GROUPS WHOSE AUTOMORPHISMS ARE ALMOST DETERMINED BY THEIR RESTRICTION TO A SUBGROUP

by MARTIN R. PETTET

(Received 10 February, 1985)

The trivial observation that every automorphism of a group is determined by its restriction to a set of generators suggests the converse question: if X is a subset of a group G such that each automorphism of G is determined (or "almost" determined) by its restriction to X, to what extent is the structure of G governed by that of the subgroup which X generates? Is this subgroup in some sense necessarily "large" in G? If the index of the subgroup is used as a measure of largeness, then in the absence of additional hypotheses, the answer to the second question is generally "no", the additive group of rationals with $X = \{1\}$ being an obvious counterexample. (More confounding is the existence of uncountable torsion-free abelian groups for which inversion is the only non-trivial automorphism. See, for example, [**2**], [**3**], and [**4**].) However, under certain finiteness assumptions, it seems that some positive conclusions are obtainable. One such example will be considered here.

Recall first that a Černikov group is one which contains an abelian subgroup of finite index and satisfies the minimum condition on subgroups. For lack of a standard name, we shall use the term *aperiodic* to refer to a group which has no non-trivial periodic homomorphic image and we shall say that G is *torsion-separable* if there is a finite subnormal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$ such that each factor G_i/G_{i-1} is either periodic or aperiodic. One motivation for working within this class of groups is the fact that a torsion-separable abelian group is necessarily periodic (see (1.3)). So one might reasonably hope to avoid the kind of pathology cited above.

If G is a group, $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ will denote, respectively, the full automorphism group and the inner automorphism group of G. G^f will denote the finite residual of G, G' the commutator subgroup, and $Z(G)$ the center. If X is a subgroup of G, X^G is the normal closure of X in G. We shall say that X is *almost normal* in G if it has only finitely many G-conjugates (that is, if $|G:N_G(X)|$ is finite).

If κ is a cardinal, the assumption that each automorphism of G is determined up to κ possibilities by its restriction to a subgroup H is, of course, equivalent to the assumption that H is fixed point-wise by at most κ elements of $\mathrm{Aut}(G)$; that is, $|C_{\mathrm{Aut}(G)}(H)| \leq \kappa$. The main result here is the following.

THEOREM. *Let G be a torsion-separable group and H be an almost normal Černikov subgroup of G. If $C_{\mathrm{Aut}(G)}(H)$ is countable and $C_{\mathrm{Inn}(G)}(H)$ is Černikov, then G is Černikov and $G^f = (H^f)^G$. In particular, $|G:H^G|$ is finite and, if H is actually finite, then G is finite.*

COROLLARY 1. *Let G be a group which satisfies the minimum condition on subnormal subgroups and H be a subnormal Černikov subgroup of G. If $C_{\mathrm{Aut}(G)}(H)$ is Černikov, then G is Černikov and $|G:H^G|$ is finite.*

Glasgow Math. J. **28** (1986) 87–93.

COROLLARY 2. *Let G be a torsion-separable group and A be an abelian divisible-by-finite subgroup of G. Suppose that each automorphism of G is uniquely determined by its restriction to A. Then $G = A \times B$ for some subgroup B with $|B| \leq 2$ and $(|a|, |B|) = 1$ for every $a \in A$.*

Periodicity assumptions on G can sometimes be dispensed with altogether if the hypotheses are extended to endomorphisms.

COROLLARY 3. *Let G be a group and H be a finite almost normal subgroup of G. If each endomorphism of G is determined up to a finite number of possibilities by its restriction to H, then G is finite.*

COROLLARY 4. *Let G be a group and A be an abelian divisible-by-finite subgroup of G. If each endomorphism of G is uniquely determined by its restriction to A, then $G = A$.*

One final consequence of the theorem worth noting is that a central-by-Černikov group whose automorphism group is countable is necessarily finite. This extends a result of Baer [**1**, p. 529] that a periodic group with finite automorphism group is finite. See also [**7**].

1. Torsion-separable groups. We begin with a few general observations about the class of torsion-separable groups.

(1.1) *Homomorphic images of torsion-separable groups are torsion-separable.*

Proof. This is clear since homomorphic images of periodic (aperiodic) groups are periodic (resp. aperiodic).

