Canad. Math. Bull. Vol. **64** (4), 2023, pp. 1411–1422
http://dx.doi.org/10.4153/S0008439523000541

Bohr operator on operator-valued polyanalytic functions on simply connected domains

Vasudevarao Allu and Himadri Halder

Abstract. In this article, we study the Bohr operator for the operator-valued subordination class $S(f)$ consisting of holomorphic functions subordinate to f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, where $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is holomorphic and $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$. We establish several subordination results, which can be viewed as the analogs of a couple of interesting subordination results from scalar-valued settings. We also obtain a von Neumann-type inequality for the class of analytic self-mappings of the unit disk $\mathbb{D}$ which fix the origin. Furthermore, we extensively study Bohr inequalities for operator-valued polyanalytic functions in certain proper simply connected domains in $\mathbb{C}$. We obtain Bohr radius for the operator-valued polyanalytic functions of the form $F(z) = \sum_{l=0}^{p-1} \bar{z}^l f_l(z)$, where f_0 is subordinate to an operator-valued convex biholomorphic function, and operator-valued starlike biholomorphic function in the unit disk $\mathbb{D}$.

1 Introduction

Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions from the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ into the complex plane $\mathbb{C}$ and denote $\|f\|_\infty := \sup_{|z|<1} |f(z)|$. In 1914, the following remarkable result for the universal constant $r = 1/3$ for functions in $H^\infty(\mathbb{D})$ was proved by Bohr [13].

Theorem A *Let $f \in H^\infty(\mathbb{D})$ with the power series $f(z) = \sum_{n=0}^\infty a_n z^n$. Then*

$$\sum_{n=0}^{\infty} |a_n| r^n \le \|f\|_\infty \tag{1.1}$$

for $|z| = r \le 1/3$, and the constant $1/3$, referred to as the classical Bohr radius, is the best possible.

The interest in the Bohr inequality has been revived when Dixon [15] used it to disprove the conjecture that if the nonunital von Neumann's inequality holds for a Banach algebra, then it is necessarily an operator algebra. In 2004, Paulsen and

Received by the editors May 19, 2023; revised June 21, 2023; accepted June 21, 2023.
Published online on Cambridge Core June 26, 2023.
The first-named author is supported by SERB-CRG, and the second-named author is supported by the Institute Postdoctoral Fellowship of IIT Bombay, India.
AMS subject classification: **47A56, 30B10, 47A63**, 30C45.
Keywords: Banach algebra, von Neumann inequality, polyanalytic functions, Bohr operator, simply connected domains.

Singh [22] extended Bohr's theorem to Banach algebras by finding a general version of Bohr inequality which is valid in the context of uniform algebras. For fixed $z \in \mathbb{D}$, we denote

$$\mathcal{G}_z := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : f \in H^\infty(\mathbb{D}) \right\}.$$

For $|z| = r$, the Bohr operator M_r on $\mathcal{G}_z$ is defined by

$$M_r(f) = \sum_{n=0}^{\infty} |a_n||z^n| = \sum_{n=0}^{\infty} |a_n| r^n.$$

The Bohr operator has the following properties, which has been established in [23].

Theorem 1.1 [2] *For each fixed $z \in \mathbb{D}$ and $|z| = r$, the Bohr operator M_r satisfies:*

(1) $M_r(f) \geq 0$, *and* $M_r(f) = 0$ *if, and only if,* $f \equiv 0$,
(2) $M_r(f + g) \leq M_r(f) + M_r(g)$,
(3) $M_r(\alpha f) = |\alpha| M_r(f)$, $\alpha \in \mathbb{C}$,
(4) $M_r(f.g) \leq M_r(f).M_r(g)$,
(5) $M_r(1) = 1$.

