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COMPACT COMPOSITION OPERATORS

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Abstract

Let (X, ζ, λ) be a σ-finite measure space, and let φ be a non-singular measurable transformation from X into itself. Then a composition transformation C_φ on $L^2(\lambda)$ is defined by $C_\varphi f = f \circ \varphi$. If C_φ is a bounded operator, then it is called a composition operator. The space $L^2(\lambda)$ is said to admit compact composition operators if there exists a φ such that C_φ is compact. This note is a report on the spaces which admit or which do not admit compact composition operators.

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1. Preliminaries

Let (X, ζ, λ) be a σ-finite measure space, and let φ be a non-singular measurable transformation (that is, one for which $\lambda\varphi^{-1}(E) = 0$ whenever $\lambda(E) = 0$) from X into itself. Then a composition transformation C_φ on $L^p(\lambda)$ $(p \geqslant 1)$ is defined as

$$C_\varphi f = f \circ \varphi \quad \text{for every } f \in L^p(\lambda).$$

If C_φ is a bounded operator on $L^p(\lambda)$, we call it a composition operator induced by φ. Every essentially bounded complex-valued measurable function θ on X induces the operator M_θ on $L^p(\lambda)$, which is defined by

$$M_\theta f = \theta . f \quad \text{for every } f \in L^p(\lambda).$$

The operator M_θ is known as the multiplication operator induced by θ. The Banach space $L^p(\lambda)$ is said to admit compact composition operators if there exists at least one non-singular measurable transformation φ such that C_φ is a compact composition operator on $L^p(\lambda)$.

The main object of this note is to describe spaces which admit and which do not admit compact composition operators in the case when $p = 2$.

For $\varepsilon > 0$, let X_ε^θ denote the set $\{x: x \in X \text{ and } |\theta(x)| > \varepsilon\}$ and let Z_ε^θ denote the subspace of $L^2(\lambda)$ consisting of all those functions which vanish outside X_ε^θ. The Banach algebra of all bounded linear operators on $L^2(\lambda)$ will be denoted by $B(L^2(\lambda))$. Now we shall prove the following lemma.

LEMMA 1.1. *Let $M_\theta \in B(L^2(\lambda))$. Then M_θ is compact if and only if Z_ε^θ is finite dimensional for every $\varepsilon > 0$.*

PROOF. Suppose M_θ is compact. Then the restriction of M_θ to Z_ε^θ is also compact. Since M_θ is invertible on Z_ε^θ (see Halmos (1967), Problem 52), we can conclude that Z_ε^θ is finite dimensional.

Conversely, if $Z_{1/n}^\theta$ is finite dimensional for every natural number n, then the operator M_{θ_n} is a compact operator, where $\theta_n = \theta$ on $X_{1/n}^\theta$ and 0 outside $X_{1/n}^\theta$. It is clear that the sequence $\{M_{\theta_n}\}$ converges to M_θ in norm. Since each M_{θ_n} is of finite rank, by Problem 138 of Halmos (1967), M_θ is compact.

The following well-known result is a corollary to the above lemma.

COROLLARY 1.1. *Let λ be a non-atomic measure, and let $M_\theta \in B(L^2(\lambda))$. Then M_θ is compact if and only if M_θ is the zero operator.*

COROLLARY 1.2. *Let $X = X_1 \cup X_2$ be the decomposition of X into non-atomic and atomic parts respectively, and let $M_\theta \in B(L^2(\lambda))$. Then M_θ is compact implies that $\theta = 0$ almost everywhere on X_1.*

PROOF. The subspace $L^2(\lambda_1)$ is invariant under M_θ, where $\lambda_1 = \lambda - \lambda_2$, λ_2 being the restriction of λ to X_2, the atomic part of X (see Zaanan (1967)). Hence M_θ is compact on $L^2(\lambda_1)$. By Corollary 1.1 $M_\theta = 0$ on $L^2(\lambda_1)$ from which it follows that $\theta = 0$ almost everywhere on X_1.