(1.2) *If G satisfies the minimum condition on subnormal subgroups, then G is torsion-separable.*

Proof. If G satisfies min-sn, then among the normal subgroups N of G with G/N periodic, there is a unique minimal element G^*. G^* also satisfies min-sn and so $G^{**}(=(G^*)^*)$ exists. Then $G^{**} \trianglelefteq G$ (since G^{**} is characteristic in G^*) and G/G^{**} is periodic. So $G^* = G^{**}$. Hence, G is aperiodic-by-periodic and, in particular, is torsion-separable.

(1.3) (a) *Every aperiodic group is perfect.*
(b) *Every solvable torsion-separable group is periodic.*

Proof. For (a), we observe that if G is aperiodic with commutator subgroup G', then since $G^n = G$ for every positive integer n, G/G' is divisible abelian. By [**8**, 4.1.5], G/G' is a direct sum of copies of the rationals and quasicyclic groups and, since both types of summands have non-trivial periodic homomorphic images, G/G' is trivial. Statement (b) follows from (a) since, if G is solvable and torsion-separable, every aperiodic factor in a torsion-separated series for G is trivial.

(1.4) *If G is torsion-separable and $N \trianglelefteq G$ such that G/N is periodic, then N is torsion-separable. Any subgroup of finite index in G is torsion-separable.*

Proof. Let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$ be a torsion-separated series for G. We will show that if $N_i = N \cap G_i$ for $0 \leq i \leq n$, then $1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_n = N$ is a torsion-separated series for N.

Certainly $N_i/N_{i-1} \cong N_iG_{i-1}/G_{i-1}$ so that, if G_i/G_{i-1} is periodic, N_i/N_{i-1} is also. Suppose now that G_i/G_{i-1} is aperiodic. Since $G_i/N_i \cong NG_i/N \leq G/N$, G_i/N_i is periodic, whence $(G_i/G_{i-1})/(N_iG_{i-1}/G_{i-1}) \cong G_i/N_iG_{i-1}$ is periodic. It follows that $G_i = N_iG_{i-1}$ and $N_i/N_{i-1} \cong G_i/G_{i-1}$ is aperiodic. This proves the first statement.

The second statement now follows immediately from the fact that any subgroup of finite index in G contains a normal subgroup of G of finite index.

The last observation in this section represents a slight extension of another result of Baer [**1**, p. 530].

(1.5) *Let H be a Černikov group and* Inn(H) *denote its group of inner automorphisms. If A is any torsion-separable group of automorphisms of H, then* $|A : A \cap \mathrm{Inn}(H)|$ *is finite (so that, in particular, A is Černikov).*

Proof. In the course of deriving this conclusion for periodic groups of automorphisms, Baer showed [**1**, pp. 534–535] that for any Černikov group H, $|C_{\mathrm{Aut}(H)}(H^f) : \hat{H}^f|$ is finite (where $\hat{H}^f$ is the subgroup of Inn(H) corresponding to the finite residual H^f of H). Thus it suffices to show that $|A : C_A(H^f)|$ is finite. Since $A/C_A(H^f)$ is isomorphic to a torsion-separable subgroup of Aut(H^f) by (1.1), we need only to settle the case that $H = H^f$.

We may also assume that A is periodic-by-aperiodic. For if A is necessarily finite in this situation then, in the general case, the lowest aperiodic factor in any torsion-separated series for A must be trivial and we can use induction on the minimal length of such a series. Hence we assume that A contains a periodic normal subgroup B such that A/B is aperiodic.

For each positive integer i, let H_i be the subgroup of $H(= H^f)$ generated by the elements of order p^i for some prime p. H_i is a finite characteristic subgroup of H by [**8**, 4.2.11]. So $A/C_A(H_i)$ is finite for every i. Therefore, $A/C_A(H_i)B$ is finite. So, since A/B is aperiodic, $A = C_A(H_i)B$ for all i. In particular, $|A : C_A(H_i)| = |B : C_B(H_i)| \leq |B|$. Since B is finite (by Baer's theorem), it follows that there is a positive integer n such that $C_A(H_n) = C_A(H_i)$ for all $i \geq n$. But H is the union of the H_i's. So we conclude that $C_A(H_n) = 1$, whence $A = A/C_A(H_n)$ is finite, as required.