By the virtue of Theorem 1.1, it is worth to mention that the space $\mathcal{G}_z$ with the norm M_r constitutes a Banach algebra. However, not all Banach spaces satisfy the Bohr phenomenon. In [7], Bénéteau *et al.* have shown that H^q, the usual Hardy spaces in $\mathbb{D}$, do not satisfy the Bohr phenomenon for any $0 < q < \infty$. A complex Banach algebra $\mathcal{A}$ satisfies the von Neumann inequality if for all polynomial $p(X)$ and for all $x \in \mathcal{A}$ with $\|x\| \leq 1$,

$$\|p(x)\| \leq \|p\|_\infty . \tag{1.2}$$

In [25], von Neumann has shown that the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ satisfies the inequality (1.2). It is well known that every Banach algebra which is an operator algebra (i.e., which is isometrically isomorphic to a closed subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$) also satisfies the von Neumann inequality (1.2). Bohr inequality has been extended to several complex variables and more abstract settings (see [8, 12, 14, 19, 21, 22, 24]).

Another interesting aspect of Bohr phenomenon thrives on considering the Bohr radius problem for subordinating families of analytic functions in $\mathbb{D}$. For two analytic functions g and f in $\mathbb{D}$, we say that g is subordinate to f, written $g \prec f$, if there exists an analytic function $\phi : \mathbb{D} \to \mathbb{D}$ with $\phi(0) = 0$ such that $g(z) = f(\phi(z))$ in $\mathbb{D}$. Let $S(f)$ be the class of analytic functions subordinate to f in $\mathbb{D}$. We say that g is quasi-subordinate to f if there exists an analytic function ψ with $|\psi(z)| \leq 1$ in $\mathbb{D}$ such that $g(z) = \psi(z) f(\phi(z))$ in $\mathbb{D}$. It is well known that if g is subordinate (or quasi-subordinate) to f in $\mathbb{D}$, then $M_r(g) \leq M_r(f)$ for $|z| = r \leq 1/3$. Bhowmik and Das [10] have studied the Bohr radius for the subordinating families, and the Bohr radius for quasi-subordination families has been studied by Alkhaleefah *et al.* [5]. In 2021, Bhowmik and Das [11] extended the Bohr phenomenon for the subordination to operator-valued analytic functions in $\mathbb{D}$. Throughout this article, $\mathcal{B}(\mathcal{H})$ stands for the space of bounded linear operators on a complex Hilbert space $\mathcal{H}$. We want to concentrate operator-valued holomorphic functions $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$. The term subordination for operator-valued functions can be defined as the scalar valued case. That is, for two holomorphic functions $g, f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$, we say that g is subordinate to f, written $g \prec f$, if there exists a holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ with $\phi(0) = 0$ such that $g(z) = f(\phi(z))$ in $\mathbb{D}$. Let $S(f)$ be the class of analytic functions subordinate to f in $\mathbb{D}$. For given two Banach spaces X and Y and a domain $\Omega \subset X$, a holomorphic function $f : \Omega \to Y$ is said to be biholomorphic on Ω if f^{-1} exists and is holomorphic in $f(\Omega) \subseteq Y$. We say that a biholomorphic function f is starlike in its domain Ω with respect to $\xi_0 \in \Omega$ if $f(\Omega)$ is a starlike domain with respect to $f(\xi_0)$, *i.e.*, $(1-t)f(\xi_0) + tf(z) \in f(\Omega)$ for all $z \in \Omega$ and $t \in [0,1]$. A biholomorphic function f is called starlike if f is starlike with respect to $0 \in \Omega$ and $f(0) = 0$. A biholomorphic function f is said to be convex if f is starlike with respect to every point in Ω. For convex or star-like biholomorphic function f in $\mathbb{D}$, Bohr phenomenon for any $g \in S(f)$ has been studied in [11].

For the rest of our discussion, we introduce some notations. Throughout this paper, $\|A\|$ stands for the operator norm of A for any $A \in \mathcal{B}(\mathcal{H})$ and $|A| = (A^*A)^{1/2}$ denotes the absolute value of A, where A^* is the adjoint of A and $T^{1/2}$ denotes the unique positive square root of a positive operator T. We denote I be the identity operator on $\mathcal{H}$.