COROLLARY 1.3. *Let $M_\theta \in B(L^2(\lambda))$ be a one-to-one operator. Then M_θ is compact implies that λ is an atomic measure.*

PROOF. If M_θ is compact, then, by Corollary 1.2, $\theta = 0$ almost everywhere on X_1 and hence $L^2(\lambda_1) \subseteq N(M_\theta)$, where $N(M_\theta)$ denotes the null space of M_θ. Since M_θ is one-to-one, it is clear that $\lambda_1 = 0$. Hence $\lambda = \lambda_2$, which shows that λ is an atomic measure.

2. Compact composition operators

THEOREM 2.1. *Let $C_\varphi \in B(L^2(\lambda))$. Then C_φ is compact if and only if $Z_\varepsilon^{f_0}$ is finite*

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dimensional for every $\varepsilon > 0$, *where* f_0 *is the Radon–Nikodym derivative of* $\lambda\varphi^{-1}$ *with respect to* λ.

PROOF. It is known that an operator A is compact if and only if $A^* A$ is compact. By a result of Singh (1974) $C_\varphi^* C_\varphi = M_{f_0}$. Hence by Lemma 1.1 the result follows.

COROLLARY 2.1. *Let* (X, ζ, λ) *be a non-atomic measure space. Then* $L^2(\lambda)$ *does not admit a compact composition operator.*

PROOF. Suppose $C_\varphi \in B(L^2(\lambda))$ is compact. Then M_{f_0} is compact. By Corollary 1.1, M_{f_0} is the zero operator, and hence C_φ is the zero operator. But no composition operator is the zero operator. Hence the proof is finished.

COROLLARY 2.2. *If* $C_\varphi \in B(L^2(\lambda))$ *is one-to-one and compact, then* λ *is atomic.*

Let $p = \{p_1, p_2, p_3, \ldots\}$ be a sequence of strictly positive numbers, and let $l^2(p)$ be the Hilbert space of all complex sequences $\{x_1, x_2, \ldots\}$ such that

$$\sum_{n=1}^{\infty} |x_n|^2 p_n < \infty.$$

Then, in this case, for a mapping φ from the set of positive integers into itself the Radon–Nikodym derivative f_0 is given by

$$f_0(m) = \frac{\lambda(\varphi^{-1}(\{m\}))}{\lambda(\{m\})} = \frac{1}{p_m} \sum_{\varphi(n)=m} p_n.$$

By Theorem 1 of Singh (1976) it follows that C_φ is bounded if and only if $\{f_0(m)\}$ is a bounded sequence. In the light of Theorem 2.1 it is obvious that C_φ is compact if and only if $f_0(m) \to 0$ as $m \to \infty$. We shall use these two facts in the following two theorems.

THEOREM 2.2. *If* $0 < \limsup_{n\to\infty} p_n < \infty$, *then* $l^2(p)$ *does not admit a compact composition operator.*

PROOF. Let $\sup p_n = \beta$. There is $\alpha > 0$ such that $p_n \geqslant \alpha$ for all $n \in K$, an infinite subset of positive integers. If C_φ is compact and K_m is the subset of K consisting of those n for which $\varphi(n) = m$,

$$f_0(m) = \frac{1}{p_m} \sum_{\varphi(n)=m} p_n \geqslant \frac{\alpha}{\beta} \sum_{n \in K_m} 1_n, \quad \text{where } 1_n = 1.$$

Thus K_m must be finite for each m and so, K being infinite, $\varphi(K)$ must be infinite. But, for each $m \in \varphi(K)$, $f_0(m) \geqslant \alpha/\beta$ and so $f_0(m) \nrightarrow 0$, which is a contradiction.

