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CONSTRUCTING LOCALLY FLAT EMBEDDINGS OF INFINITE DIMENSIONAL MANIFOLDS WITHOUT TUBULAR NEIGHBORHOODS

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1. Introduction. All spaces in this paper will be separable and metric. A closed embedding $i: M \to N$ is said to be *locally flat* (of codimension n) if for each $x_0 \in M$ there is an open set U in M containing x_0 and an open embedding $h: U \times R^n \to N$ for which $h(x, 0) = i(x)$, for all $x \in U$. When there is no confusion we will identify M with its image and regard i as the inclusion $M \hookrightarrow N$. We say that a closed subset M of N has a *tubular neighborhood* (of codimension n) if there is a fiber bundle $E \to M$, with fiber R^n and 0-section M, and an open embedding $\varphi: E \to N$ for which $\varphi|M = \text{id}$.

The following is known for M, N finite dimensional manifolds without boundary and $M \hookrightarrow N$ locally flat.

1. M always has a tubular neighborhood in codimension 1 [**1**].
2. If dim $M \neq 2$, then M always has a tubular neighborhood in codimension 2 [**8**].
3. If $n \geqq \dim M - j - 1$ and $n \geqq 5 + j$, where $j = 0, 1, 2$, then M always has a tubular neighborhood in codimension n [**15**]. (This implies Milnor's result on the *stable* existence of tubular neighborhoods, i.e. if $M \hookrightarrow N$ is locally flat, then $M \times \{0\}$ has a tubular neighborhood in $M \times R^k$, for some $k \geqq 0$ [**11**].)
4. There is a locally flat embedding $S^{19} \subset S^{19} \times S^9$ with no tubular neighborhood [**13**].

Now switching categories let M, N both be *Q-manifolds*, i.e. manifolds modeled on the Hilbert cube Q, or *l_2-manifolds*, i.e. manifolds modeled on separable Hilbert space l_2. The following results are known about locally flat embeddings $M \hookrightarrow N$.

1. M always has a tubular neighborhood in codimension 1 [**1**].
2. M always has a tubular neighborhood in codimension 2 [**12**].

The purpose of this paper is to construct a codimension 3 counterexample. Specifically we construct a codimension 3 locally flat embedding $S^3 \times Q \subset S^3 \times Q$ (or $S^3 \times l_2 \subset S^3 \times l_2$) which has no tubular neighborhood. Moreover no stabilization, $S^3 \times Q \times \{0\} \subset S^3 \times Q \times R^k$ (or $S^3 \times l_2 \times \{0\} \subset S^3 \times l_2 \times R^k$), has a tubular neighborhood. Our main result is the theorem stated

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below which gives a general method of constructing examples of this sort. In the corollary we use this result to calculate the specific example mentioned above.

Let Top_n denote the group of all origin-preserving homeomorphisms of R^n. Topological fiber bundles over a base B with fiber R^n and 0-section B are classified by maps of B to the classifying space $B\,\mathrm{Top}_n$ [**10**]. Let G_n be the H-space of all self-homotopy equivalences of S^n. Hurewicz fibrations over B with fiber S^n are classified by maps of B to the classifying space BG_n[**14**]. There is a natural map e: $\mathrm{Top}_n \to G_{n-1}$ given by sending each $h \in \mathrm{Top}_n$ to the composition

$$e(h): S^{n-1} \hookrightarrow R^n - \{0\} \xrightarrow{h} R^n - \{0\} \xrightarrow{r} S^{n-1},$$

where r is the radial retraction. This induces a map of classifying spaces, $\tilde{e}$: $B\,\mathrm{Top}_n \to BG_{n-1}$. We say that a map φ: $X \to BG_{n-1}$ *lifts* to $B\,\mathrm{Top}_n$ if there is a map $\tilde{\varphi}$: $B \to B\,\mathrm{Top}_n$ for which $\tilde{e}\tilde{\varphi}$ is homotopic to φ. Also for each $k \geqq 0$ the natural inclusion map $S^n \hookrightarrow S^{n+k}$ induces a map $G_n \to G_{n+k}$, given by suspension of maps, which in turn induces a map of classifying spaces, $BG_n \to BG_{n+k}$. Here is our main result. See § 4 for a proof.