2. The theorem. Before proving the theorem, it is convenient to isolate two simple lemmas, the first of which is a straightforward adaptation of Maschke's theorem and the second of which is a rather obvious device for extending automorphisms.

(2.1) *Let V be an abelian normal subgroup of finite index n in G, and suppose that D is a divisible subgroup of V with $D \trianglelefteq G$. Then there exists a subgroup E of V with $E \trianglelefteq G$, $V^n = DE$, and $(D \cap E)^n = 1$.*

https://doi.org/10.1017/S0017089500006388 Published online by Cambridge University Press

Proof. Let $V = D \times L$ (since D is divisible) and π be the corresponding projection $V \to D$. Define the endomorphism $\pi^*: V \to D$ by

$$\pi^*(v) = \prod_t t\pi(t^{-1}vt)t^{-1}$$

where the product is taken over a transversal for V in G. Then $\pi^*(g^{-1}vg) = g^{-1}\pi^*(v)g$ for all $g \in G$, $v \in V$ and, in particular, $\ker(\pi^*) \trianglelefteq G$. If $d \in D$, then $\pi^*(d) = d^n$. So, for any $v \in V$, $\pi^*(v^{-n}\pi^*(v)) = \pi^*(v)^{-n}\pi^*(v)^n = 1$. It follows that $V^n \subseteq D\ker(\pi^*)$. So since $D = D^n \leqslant V^n$, $V^n = DE$ where $E = V^n \cap \ker(\pi^*)$. If $v \in D \cap E$, then $v^n = \pi^*(v) = 1$, so that $(D \cap E)^n = 1$.

NOTE. Although the factorization of V^n in (2.1) will suffice for our purposes, the referee has pointed out that since $V^n/E = D/D \cap E$ is divisible, $V/E = (V^n/E) \times (W/E)$ so that if $E_1 = \{x \in V : x^n \in E\}$, then $E_1 \trianglelefteq G$, $V = DE_1$, and $(D \cap E_1)^{n^2} = 1$. In fact, he refers to a short cohomological argument [**6**, Lemma 10 (ii)] which yields $G = XD$ with $(X \cap D)^n = 1$. If $E = V \cap X \trianglelefteq G$, we have $V = DE$ and $(D \cap E)^n = 1$.

(2.2) *Suppose $G = AB$ where $A \trianglelefteq G$. Let $\hat{B}$ denote the subgroup of* $\mathrm{Aut}(A)$ *induced by conjugation by elements of B. If σ is an element of $C_{\mathrm{Aut}(A)}(A \cap B) \cap C_{\mathrm{Aut}(A)}(\hat{B})$, then the map $\sigma^*: G \to G$ defined by*

$$\sigma^*(ab) = \sigma(a)b \quad \textit{for all} \quad a \in A, b \in B$$

is an element of $C_{\mathrm{Aut}(G)}(B)$. The map $\sigma \mapsto \sigma^*$ is injective.

Proof. This is a routine calculation. Suffice it to say that the fact that $\sigma \in C_{\mathrm{Aut}(A)}(A \cap B)$ ensures that σ^* is well-defined and injective, while the assumption that it commutes with the action of B is required for σ^* to be an endomorphism.

Proof of the theorem. Let G be a torsion-separable group with a subgroup H which satisfies the hypotheses of the theorem. By (1.4) and (1.1), $N_G(H)/C_G(H)$ is isomorphic to a torsion-separable group of automorphisms of H and hence, by (1.5), it is Černikov and $N_G(H)/HC_G(H)$ is finite. (This is the only point in the argument where the full force of (1.5) is used. Henceforth, Baer's version (for periodic automorphism groups) will suffice.) Since $|G:N_G(H)|$ is finite and $C_G(H)/Z(G) \cong C_{\mathrm{Inn}(G)}(H)$ is Černikov, $|G:HC_G(H)|$ is finite and $G/Z(G)$ is Černikov. It follows from (1.4) that $Z(G)$ is torsion-separable and from (1.3) that it is periodic; so G is locally finite. Also, by [**5**, Lemma 4.23], G' is Černikov so that, if $K = HG'$, then K is Černikov, $K \trianglelefteq G$ and G/K is abelian.