COROLLARY 2.3. *The Hilbert space l^2 does not admit a compact composition operator.*

THEOREM 2.3. *If* $\sup p_n = \infty$, *then* $l^2(p)$ *admits a compact composition operator.*

PROOF. Define inductively a strictly increasing sequence $\{\varphi(n)\}$ such that, for each n, $p_{\varphi(n)} > n.p_n$. Then for this C_φ, $f_0(m) = 0$ if m is not $\varphi(n)$ for some n, while $f_0(\varphi(n)) = p_n/p_{\varphi(n)} < 1/n$. Thus C_φ is compact. This proves the theorem.

THEOREM 2.4. *Let* $p = \{p_1, p_2, \dots\}$ *and* $\sum_{i=1}^{\infty} p_i < \infty$. *Then* $l^2(p)$ *admits a compact composition operator.*

PROOF. Let m be an arbitrary fixed positive integer. Then define the function φ as

$$\varphi(n) = \begin{cases} n & \text{if } n < m, \\ m & \text{if } n \geqslant m. \end{cases}$$

The operator C_φ is bounded and compact.

In the proof of the above theorem we have obtained a finite rank composition operator. But, in this case, there do exist compact composition operators which are not of finite rank. This is shown in the following example.

EXAMPLE 2.1. If $p_n = a^{2n}$, where $0 < a < 1$, and $\varphi(n) = n/2$ in case n is even and $\varphi(n) = (n+1)/2$ in case n is odd, then C_φ is one-to-one and compact on $l^2(p)$.

In support of Theorem 2.3 we cite the following example.

EXAMPLE 2.2. Let $p_n = 1/n$ if n is even and $p_n = n$ if n is odd. Let $\varphi(n) = (n-1)$ if n is even and $\varphi(n) = n^2$ if n is odd. Then C_φ is bounded and compact.

Let (X, ζ, λ) be a σ-finite measure space, and let $X = X_1 \cup X_2$ be the decomposition of X into non-atomic and atomic parts respectively. From now on we shall assume that X_1 and X_2 are non-null measurable subsets of X. Without any loss of generality we can assume that atoms are points. First, we shall prove the following lemma.

LEMMA 2.5. *If* $C_\varphi \in B(L^2(\lambda))$, *then* C_φ *is compact implies that* $X = \varphi^{-1}(X_2)$.

PROOF. If C_φ is compact, then M_{f_0} is compact, and hence by Corollary 1.2 $f_0 = 0$ almost everywhere on X_1. Therefore $\lambda\varphi^{-1}(X_1) = 0$. Since

$$X = \varphi^{-1}(X_1) \cup \varphi^{-1}(X_2),$$

we have $X = \varphi^{-1}(X_2)$.

COROLLARY 2.4. *Let $X = X_1 \cup X_2$ be the decomposition of X, and let $\lambda(X) = \infty$ and $\lambda(X_2) < \infty$. Then $L^2(\lambda)$ does not admit a compact composition operator.*

PROOF. The proof follows from Lemma 2.5 and Theorem 1 of Singh (1976).

COROLLARY 2.5. *Let $X = X_1 \cup X_2$ be the decomposition of X, and let $\lambda(X) = \infty$. Then $L^2(\lambda)$ admits a compact composition operator only if $\lambda(X_2) = \infty$.*

THEOREM 2.6. Let (X, ζ, λ) be a σ-finite measure space, and let

$$0 \leqslant \alpha \leqslant \lambda(\{x\}) \leqslant \beta < \infty \quad \text{for every } x \in X_2.$$

Then $L^2(\lambda)$ does not admit a compact composition operator.

PROOF. If X contains finitely many atoms and there exists a compact composition operator C_φ, then by Lemma 2.5 $\varphi^{-1}(X_2) = X$, which contradicts the boundedness of C_φ (see Singh (1976)).

If X has infinitely many atoms and $L^2(\lambda)$ admits a compact composition operator C_φ, then $Z_\delta^{f_0}$ is finite dimensional for every $\delta > 0$. This implies that $X_\delta^{f_0}$ has finitely many atoms. Since $\lambda(X_2) = \infty$, C_φ cannot be bounded, which is a contradiction.

