/n = O[L_1(x)]. \tag{8}$$

Then we have the following.

(a) *If* L_2 *is slowly varying and decreasing and*

$$\vartheta_\alpha(x) = \rho x + O[xL_2(x)] \tag{9}$$

as $x \to \infty$, *then as* $x \to \infty$

$$\sum_{n \leq x} \alpha(n) = O[xL_3(x)/\log x], \tag{10}$$

where

$$L_3(x) = \max\,[\rho L_1(x), L_2^*(x)].$$

If $\rho = 0$ *then* (10) *follows without* (8).

(b) *If all values of* $\alpha(n)$ *lie in a right angle region* $\beta \leqq \arg z \leqq \beta + \pi/2$ *then as* $x \to \infty$

$$\sum_{n \leq x} \alpha(n) = O[xL_1(x)/\log x] + O[x/\log x].$$

Proof. By (9) and the boundedness of $\alpha(n)$

$$\begin{aligned}\sum_{n \leq x} \alpha(n)\vartheta(x/n) &= \rho x \sum_{n \leq x} \alpha(n)/n + O[\sum_{n \leq x}(x/n)L_2(x/n)] \\ &= O[\rho x L_1(x)] + O[xL_2^*(x)]\end{aligned}$$

by (8) and by approximating the sum in the O-term by an integral. Substitution into lemma 1 yields (10) since L_2^* is increasing. If $\rho = 0$ then (8) is irrelevant. In this case theorem 1 follows immediately with the aid of lemma 2(a).

Now if for all n, $\beta \leqq \arg \alpha(n) \leqq \beta + \pi/2$, let $\alpha_1(n) = \mathscr{R}[e^{-i\beta}\alpha(n)]$ and $\alpha_2(n) = \mathscr{I}[e^{-i\beta}\alpha(n)]$ so that for $j = 1$ or 2 and all $n \geqq 1$, $\alpha_j(n) \geqq 0$ and

$$\sum_{n \leq x} \alpha_j(n)/n \leqq |\sum_{n \leq x} \alpha(n)/n| = O[L_1(x)].$$

Hence since $\vartheta(x) = O(x)$ then

$$\sum_{n \leq x} \alpha_j(n)\vartheta(x/n) = O[x \sum_{n \leq x} \alpha_j(n)/n] = O[xL_1(x)].$$

Thus

$$\begin{aligned}\sum_{n \leq x} \alpha(n)\vartheta(x/n) &= e^{i\beta}[\sum_{n \leq x} \alpha_1(n)\vartheta(x/n) + i\sum_{n \leq x} \alpha_2(n)\vartheta(x/n)] \\ &= O[xL_1(x)].\end{aligned}$$

By lemma 1 this completes the proof of lemma 3.

Lemma 4. *Under hypotheses* (H1), (H2) *and* (H3a *or* b) *for* $0 < \omega < 1$

$$\sum_{n=1}^{\infty} \alpha(n)/n^{\omega+1} = O[\omega^{-\rho}L_*(1/\omega)]$$

where $L_*(x) = \exp K \int_1^x t^{-1}L(e^t)dt$, *K being the O-constant in* (H3).

We remark that ρ is a complex number and due to the prime number theorem since $|\alpha(n)| \leqq 1$ we have $|\rho| \leqq 1$. Since $\rho \neq 1$ then $\mathscr{R}\rho < 1$. In a O-term involving a complex valued function we always take the absolute value. Thus $f(\omega) = O(\omega^{-\rho})$ means $|f(\omega)| \leqq K_2 \omega^{-\mathscr{R}\rho}$.

PROOF OF LEMMA 4. As in [1] § 4.2.5 we have for $\mathscr{R}s > 1$

$$\sum_{n=1}^{\infty} \alpha(n)/n^s = g(s)\zeta(s) \exp\left[-\sum_p (1-\alpha(p))/p^s\right] \tag{11}$$

where $g(s)$ is bounded in the neighborhood of 1. In fact $g(s)$ is analytic for $\mathscr{R}s > \log 2/\log 3$.