(1) *Every divisible subgroup of $Z(G)$ is contained in K.*

If Z is a quasicyclic p-subgroup of $Z(G)$, then ZK/K is divisible so that $G/K = (ZK/K) \times (L/K)$ for some subgroup L. Thus, $G = ZKL = ZL$ and $ZK \cap L = K$. Since $Z \leqslant Z(G)$, it follows from (2.2) that $C_{\mathrm{Aut}(Z)}(Z \cap L)$ is isomorphic to a subgroup of $C_{\mathrm{Aut}(G)}(L)$. Since $H \leqslant K \leqslant L$, it follows that $C_{\mathrm{Aut}(Z)}(Z \cap L)$ is countable. Because all subgroups of a quasicyclic group are characteristic, $\mathrm{Aut}(Z)/C_{\mathrm{Aut}(Z)}(Z \cap L)$ is isomorphic to a subgroup of $\mathrm{Aut}(Z \cap L)$. But $\mathrm{Aut}(Z)$ is isomorphic to the group of p-adic units and hence is uncountable. Thus, $Z \cap L$ is not finite so that $Z = Z \cap L \leqslant K$.

(2) *G/K is reduced (that is, contains no non-trivial divisible subgroups).*

Suppose that X/K is a quasicyclic subgroup of G/K. Because K is a normal Černikov subgroup of X, (1.5) implies that $|X:KC_X(K)|$ is finite. So $X=KC_X(K)$ and $C_X(K)/Z(K)\cong X/K$. Since $Z(K)\leq Z(C_X(K))$ and since every proper subgroup of a quasicyclic group is cyclic, we conclude that $C_X(K)$ is abelian. X (and hence $C_X(K)$) is Černikov so that $C_X(K)=A\times B$, where $A=C_X(K)^f$ is divisible and B is finite. Since $A\trianglelefteq G$, (1.5) yields that $G/C_G(A)$ is finite. So from [**5**, Lemma 3.29.1],

$$A=[A,G]C_A(G)=[A,G]\times(A\cap Z(G)).$$

Now $[A,G]\leq G'\leq K$ and (1) implies that $(A\cap Z(G))^f\leq K$, so that $|A:A\cap K|$ is finite. Since A is divisible, this forces $A\leq K$ so that $X=KC_X(K)=KB$ and $X/K\cong B/B\cap K$. This is absurd since B is finite.

(3) *Every primary component of G/K is finite.*

Suppose that R/K is the p-component of G/K for some prime p. If $R\cap Z(G)$ is a p'-group, then obviously $R\cap Z(G)\leq K$ so that R/K is a homomorphic image of $R/R\cap Z(G)\cong RZ(G)/Z(G)$ which is Černikov. But by (2), R/K is also reduced so that it must, in fact, be finite. We may, therefore, assume that $R\cap Z(G)$ contains a subgroup V of order p. R/KV is then a direct summand of G/KV and so

$$\mathrm{Hom}(R/KV,V)\subseteq\mathrm{Hom}(G/KV,V).$$

Now it is again a straightforward calculation to verify that if $f\in\mathrm{Hom}(G/KV,V)$, then the map $\sigma:G\to G$ defined by

$$\sigma(x)=xf(KVx)\quad\text{for every } x\in G$$

belongs to $C_{\mathrm{Aut}(G)}(KV)$. Thus, if $\bar{R}=R/KV$, $\mathrm{Hom}(\bar{R}/\bar{R}^p,V)=\mathrm{Hom}(\bar{R},V)$ is countable. Since $\bar{R}/\bar{R}^p$ is elementary abelian and since the functor $\mathrm{Hom}(\cdot,V)$ takes direct sums to direct products, $\bar{R}/\bar{R}^p$ must be finite, whence $(R/K)/(R/K)^p$ is finite. But since R/K is reduced, it is clear from [**8**, 4.3.11] that R/K must be finite.