In 2010, Fournier and Ruscheweyh [16] extensively studied the Bohr radius problem for arbitrary simply connected domains containing $\mathbb{D}$. Let $\mathcal{H}(\Omega)$ be the class of analytic functions $f : \Omega \to \mathbb{C}$, and let $\mathcal{B}(\Omega)$ denote the class of functions $f \in \mathcal{H}(\Omega)$ such that $f(\Omega) \subseteq \overline{\mathbb{D}}$. For the class $\mathcal{B}(\Omega)$, the Bohr radius $\mathcal{B}_\Omega$ is defined by (see [4, 16])

$$B_\Omega := \sup\left\{r \in (0,1) : M_r(f) \le 1 \text{ for all } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}(\Omega),\ z \in \mathbb{D}\right\},$$

where $M_r(f) := \sum_{n=0}^{\infty} |a_n| r^n$ is the Bohr operator for $f \in \mathcal{B}(\Omega)$ in $\mathbb{D}$. For $\Omega = \mathbb{D}$, $\mathcal{B}(\Omega)$ reduces to $B_{\mathbb{D}} = 1/3$, which is the classical Bohr radius for the class $\mathcal{B}(\mathbb{D})$.

For $0 \le \gamma < 1$, Fournier and Ruscheweyh [16] have estimated the Bohr radius for the class $\mathcal{B}(\Omega_\gamma)$ and proved that $\mathcal{B}_{\Omega_\gamma} = (1+\gamma)/(3+\gamma)$, where

$$\Omega_\gamma := \left\{z \in \mathbb{C} : \left|z + \frac{\gamma}{1-\gamma}\right| < \frac{1}{1-\gamma}\right\}.$$

Let $H^\infty(\Omega, X)$ be the space of bounded analytic functions from Ω into a complex Banach space X and $\|f\|_{H^\infty(\Omega,X)} = \sup_{z\in\Omega} \|f(z)\|$. The Bohr phenomenon for operator-valued functions on simply connected domains has been studied in [6]. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For the class $H^\infty(\Omega, \mathcal{B}(\mathcal{H}))$, we denote (see [6])

(1.3)
$$\lambda_{\mathcal{H}} := \lambda_{\mathcal{H}}(\Omega) := \sup_{\substack{f \in H^\infty(\Omega, \mathcal{B}(\mathcal{H})) \\ \|f(z)\| \le 1}} \left\{\frac{\|A_n\|}{\|I - |A_0|^2\|} : A_0 \not\equiv f(z) = \sum_{n=0}^{\infty} A_n z^n,\ z \in \mathbb{D}\right\}.$$

https://doi.org/10.4153/S0008439523000541 Published online by Cambridge University Press

Recently, Allu and Halder [6] have established the Bohr theorem for the functions in $H^\infty(\Omega,\mathcal{B}(\mathcal{H}))$.

Theorem 1.2 [6] *Let $f\in H^\infty(\Omega,\mathcal{B}(\mathcal{H}))$ with $\|f(z)\|_{H^\infty(\Omega,\mathcal{B}(\mathcal{H}))}\le 1$ such that $f(z)=\sum_{n=0}^\infty A_nz^n$ in $\mathbb{D}$, where $A_0=\alpha_0 I$ for $|\alpha_0|<1$ and $A_n\in\mathcal{B}(\mathcal{H})$ for all $n\in\mathbb{N}\cup\{0\}$. Then*

$$\sum_{n=0}^\infty \|A_n\|\,r^n\le 1 \quad \textit{for}\quad r\le \frac{1}{1+2\lambda_{\mathcal{H}}}. \tag{1.4}$$

For $\Omega=\Omega_\gamma$ and $p=1$ in [6, Corollary 1.52], we have the following result.