THEOREM. *For every Q-manifold (or l_2-manifold) M and map φ: $M \to BG_{n-1}$, there exists a codimension n locally flat embedding h: $M \to M$ such that if $h(M)$ has a tubular neighborhood, then φ can be lifted to B* Top_n. *Moreover if any stabilization $h(M) \times \{0\} \hookrightarrow M \times R^k$ has a tubular neighborhood, then*

$$M \xrightarrow{\varphi} BG_{n-1} \to BG_{n+k-1}$$

can be lifted to B Top_{n+k}.

We now use this result to produce our example. See § 5 for details.

COROLLARY. *There exists a codimension* 3 *locally flat embedding h: $S^3 \times Q \to S^3 \times Q$ (or h: $S^3 \times l_2 \to S^3 \times l_2$) so that no stabilization has a tubular neighborhood.*

Finally we remark that the above theorem raises the following interesting question: *Given a Q-manifold (or l_2-manifold) M, can we classify all locally flat embeddings $M \hookrightarrow N$ by maps of M to BG_{n-1}?* Theorem 1 essentially answers the "realization" part of this question.

In the sequel we will only give the details for the proof of the theorem for the Q-manifold case. Anyone familiar with basic l_2-manifold apparatus can easily make the minor alterations in the argument necessary to give a proof for the l_2-manifold case. The proof of the Corollary requires no manifold apparatus at all. It follows mechanically from the theorem by recalling some well-known calculations involving the homotopy groups of BO_n, $B\,\mathrm{Top}_n$ and BG_n.

2. An infinite dimensional lemma. The purpose of this section is to establish a result which will be needed in the proof of the theorem. For its proof we have the first recall some material from [**6**].

A map $f: B \times X \to B \times Y$ is said to be *fiber preserving* (f.p.) if $f(\{b\} \times X) \subset \{b\} \times Y$, for all $b \in B$. A closed set $A \subset B \times M$ is said to be a *fibered Z-set* if there are f.p. maps $f: B \times M \to B \times M - A$ which are arbitrarily close to id. An f.p. embedding $f: B \times A \to B \times M$ is said to be a *fibered Z-embedding* if $f(B \times A)$ is a fibered Z-set. We now state two results which will be needed in the proof of Lemma 2.1 below.

FIBERED MAPPING APPROXIMATION [**6**]. *Let M be a Q-manifold, let (B, B_0) be a compact pair, and let A be compact. If there is an* f.p. *map $f: B \times A \to B \times M$ which is a fibered Z-embedding from $B_0 \times A$ to $B_0 \times M$, then f is* f.p. *homotopic to a fibered Z-embedding* rel $B_0 \times A$.

FIBERED Z-SET UNKNOTTING [**6**]. *Let M be a Q-manifold and let A, B be compact. If there are fibered Z-embeddings f, $g: B \times A \to B \times M$ which are* f.p. *homotopic, then there is an* f.p. *homeomorphism $h: B \times M \to B \times M$ for which $hf = g$.*

Remark. Both of these results come from § 4 of [**6**]. See also [**5**] for related results.

We are now ready for our main result. For notation let M be a compact Q-manifold, let H be the homeomorphism group of $M \times [0, 1)$, and let G be the space of self homotopy equivalences of $M \times [0, 1)$. (All function spaces will have the compact-open topology.)

LEMMA 2.1. *The inclusion $H \hookrightarrow G$ is a weak homotopy equivalence.*

Proof. Using the homotopy sequence of the pair (G, H) it suffices to prove that the relative groups $\pi_n(G, H)$ all vanish. An element of $\pi_n(G, H)$ may be represented by an f.p. map $f: \Delta^n \times M \times [0, 1) \to \Delta^n \times M \times [0, 1)$ for which $f|\partial\Delta^n \times M \times [0, 1)$ is a homeomorphism and $f|\{*\} \times M \times [0, 1) = \text{id}$. (Here Δ^n is the standard n-simplex and $* \in \partial\Delta^n$ is a basepoint.) We want to prove that f is f.p. homotopic to a homeomorphism rel $\partial\Delta \times M \times [0, 1)$.