(4) *G is Černikov.*

In view of (3), it is enough to show that G/K has only finitely many non-trivial primary components. Now by (1.5), $|G:KC_G(K)|$ is finite. So if F is generated by a transversal for $KC_G(K)$ in G, then because G is locally finite, F is finite. Let π be the set of prime divisors of orders of elements of FK, so that π is finite. $C_G(K)/Z(K)\cong KC_G(K)/K$ which is abelian. So $C_G(K)/Z(K)=(S/Z(K))\times(T/Z(K))$, where the factors are π- and π'-groups respectively. Since $Z(K)\leq Z(T)$, T is nilpotent and thus is the direct product of its Sylow π-subgroup $Z(K)$ and a π'-group Q. Now $Q\trianglelefteq G$, $G=FKC_G(K)=FKSQ$, and $FKS\cap Q=1$ (since FKS is a π-group) so that $G=FKS\times Q$. $\mathrm{Aut}(Q)$ is, therefore, isomorphic to a subgroup of $C_{\mathrm{Aut}(G)}(FKS)$ which is countable. But $\mathrm{Aut}(Q)$ is an unrestricted direct product of the automorphism groups of the primary components of Q, so there are only finitely many non-trivial such components. It follows easily that G/K has only finitely many non-trivial primary components.

(5) $G^f=(H^f)^G$.

https://doi.org/10.1017/S0017089500006388 Published online by Cambridge University Press

Let $D=(H^f)^G$. Since $H^f \leqslant G^f$, $D \leqslant G^f$. Now since $|G:N_G(H)|$ is finite and $|HD:D|=|H:H\cap D| \leqslant |H:H^f|$, it follows from Dicman's lemma [**8**, 14.5.7] that $H^G/D=(HD/D)^G$ is finite. Thus, $(H^G)^f \leqslant D=(H^f)^G \leqslant G^f$. So it suffices to show that $(H^G)^f=G^f$. The upshot of this is that, replacing H by H^G if necessary, we may assume from now on that $H \trianglelefteq G$.

By (2.1), $G^f=H^fE$ for some $E \trianglelefteq G$ with $H^f\cap E$ bounded (and hence finite). Let F be the finite subgroup generated by a transversal for G^f in G. Then $|FH\cap E:H^f\cap E| \leqslant |FH:H^f| \leqslant |F|\,|H:H^f|$ so that $FH\cap E$ is finite.

Suppose that some primary component E_p of E is unbounded. Then E_p is a faithful module over the ring of p-adic integers and multiplication by p-adic units induces an uncountable subgroup A of $Z(\mathrm{Aut}(E_p))$. Since each subgroup of E_p is invariant under A, $A/C_A((FH\cap E)_p)$ is isomorphic to a subgroup of $\mathrm{Aut}((FH\cap E)_p)$. Since $FH\cap E$ is finite, we conclude that $C_A((FH\cap E)_p)$ is uncountable. Each element of $C_A((FH\cap E)_p)$ extends to an automorphism of E (acting trivially on $E_{p'}$), thence to an automorphism of $G^f=H^fE$ (since $H^f\cap E \leqslant FH\cap E$). The result is an uncountable subgroup A^* of $\mathrm{Aut}(G^f)$. Since $FH\cap G^f=FH\cap H^fE=H^f(FH\cap E) \leqslant H^fE_{p'}(FH\cap E)_p$, we have $A^* \leqslant C_{\mathrm{Aut}(G^f)}(FH\cap G^f)$. Moreover, since H^f and E are each normal in G, it is clear that A^* commutes with the conjugation action of G on G^f. Since $G=G^fF=G^fFH$, (2.2) implies that A^* can be extended to produce an uncountable subgroup of $C_{\mathrm{Aut}(G)}(FH)$, contradicting the assumption that $C_{\mathrm{Aut}(G)}(H)$ is countable.

Thus, each primary component of E is bounded so that, since E is Černikov, E is finite. But then $|G^f:H^f| \leqslant |E|$ so that, since G^f has no proper subgroups of finite index, $G^f=H^f$ as required.

3. The corollaries. Corollary 1 is an immediate consequence of (1.2) and an observation of Robinson and Roseblade [**8**, 13.3.8] that if G satisfies the minimum condition for subnormal subgroups, then every subnormal subgroup of G is almost normal.