By the simplest O-type prime number theorem

$$\pi(x) = li\, x + O(x/\log^2 x) = x/\log x + O(x/\log^2 x).$$

Let $B(x) = \pi(x) - \Phi(x)$. Then (H3a) implies

$$B(x) = (1-\rho) li\, x + K\theta_1 x L(x)/\log x + O[x/\log^2 x] \tag{12}$$

and (H3b) implies

$$B(x) = (1-\rho)x/\log x + K\theta_2 x L(x)/\log x + O[x/\log^2 x]. \tag{13}$$

Under (12), if $s = \omega+1 > 1$, by partial summation

$$\sum_p (1-\alpha(p))/p^s = s\int_2^\infty u^{-s-1} B(u)du \tag{14}$$

$$= (1-\rho)s\int_2^\infty u^{-s-1}\, li\, u du + sK\theta_3 \int_e^\infty (L(u)/u^{\omega+1}\log u)du + O(1) \tag{15}$$

since $\int_2^\infty u^{-s}\log^{-2} u du \leqq \int_2^\infty u^{-1}\log^{-2} u du < +\infty$. In the first term in (15) substitute $li\, u = \int_2^u (1/\log t)dt$ and invert the order of integration. For the second term since L is non-increasing and $o(1)$

$$\int_{e^{1/\omega}}^\infty (L(u)/u^{\omega+1}\log u)du \leqq L(e^{1/\omega})\omega \int_{e^{1/\omega}}^\infty u^{-\omega-1}du = L(e^{1/\omega})e^{-1} = o(1), \tag{16}$$

and

$$\int_e^{e^{1/\omega}} (L(u)/u^{\omega+1}\log u)du \leqq \int_e^{e^{1/\omega}} (L(u)/u\log u)du = \int_1^{1/\omega} t^{-1}L(e^t)dt = L'(1/\omega).$$

We are led to

$$\sum_p (1-\alpha(p))/p^s = (1-\rho)\int_2^\infty (t^{-s}/\log t)dt + K\theta_4 L'(1/\omega) + O(1). \tag{17}$$

Similarly (13) leads to (17) and so (17) is valid under either (H3a) or (H3b).

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Now substituting e^t for t

$$\int_2^\infty (t^{-s}/\log t)dt = \int_1^\infty t^{-1} e^{\omega t} dt + O(1). \tag{18}$$

By integrating $e^{-\omega z}/z$ over the contour made up of the real interval $[1, R]$, the quarter circle $|z| = R$ from R to iR, the straight segment from iR to i and the quarter circle $|z| = 1$ from i to 1 and letting $R \to \infty$ we find that

$$\int_1^\infty t^{-1} e^{-\omega t} dt = -\log \omega + O(1) \text{ for } 0 < \omega < 1. \tag{19}$$

Thus by (17), (18) and (19) for $0 < \omega < 1$

$$\begin{aligned} \sum_p (1-\alpha(p))/p^{\omega+1} = &-(1-\rho)\log\omega \\ &+\theta K L'(1/\omega)+O(1). \end{aligned} \tag{20}$$

We remark that we can also derive

$$\begin{aligned} \sum_p (1-\alpha(p))/p^{\omega+1+i\tau} = &-(1-\rho)\log\omega \\ &+\theta' K L'(1/\omega)+O(1) \end{aligned} \tag{21}$$

uniformly for $0 < \omega < 1$, $-K_3\omega \leqq \tau \leqq K_3\omega$.

Since $\zeta(\omega+1) \sim 1/\omega$ as $\omega \to 0$ then (11) and (20) lead us to the conclusion of lemma 4.

Lemma 5. *If $f(t) \geqq 0$ and $\tau(t)$ is non-decreasing for $t \geqq 0$ and if*

$$F(\omega) = \int_0^\infty e^{-\omega t} f(t) d\tau(t)$$

converges for $\omega > 0$, then for $x > 0$

$$\int_0^x f(t)d\tau(t) \leqq e\, F(1/x).$$

Proof. For $0 \leqq t \leqq x$, $e^{1-t/x} \geqq 1$ and so

$$\int_0^x f(t)d\tau(t) \leqq \int_0^x e^{1-t/x} f(t)d\tau(t) \leqq \int_0^\infty e^{1-t/x} f(t)d\tau(t).$$

The following lemma 6 was proved by G. Freud [2] using polynomial approximations. It is a Tauberian theorem with error term.