Consider the restriction $f|\Delta \times M \times \{0\}$. By Fibered Mapping Approximation there is a fibered Z-embedding $g: \Delta \times M \times \{0\} \to \Delta \times M \times [0, 1)$ which agrees with f on $\partial\Delta \times M \times \{0\}$ and which is f.p. homotopic to f rel $\partial\Delta \times M \times \{0\}$. Now compare g with id: $\Delta \times M \times \{0\} \to \Delta \times M \times [0, 1)$ and note that $g = \text{id}$ on $\{*\} \times M \times \{0\}$. It easily follows that g is f.p. homotopic to id because Δ is contractible.

Using Fibered Z-Set Unknotting there is an f.p. homeomorphism $h_1: \Delta \times M \times [0, 1) \to \Delta \times M \times [0, 1)$ for which $h_1|\Delta \times M \times \{0\} = g$. Then $h_1^{-1}f: \partial\Delta \times M \times [0, 1) \to \partial\Delta \times M \times [0, 1)$ is an f.p. homeomorphism which is the identity on $\partial\Delta \times M \times \{0\}$. By a variation of the usual Alexander trick this extends to an f.p. homeomorphism $h_2: \Delta \times M \times [0, 1) \to \Delta \times M \times [0, 1)$ for which $h_2 = \text{id}$ on $\Delta \times M \times \{0\}$. Then $h = h_1h_2: \Delta \times M \times [0, 1) \to \Delta \times M \times [0, 1)$ is an f.p. homeomorphism and $h_2^{-1}h_1^{-1}f$ is the identity on $\partial\Delta \times M \times [0, 1) \cup (\Delta \times M \times \{0\})$. Thus $h_2^{-1}h_1^{-1}f$ is clearly f.p. homotopic to id rel $\partial\Delta \times M \times$

$[0, 1)$ by again using the Alexander trick. This implies that f is f.p. homotopic to h rel $\partial\Delta \times M \times [0, 1)$.

Remark 1. Lemma 2.1 is not in the exact form that is needed in the proof of the theorem. If G_M denotes the space of all self-homotopy equivalences of M, then the map α: $G_M \to G$, defined by sending $f \in G_M$ to

$$M \times [0, 1) \xrightarrow{f \times \mathrm{id}} M \times [0, 1),$$

is a homotopy equivalence. A homotopy inverse β: $G \to G_M$ is easily defined by sending $f \in G$ to the composition

$$M \xrightarrow{\times 0} M \times [0, 1) \xrightarrow{f} M \times [0, 1) \xrightarrow{\mathrm{proj}} M.$$

Then Lemma 2.1 implies that the composition

$$H \hookrightarrow G \xrightarrow{\beta} G_M$$

is a weak homotopy equivalence.

Remark 2. The l_2-manifold version of Lemma 2.1 was established by Wong [**17**] by a different argument. The proof given here is modeled on the proof given in [**2**], which was designed to show that any homeomorphism on an l_2-manifold which is homotopic to id is isotopic to id.

3. The basic construction. Our main result here is Theorem 3.3, which is the main tool needed for the proof of the theorem. It is based on the following notation.

Let M be a Q-manifold and let

$$E \xrightarrow{p} M$$

be a fiber bundle with fiber $S^{n-1} \times Q \times [0, 1)$. Let $\tilde{E}$ be the space formed from E by taking the one-point compactification of each fiber in E. Then there is a natural map $\tilde{p}$: $\tilde{E} \to M$ so that each $\tilde{p}^{-1}(x)$ is the one-point compactification of $p^{-1}(x)$. Identifying M with $\tilde{E} - E$ we have $\tilde{E} = E \cup M$ and $\tilde{p}|M = \mathrm{id}$. Also

$$\tilde{E} \xrightarrow{\tilde{p}} M$$

is a fiber bundle with fiber $F = C(S^{n-1} \times Q)$ (the cone over $S^{n-1} \times Q$) and 0-section M. We may write

$$C(S^{n-1} \times Q) = S^{n-1} \times Q \times [0, 1) \cup \{*\},$$

where $*$ is the cone point.