Proof of Corollary 2. By hypothesis, the inner automorphism of G induced by any element of A is the identity, so that $A \leqslant Z(G)$, whence $\mathrm{Inn}(G) \leqslant C_{\mathrm{Aut}(G)}(A)=1$. Thus, G is abelian and the theorem yields that it is Černikov and $G^f=A^f$.

First we observe that it suffices to prove that A is a direct factor of G. For if $G=A\times B$, then clearly $\mathrm{Aut}(B)$ is trivial so that $|B| \leqslant 2$. Moreover, $(|a|,|B|)=1$ for every $a\in A$, for otherwise $B=\langle b\rangle$ has order 2 and A contains an element a_0 of order 2. But then the map

$$a\mapsto a \quad \text{for all } a\in A, \quad b\mapsto a_0b$$

defines a non-trivial element of $C_{\mathrm{Aut}(G)}(A)$, a contradiction.

If $A=A^f\times F$, then A/F is divisible so that $G/F=(A/F)\times(C/F)$ for some finite subgroup C. Then $G=AC$ and $A\cap C=F$ so that $C_{\mathrm{Aut}(C)}(F)$ is trivial. If $C=F\times B$ for some B, then $G=A\times B$ and we are done. Thus we are reduced to the case that G is finite and, in fact, a p-group for some prime p.

The proof is completed by induction on $|G|$. Let a be an element of maximal order in A, so that $A = \langle a \rangle \times K$ for some K by [**8**, 4.2.7]. The map $x \mapsto x^{|a|+1}$ defines an element of $C_{\mathrm{Aut}(G)}(A)$ so that, in fact, a has maximal order in G. If $\bar{G} = G/K$, then $\bar{G} = \langle \bar{a} \rangle \times \bar{L}$. So $G = \langle a \rangle \times L$ for some L. Then $C_{\mathrm{Aut}(L)}(K)$ must be trivial so that, since $|L| < |G|$, $L = K \times B$ for some B. Then $G = A \times B$ and the proof is complete.

Proof of Corollary 3. The hypotheses imply that $N_G(H)/C_G(H)$ and $C_G(H)/Z(G)$ are finite. So $G/Z(G)$ is finite. If $n = |G : Z(G)|$, the transfer homomorphism $G \to Z(G)$ is just the map $x \mapsto x^n$ [**5**, Theorem 4.11]. Therefore, if h is the exponent of H, then for any integer k, the map $x \mapsto x^{nhk}$ is an endomorphism of G whose kernel contains H. Since, by hypothesis, there are only finitely many such endomorphisms, G must be periodic and the theorem applies.

Proof of Corollary 4. As in the proof of Corollary 2, G is abelian, $A = A^f \times F$ for some finite subgroup F, and $G = AC$ for some C with $A \cap C = F$. An endomorphism of C which fixes F extends, therefore, to one of G fixing A. If $C = F$ then $G = A$, so that it suffices to prove the corollary in the case $A = F$. In this case, Corollary 3 yields that G is finite so that, by Corollary 2, $G = A \times B$ with $|B| \leq 2$. But the hypothesis forces the projection map from G onto A to be the identity map, so that $G = A$ as required.

REFERENCES

1. R. Baer, Finite extensions of abelian groups with minimum conditions, *Trans. Amer. Math. Soc.* **79** (1955), 521–540.

2. A. L. S. Corner, Endomorphism algebras of large modules with distinguished submodules, *J. Algebra* **11** (1969), 155–185.

3. L. Fuchs, The existence of indecomposable abelian groups of arbitrary power, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 453–457.

4. J. de Groot, Indecomposable abelian groups, *Nederl. Akad. Wetensch. Proc. Ser. A* **60** = *Indag. Math.* **19** (1957), 137–145.

5. D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups I* (Springer-Verlag, 1972).

6. D. J. S. Robinson, Splitting theorems for infinite groups, *Symposia Mathematica*, *Vol. XVII* (Academic Press, 1976), 441–470.

7. D. J. S. Robinson, A contribution to the theory of groups with finitely many automorphisms, *Proc. London Math. Soc.* (3) **35** (1977), 34–54.

8. D. J. S. Robinson, *A Course in the Theory of Groups* (Springer-Verlag, 1982).

Department of Mathematics
University of Toledo
Toledo
Ohio 43606
U.S.A.