THEOREM 2.7. *Let X be a σ-finite measure space of infinite measure with infinitely many atoms, and let* $\inf\{\lambda(\{x\}) : x \in X_2\} = \alpha > 0$. *Then $L^2(\lambda)$ admits a compact composition operator if and only if* $\sup\{\lambda(\{x\}) : x \in X_2\} = \infty$.

PROOF. The necessary part follows from Theorem 2.5 and the proof of the sufficient part is analogous to the proof of Theorem 2.3.

THEOREM 2.8. *Let X be a σ-finite measure space of infinite measure with infinitely many atoms, and let* $\sup\{\lambda(\{x\}) : x \in X_2\} = \beta > 0$. *Then $L^2(\lambda)$ admits a compact composition operator only if* $\inf\{\lambda(\{x\}) : x \in X_2\} = 0$.

PROOF. The proof follows from Theorem 2.6.

The converse of the above theorem is not true as is shown in the following example.

EXAMPLE 2.3. Let $X = [0, \frac{1}{2}] \cup N$, where N is the set of natural numbers, and let λ be the Lebesgue measure on $[0, \frac{1}{2}]$ and $\lambda(\{n\}) = 1$ if n is odd and $1/2^n$ if n is even. Then $L^2(\lambda)$ does not admit a compact composition operator.

THEOREM 2.9. *Let X be a σ-finite measure space with infinitely many atoms and* $\sup\{\lambda(\{x\}) : x \in X_2\} = \infty$. *Then $L^2(\lambda)$ admits a compact composition operator.*

PROOF. The proof is analogous to the proof of Theorem 2.3.

THEOREM 2.10. *Let (X,ζ,λ) be a totally finite measure space with finitely many atoms, and let $C_\varphi \in B(L^2(\lambda))$. Then C_φ is compact if and only if $X=\varphi^{-1}(X_2)$.*

PROOF. The proof is obvious.

The following two examples illustrate Theorems 2.9 and 2.10.

EXAMPLE 2.4. Let $X=]-\infty,0]\cup N$, where N is the set of natural numbers. Let λ be the Lebesgue measure on $]-\infty,0]$, and $\lambda(\{n\})=n^2$ if n is odd and $1/2^n$ if n is even. Then, if $\varphi(x)=2n+1$ for $x\in]-\infty,0]$ and $(n-1)\leqslant -x<n$, $\varphi(x)=x^2$ in case x is an odd positive integer and $\varphi(x)=(x+1)^2$ in case x is an even positive integer, C_φ is a compact composition operator.

EXAMPLE 2.5. Let $X=[0,1]\cup\{2,3\}$. Let λ be the Lebesgue measure on $[0,1]$ and $\lambda(\{2\})=\lambda(\{3\})=1$. Let $\varphi(x)=2$ if $x\in[0,1]\cup\{3\}$, and $\varphi(2)=3$. If ζ is the σ-algebra of λ-measurable subsets of X, then $\varphi^{-1}(\zeta)$ has finitely many elements and $L^2(X,\varphi^{-1}(\zeta),\lambda)$ is the range of C_φ. Hence C_φ is of finite rank and therefore it is compact.

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References

P. R. Halmos (1967), *A Hilbert space problem book* (Van Nostrand, Princeton, N.J., 1967).
W. C. Ridge (1973), 'Spectrum of a composition operator', *Proc. Amer. Math. Soc.* **37**, 121–127.
R. K. Singh (1974), 'Compact and quasinormal composition operators', *Proc. Amer. Math. Soc.* **45**, 80–82.
R. K. Singh (1976), 'Composition operators induced by rational functions', *Proc. Amer. Math. Soc.* **59**, 329–333.
A. C. Zaanan (1967), *Integration*, completely revised edition of *An introduction to the theory of integration* (Interscience, New York, 1967).

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