LEMMA 3.1. *There is a homeomorphism* φ: $Q \times F \to Q \times B^n$ *such that* $\varphi(q, *) = (q, 0)$ *for all* $q \in Q$.

(B^n is the n-ball in R^n and $\partial B^n = S^{n-1}$.)

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Proof. It will suffice to construct a homeomorphism $\varphi': Q \times (F - \{*\}) \rightarrow Q \times (B^n - \{0\})$ which extends to our desired φ. To do this we first need a map $\theta: Q \times Q \times [0, 1] \rightarrow Q$ such that for each t, $\theta_t: Q \times Q \rightarrow Q$ is a map which is projection onto the first factor for $t = 1$, and is a homeomorphism for $t < 1$. Such maps are easily obtained by using Wong's coordinate-switching trick [**3**, p. 18]. Now replace $B^n - \{0\}$ by $S^{n-1} \times [0, 1)$ and define

$$\varphi': Q \times S^{n-1} \times Q \times [0, 1) \rightarrow Q \times S^{n-1} \times [0, 1)$$

by letting $\varphi' | Q \times S^{n-1} \times Q \times \{t\}$ be the homeomorphism taking $Q \times S^{n-1} \times Q \times \{t\}$ to $Q \times S^{n-1} \times \{t\}$ for which $\varphi'(q_1, s, q_2, t) = (\theta_t(q_1, q_2), s, t)$. It is clear that φ' extends in our required manner.

LEMMA 3.2. *$\tilde{E}$ is homeomorphic to M and M is locally flat in $\tilde{E}$.*

Proof. By stability [**3**, p. 22] we may write $M = M' \times Q$, where M' is a copy of M. Then

$$\tilde{E} \xrightarrow{\tilde{p}} M' \times Q \xrightarrow{\text{proj}} M'$$

is a fiber bundle with fiber $F \times Q$, and this is Q by Theorem 3.1. Since the homeomorphism group of Q is contractible [**16**] we conclude that

$$\tilde{E} \xrightarrow{\tilde{p}} M' \times Q \rightarrow M'$$

is trivial, and therefore $\tilde{E} \cong M' \times Q = M$.

To see that M is locally flat choose $U \subset M'$ which is open and contractible. Then there exists a homeomorphism $h: \tilde{p}^{-1}(U \times Q) \rightarrow U \times Q \times F$ such that $h(u, q) = (u, q, *)$, for all (u, q) in the 0-section $M' \times Q$, and such that the following commutes:

$$\begin{array}{ccc} \tilde{p}^{-1}(U \times Q) & \xrightarrow{\ h\ } & U \times Q \times F \\ & {\scriptstyle \tilde{p}} \searrow \quad \swarrow {\scriptstyle \text{proj}} & \\ & U \times Q & \end{array}$$

Using Lemma 3.1 let $\bar{\varphi}: U \times Q \times F \rightarrow U \times Q \times B^n$ be a homeomorphism such that $\bar{\varphi}(u, q, *) = (u, q, 0)$, for all $(u, q) \in U \times Q$. Then

$$h' = \bar{\varphi}h: \tilde{p}^{-1}(U \times Q) \rightarrow U \times Q \times B^n$$

is a homeomorphism such that $h'(u, q) = (u, q, 0)$. The inverse,

$$(h')^{-1}: U \times Q \times B^n \rightarrow \tilde{E}$$

is an open embedding such that $(h')^{-1}(u, q, 0) = (u, q)$.

THEOREM 3.3. *If M has a tubular neighborhood in $\tilde{E} \times R^k$, then there is a fiber bundle*

$$\mathscr{E} \xrightarrow{q} M,$$

with fiber R^{n+k} and 0-section M, and a fiber homotopy equivalence $f: \mathscr{E} - M \to \tilde{E} \times R^k - M$. (Here we identify M with $M \times \{0\}$ in $\tilde{E} \times R^k$.)