Lemma 6. *Under the hypotheses of lemma 5 if $\kappa < 1$ and*

$$F(\omega) = \omega^{-1} + O(\omega^{-\kappa})$$

as $\omega \to 0^+$, then as $x \to \infty$

$$\int_0^x f(t)d\tau(t) = x + O(x/\log x).$$

4. Proof of theorem 2

Suppose for all n, $\beta \leqq \arg \alpha(n) \leqq \beta+\pi/2$ and let $\alpha_1(n) = \mathscr{R}[e^{-i\beta}\alpha(n)]$, $\alpha_2(n) = \mathscr{I}[e^{-i\beta}\alpha(n)]$ so that $\alpha_j(n) \geqq 0$ $(j = 1, 2)$. Given j let $f(t) = 1$, $\tau(t) = \sum_{n<e^t} \alpha_j(n)/n$. Then lemma 5 applies and

$$
\begin{aligned}
\sum_{n<e^x} \alpha_j(n)/n &= \int_0^x d\tau(t) \leqq eF(1/x) \\
&= e\sum_{n=1}^{\infty} \alpha_j(n)/n^{1+1/x} \leqq e\left|\sum_{n=1}^{\infty} \alpha(n)/n^{1+1/x}\right| \qquad (1)\\
&= O[x^\rho L_*(x)]
\end{aligned}
$$

by lemma 4. Since $\alpha(n) = e^{i\beta}[\alpha_1(n)+i\alpha_2(n)]$ upon substituting $\log x$ for x in (1)

$$\sum_{n\leqq x} \alpha(n)/n = O[\log^\rho x\, L_*(\log x)] \qquad (2)$$

since $|\alpha(n)/n| \leqq 1/n < 1/\log n \leqq (\log n)^{\mathscr{R}\rho}$.

We can now apply lemma 3(b) with

$$L_1(x) = \log^{\mathscr{R}\rho} x L_*(\log x)$$

and we have conclusion C2.

5. Proof of theorem 1

Let $\alpha_1(n)$ and $\alpha_2(n)$ denote the real and imaginary parts of $\alpha(n)$. Given $j(= 1$ or $2)$ let $f(t) = 1$, $\tau(t) = \sum_{n<e^t} (1+\alpha_j(n))/n$. Then for $0 < \omega < 1$

$$
\begin{aligned}
F(\omega) &= \sum_{n=1}^{\infty} (1+\alpha_j(n))/n^{\omega+1} \\
&= \zeta(\omega+1) + \sum_{n=1}^{\infty} \alpha_j(n)/n^{\omega+1} \qquad (1)\\
&= 1/\omega+O(\omega^{-\rho-\varepsilon})
\end{aligned}
$$

where $0 < \varepsilon < 1-\mathscr{R}\rho$, due to lemma 4, since

$$L_*(x) = O(x^\varepsilon).$$

Lemma 6 is applicable with $\kappa = \mathscr{R}\rho+\varepsilon < 1$. We have

$$\sum_{n\leqq e^x} (1+\alpha_j(n))/n = \int_0^x d\tau(t) = x+O(x/\log x).$$

But $\sum_{n<y} 1/n = \log y+O(1)$ and so

$$\sum_{n<y} \alpha_j(n)/n = O(\log y/\mathrm{loglog}\, y). \qquad (2)$$

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As before we can drop the subscript j and include the term with $n = y$ if it exists. Thus we can apply lemma 3(a) with

$$L_1(x) = \log x/\operatorname{loglog} x$$

by (2) and

$$L_2(x) = \max\,[L(x),\, \eta/\log(x+1)$$

by lemma 2, where $\eta = 0$ under (H3a), $\eta = 1$ under (H3b). Since

$$\int_e^x (t \log t)^{-1} dt = \operatorname{loglog} x = O(\log x/\operatorname{loglog} x)$$

we have conclusion C1.

6. A generalization

Now suppose in place of hypothesis (H2) we have

$$\text{(H2, } \kappa) \qquad |\alpha(n)| \leqq n^\kappa$$

where $\kappa > -1$. Then we can derive from the prime number theorem that

$$\text{(1)} \qquad \sum_{p \leqq x} p^\kappa \sim li_\kappa\,(x) = \int_2^x (t^\kappa/\log t)dt \sim x^{\kappa+1}/(\kappa+1) \log x.$$

Thus suppose

$$\text{(H3, } \kappa) \qquad \sum_{p \leqq x} \alpha(p) = \rho\, li_\kappa\,(x) + O[x^{\kappa+1}L(x)/\log x]$$

where $L(x)$ is slowly varying and decreasing to 0. Then we readily find by partial summation that

$$\text{(2)} \qquad \sum_{p \leqq x} \alpha(p)/p^\kappa = \rho\, li\, x + O[xL(x)/\log x]$$

and thus if $\rho \neq 1$ $\alpha(n)/n^\kappa$ satisfies the hypotheses of theorem 1. Hence

$$\sum_{n \leqq x} \alpha(n)/n^\kappa = O[xL^*(x)/\log x] + O[\rho x/\operatorname{loglog} x].$$