Proof. Assuming that M has a tubular neighborhood in $\tilde{E} \times R^k$ means that there is a fiber bundle

$$\mathscr{E} \xrightarrow{q} M,$$

with fiber R^{n+k} and 0-section M, and an open embedding $i: \mathscr{E} \to \tilde{E} \times R^k$ for which $i|M = \text{id}$. We will construct a homotopy equivalence $f': \mathscr{E} - M \to \tilde{E} \times R^k - M$ for which $p_1 f' \simeq q$, where $p_1 = \tilde{p} \circ \text{proj}: \tilde{E} \times R^k \to M$. Theorem 2.2 of [**4**] then implies that f' must be homotopic to a fiber homotopy equivalence f as desired.

Our map $f': \mathscr{E} - M \to \tilde{E} \times R^k - M$ is easily defined to be $f' = i|\mathscr{E} - M$. Define $g: \tilde{E} \times R^k - M \to \mathscr{E} - M$ by

$$g: \tilde{E} \times R^k - M \xrightarrow{d} i(\mathscr{E} - M) \xrightarrow{i^{-1}} \mathscr{E} - M,$$

where d is just the end result of an f.p. deformation of $\tilde{E} \times R^k$ down to a neighborhood of the 0-section. We need homotopies $f'g \simeq \text{id}$ and $gf' \simeq \text{id}$.

The composition $f'g$ is just

$$f'g: \tilde{E} \times R^k - M \xrightarrow{d} i(\mathscr{E} - M) \xrightarrow{i^{-1}} \mathscr{E} - M \xrightarrow{i} i(\mathscr{E} - M) \hookrightarrow \tilde{E} \times R^k - M.$$

This is homotopic to id because $d \simeq \text{id}$. The composition gf' is just

$$gf': \mathscr{E} - M \xrightarrow{i} \tilde{E} \times R^k - M \xrightarrow{d} i(\mathscr{E} - M) \xrightarrow{i^{-1}} \mathscr{E} - M.$$

Let $\theta_t: \mathscr{E} - M \to \mathscr{E} - M$ be an f.p. homotopy so that $\theta_0 = \text{id}$ and $\theta_1(\mathscr{E} - M)$ lies in a small neighborhood around M. Clearly

$$gf' \simeq gf'\theta_1 \simeq \theta_1 \simeq \text{id},$$

where the first homotopy uses θ_t and the second uses $d \simeq \text{id}$. Thus f' is a homotopy equivalence.

Finally we need to check that $p_1 f' \simeq q$. This arises from

$$p_1 f' = p_1 i|\mathscr{E} - M \simeq p_1 iq|\mathscr{E} - M = q|\mathscr{E} - M,$$

where we have used the fiber homotopy $\text{id} \simeq q$.

Remark. To make effective use of Theorem 3.3 in § 4 below the reader should notice that $\tilde{E} \times R^k - M$ is fiber homotopy equivalent to the iterated suspension bundle, $\sum^k (E) \to M$. This means that if $E \to M$ (thought of as a Hurewicz fibration) is classified by a map $\varphi: E \to BG_{n-1}$, then $\tilde{E} \times R^k - M \to M$ is classified by

$$E \xrightarrow{\varphi} BG_{n-1} \to BG_{n+k-1}.$$

4. Proof of the theorem. Let $\varphi: M \to BG_{n-1}$ be a map. Since BG_{n-1} is a classifying space for Hurewicz fibrations with fibers S^{n-1}, this determines a Hurewicz fibration

$$E_1 \xrightarrow{p_1} M$$

whose fibers are homotopy equivalent to S^{n-1}. This implies that if there is a fiber bundle $\mathscr{E} \to M$ with fiber R^{n+k} and 0-section M, and a fiber homotopy equivalence of $\mathscr{E} - M$ to the iterated suspension fibration $\sum^k (E_1) \to M$, then

$$M \xrightarrow{\varphi} BG_{n-1} \to BG_{n+k-1}$$

can be lifted to $B\ \mathrm{Top}_{n+k}$.

Let H_{n-1} be the group of all homeomorphisms of $S^{n-1} \times Q \times [0, 1)$. There is a map $\beta: H_{n-1} \to G_{n-1}$ defined by sending h in H_{n-1} to

$$\beta(h): S^{n-1} \xrightarrow{\times 0} S^{n-1} \times Q \times [0, 1) \xrightarrow{h} S^{n-1} \times Q \times [0, 1) \xrightarrow{\mathrm{proj}} S^{n-1}.$$

It follows from Lemma 2.1 that β is a weak homotopy equivalence. Thus β induces a map of classifying spaces, $\tilde{\beta}: BH_{n-1} \to BG_{n-1}$, which is a homotopy equivalence. This means that $\varphi: M \to BG_{n-1}$ can be lifted to BH_{n-1}, and thus there is a fiber bundle $E \to M$ with fiber $S^{n-1} \times Q \times [0, 1)$ which is fiber homotopy equivalent to $E_1 \to M$.

Associated with the bundle $E \to M$ there is a codimension n locally flat embedding $h: M \to M$. This is just the construction of § 3. Now suppose that $h(M) \equiv h(M) \times \{0\}$ has a tubular neighborhood in $M \times R^k$. Then Theorem 3.3 implies that there is a fiber bundle $\mathscr{E} \to M$, with fiber R^{n+k} and 0-section M, and a fiber homotopy equivalence $\mathscr{E} - M \simeq \sum^k (E)$. Since E is fiber homotopy equivalent to E_1 we conclude that

$$M \xrightarrow{\varphi} BG_{n-1} \to BG_{n+k-1}$$

can be lifted to $B\ \mathrm{Top}_{n+k}$, thus giving our desired result.

5. Proof of the corollary. By the theorem it suffices to find a map $\varphi: S^3 \to BG_2$ such that

$$S^3 \xrightarrow{\varphi} BG_2 \to BG_{2+k}$$

does not lift to $B\ \mathrm{Top}_{3+k}$, for any $k \geqq 0$. Since $\pi_3(BG_n) \approx \pi_2(G_n)$ and $\pi_3(B\ \mathrm{Top}_n) \approx \pi_2(\mathrm{Top}_n)$, we are looking for a map $\psi: S^2 \to G_2$ which does not stably lift.

Let F_2 be the subset of G_2 consisting of those maps which fix the north pole. Evaluation at the north pole clearly gives us a fibration sequence, $F_2 \to G_2 \to S^2$. Now F_2 is just $\Omega^2(S^2)$, and therefore $\pi_2(F_2) \approx \pi_4(S^2) \approx Z_2$. (Compare with [**7**, p. 211].) To see that $\pi_2(G_2) \neq 0$ we just consider the commuting diagram,

$$\begin{array}{ccccccccc} \ldots \to & \pi_3(S^2) & \to & \pi_2(O_2) & \to & \pi_2(O_3) & \to & \pi_2(S^2) & \to \ldots \\ & \text{id}\downarrow & & e_*\downarrow & & e_*\downarrow & & \downarrow\text{id} & \\ \ldots \to & \pi_3(S^2) & \to & \pi_2(F_2) & \to & \pi_2(G_2) & \to & \pi_2(S^2) & \to \ldots . \end{array}$$

(Here the top row arises from the fibration sequence $O_2 \to O_3 \to S^2$, where O_n is the orthogonal group, and the map e: $\text{Top}_n \to G_{n-1}$ comes from § 1.) Since $\pi_2(O_2) = 0$ we must have $\pi_2(F_2) \to \pi_2(G_2)$ injective. Our desired map ψ: $S^2 \to G_2$ comes from the non-trivial element of $\pi_2(F_2)$ injected into $\pi_2(G_2)$. We have an inclusion and/or suspension-induced commuting diagram,

$$\begin{array}{ccccc} \pi_2(F_2) & \to & \pi_2(F_3) & \to & \pi_2(F_4) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_2(G_2) & \to & \pi_2(G_3) & \to & \pi_2(G_4) \end{array} .$$

The computation $\pi_2(F_2) \approx Z_2$ given above indicates that we are already in the stable range, so the arrows in the top row are isomorphisms. Moreover it easily follows from the fibration sequence $F_4 \to G_4 \to S^4$ that the vertical arrow, $\pi_2(F_4) \to \pi_2(G_4)$, is an isomorphism. Thus

$$\pi_2(F_2) \to \pi_2(G_2) \to \pi_2(G_{2+k})$$

is injective, for any $k \geqq 0$.