By partial summation, we easily find $\sum_{n \leqq x} \alpha(n)$ in terms of this last sum and we have

THEOREM 3. *Under hypotheses* (H1), (H2, κ) *and* (H3, κ), *if* $\kappa > -1$ *and* $\rho \neq 1$, *then as* $x \to \infty$

$$\sum_{n \leqq x} \alpha(n) = O[x^{\kappa+1}L^*(x)/\log x] + O[\rho x^{\kappa+1}/\operatorname{loglog} x].$$

Similarly we have

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THEOREM 4. *Under hypotheses* (H1), (H2, κ), (H3, κ) *and* (H4) *if* $\kappa > -1$ *and* $\rho \neq 1$ *then as* $x \to \infty$

$$\sum_{n \leqq x} \alpha(n) = O[x^{\kappa+1} \log^{\rho-1} x L_*(\log x)] + O[x^{\kappa+1}/\log x].$$

7. Remarks on applications

In Delange's paper one can find applications of his theorem to several general classes of arithmetic functions. If we insert error terms into his hypotheses we easily derive results similar to his but with error terms.

For example, let $g(n)$ be an additive integer-valued function and let $\mathscr{A}$ be a set of natural numbers whose characteristic function $\chi_{\mathscr{A}}$ is multiplicative. Suppose $\mathscr{A}$ has density $D > 0$, in fact suppose

$$(1) \qquad \sum_{n \leqq x} \chi_{\mathscr{A}}(n) = Dx + O[xL_1(x)]$$

where $L_1(x)$ is slowly varying and $o(1)$. Suppose for $0 < \theta < 2\pi$ there is a $\rho \neq 1$ with

$$(2) \qquad \sum_{p \leqq x} \chi_{\mathscr{A}}(p) e^{i\theta g(p)} = \rho\, li\, x + O[xL(x)/\log x].$$

Then for $0 \leqq r < q$, we have

$$(3) \qquad \sum_{\substack{n \leqq x \\ n \in \mathscr{A} \\ g(n) \equiv r \pmod q}} 1 = Dx/q + O[xL_2(x)]$$

with

$$L_2(x) = \max\,[L_1(x), L^*(x)/\log x, 1/\log\log x].$$

This results from the fact that the sum in (3) is equal to

$$(4) \qquad \frac{1}{q} \sum_{j=0}^{q-1} \sum_{\substack{n \leqq x \\ n \in \mathscr{A}}} \exp\left\{j\, \frac{2\pi i}{q}\, [g(n) - r]\right\}.$$

By taking $\theta = 2\pi i j/q$ in (2) for $1 \leqq j \leqq q-1$ and multiplying by $\exp\{-\theta r\}$, theorem 1 gives

$$(5) \qquad \sum_{n \leqq x} \chi_{\mathscr{A}}(n) \exp\left\{j\, \frac{2\pi i}{q}\, [g(n) - r]\right\} = O[xL_2(x)].$$

The term in (4) with $j = 0$ is equal to $1/q$ times the left hand side of (1). Thus we have (3).

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[*Added in proof:* 18 March 1965]. Professor Delange has improved Theorem 1 in the following two ways. First he observes that $L(x)$ need not be slowly varying. It suffices that $L(x)$ be non-increasing. Second he replaces the term $O[\rho x/\log \log x]$ in (C1) by

$$O\left[\rho x \log^{\rho-1} x \int_2^x L^*(t)/(t \log^{a\rho+1} t) dt\right].$$

Delange's original theorem [1] is now a special case of this improved theorem.

References

[1] H. Delange, Un théorème sur les fonctions arithmétiques multiplicatives et ses applications, Ann. scient. Éc. Norm. Sup. (3) 78 (1961), 1—29.
[2] G. Freud, Acta Math. Acad. Sci. Hungar. 2 (1951), 299—308.
[3] J. Karamata, Math. (Cluj) 4 (1930), 38—53.
[4] J. Korevaar, et al., Nieuw Arch. Wiskunde 23 (1949), 77—86.

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