To see that

$$S^2 \xrightarrow{\psi} G_2 \to G_{2+k}$$

cannot be stably lifted it suffices to show that the image of $\pi_2(F_2)$ in $\pi_2(G_{2+k})$ does not lie in the image of $\pi_2(\text{Top}_{3+k})$ in $\pi_2(G_{2+k})$, for all $k \geqq 0$. Let Top_3/O_3 be the homotopy fiber of the inclusion, $BO_3 \hookrightarrow B\ \text{Top}_3$, where BO_3 is the classifying space of O_3. It follows from [**9**, 253] that $\pi_i(\text{Top}_3/O_3) = 0$, $i \leqq 4$. Thus $\pi_2(\text{Top}_3) \approx \pi_2(O_3) = 0$, and it follows that $\pi_2(\text{Top}_3) \to \pi_2(G_2)$ is the 0-map. Since Top/O has homotopy groups $\pi_i K(Z_2, 3)$, $i \leqq 6$ [**9**, p. 251], we conclude that $\pi_2(\text{Top}) = 0$. This means that the sequence

$$\pi_2(\text{Top}_3) \to \pi_2(\text{Top}_4) \to \ldots \quad \text{(inclusion-induced)}$$

stabilizes to 0. So on considering the commutative diagram,

$$\begin{array}{cccc} & \pi_2(Top_3) & \to & \pi_2(Top_4) \to \ldots \\ (\tilde{e})_* & \downarrow & & (\tilde{e})_*\downarrow \\ & \pi_2(G_2) & \to & \pi_2(G_3) \to \ldots , \end{array}$$

we conclude that Image $(\pi_2(F_2) \to \pi_2(G_{2+k}))$ does not lie in Image $(\pi_2(\text{Top}_{3+k}) \to \pi_2(G_{2+k}))$, for any $k \geqq 0$.

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References

1. M. Brown, *Locally flat embeddings of topological manifolds*, Topology of 3-manifolds (Prentice-Hall, 1962).
2. T. A. Chapman, *Homotopic homeomorphisms of infinite-dimensional manifolds*, Compositio Math. *27* (1973), 138–140.
3. ——— *Lectures on Hilbert cube manifolds*, C.B.M.S. Regional Conference Series in Math. *28*, 1975.
4. T. A. Chapman and Steve Ferry, *Fibering Hilbert cube manifolds over ANRs*, Compositio Math., *36* (1978), 7–35.
5. T. A. Chapman and R. Y. T. Wong, *On homeomorphisms of infinite-dimensional bundles, III*, Trans. *A.M.S. 191* (1974), 269–276.
6. Steve Ferry, *The homeomorphism group of a compact Hilbert cube manifold is an ANR*, Annals of Math., *106* (1977), 101–119.
7. D. Husemoller, *Fiber bundles* (McGraw-Hill, 1966).
8. R. C. Kirby and L. C. Siebenmann, *Normal bundles for codimension 2 locally flat embeddings*, Proc. Topology Conf., Park City, Utah, Feb. 1974, Springer Lecture Notes in Math. *438*, 1975.
9. ——— *Foundational essays on topological manifolds, smoothings and triangulations*, Annals of Math. Studies (Princeton University Press, 1977).
10. J. Milnor, *Construction of universal bundles I*, Annals of Math. *63* (1956), 272–284.
11. ——— *Topological manifolds and smooth manifolds*, Proc. Int. Cong. Math., Stockholm, 1962.
12. W. Nowell, *Locally flat embeddings of Hilbert cube manifolds*, Doctoral Dissertation, University of Kentucky, 1977.
13. C. P. Rourke and B. J. Sanderson, *An embedding without a normal microbundle*, Invent. Math. *3* (1967), 293–299.
14. J. Stasheff, *A classification theorem for fiber spaces*, Topology *2* (1963), 239–246.
15. Ronald J. Stern, *On topological and piecewise linear vector fields*, Topology *14* (1975), 257–269.
16. R. Y. T. Wong, *On homeomorphisms of certain infinite-dimensional spaces*, Trans. A.M.S. *128* (1967), 148–154.
17. ——— *Parametric extensions of homeomorphisms on s-manifolds*, Bull. of the Inst. Math. Academic Sinica *2* (1974), 121–126.

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