Theorem 1.3 [6] *Let $f\in H^\infty(\Omega_\gamma,\mathcal{B}(\mathcal{H}))$ with $\|f(z)\|_{H^\infty(\Omega_\gamma,\mathcal{B}(\mathcal{H}))}\le 1$ such that $f(z)=\sum_{n=0}^\infty A_nz^n$ in $\mathbb{D}$, where $A_0=\alpha_0 I$ for $|\alpha_0|<1$ and $A_n\in\mathcal{B}(\mathcal{H})$ for all $n\in\mathbb{N}\cup\{0\}$. Then*

$$\sum_{n=0}^\infty \|A_n\|\,r^n\le 1 \quad \textit{for}\quad r\le \frac{1+\gamma}{3+\gamma}. \tag{1.5}$$

When $\Omega_\gamma=\mathbb{D}$, i.e., $\gamma=0$, under the same assumptions as in Theorem 1.3, we have

$$\sum_{n=0}^\infty \|A_n\|\,r^n\le 1 \quad \text{for}\quad r\le \frac{1}{3}. \tag{1.6}$$

2 Bohr operator on operator-valued subordination classes

In this section, we study subordination results for Bohr operator on operator-valued analytic functions in $\mathbb{D}$. Recall that $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For analytic functions $f:\mathbb{D}\to\mathcal{B}(\mathcal{H})$ with $f(z)=\sum_{n=0}^\infty A_nz^n$ in $\mathbb{D}$ and $A_n\in\mathcal{B}(\mathcal{H})$ for $n\in\mathbb{N}\cup\{0\}$, we define the Bohr operator $M_r(f)$ as the scalar valued case. That is, $M_r(f)=\sum_{n=0}^\infty\|A_n\|\,|z|^n$. It can be easily seen that the operator M_r satisfies the same property as in Theorem 1.1. In fact, for $f,g:\mathbb{D}\to\mathcal{B}(\mathcal{H})$ with $f(z)=\sum_{n=0}^\infty A_nz^n$ and $g(z)=\sum_{n=0}^\infty B_nz^n$ in $\mathbb{D}$ with $A_n,B_n\in\mathcal{B}(\mathcal{H})$ for $n\in\mathbb{N}\cup\{0\}$, we have

$$M_r(f+g)=\sum_{n=0}^\infty\|A_n+B_n\|\,r^n\le\sum_{n=0}^\infty\|A_n\|\,r^n+\sum_{n=0}^\infty\|B_n\|\,r^n=M_r(f)+M_r(g). \tag{2.1}$$

Using (2.1), it is easy to see that if $F(z)=\sum_{k\in\mathbb{Z}}f_k(z)$ is analytic function in $\mathbb{D}$, then $M_r(F)\le\sum_{k\in\mathbb{Z}}M_r(f_k)$, where $f_k:\mathbb{D}\to\mathcal{B}(\mathcal{H})$ is analytic function in $\mathbb{D}$ for each $k\in\mathbb{Z}$. On the other hand, we note that $M_r(\beta f)=|\beta|M_r(f)$ for any $\beta\in\mathbb{C}$ and $M_r(z^pf)=r^pM_r(f)$. We observe that $(fg)(z)=\sum_{n=0}^\infty A_n(z^ng(z))$ and hence

$$M_r(fg)\le\sum_{n=0}^\infty\|A_n\|\,r^nM_r(g)=M_r(f)M_r(g). \tag{2.2}$$

Clearly, $M_r(I)=1$. The following result has been established in [2].

https://doi.org/10.4153/S0008439523000541 Published online by Cambridge University Press

Lemma 2.3 [2] *Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic function with $\phi(0) = 0$. Then $M_r(\phi) \le |z|$ for $|z| = r \le 1/3$.*

The following result is the operator-valued subordination result for Bohr operator, which has been first proved in [11]. By using Lemma 2.3, we give an alternative proof.

Theorem 2.1 *Let $f, g : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be holomorphic functions such that $f \prec g$. Then*

$$M_r(f) \le M_r(g) \quad \text{for } |z| = r \le \frac{1}{3}. \tag{2.4}$$

Proof Let $f(z) = \sum_{n=0}^{\infty} A_n z^n$ and $g(z) = \sum_{n=0}^{\infty} B_n z^n$ in $\mathbb{D}$ with $A_n, B_n \in \mathcal{B}(\mathcal{H})$ for $n \in \mathbb{N} \cup \{0\}$. Since $f \prec g$ in $\mathbb{D}$, then there exists an analytic function $\phi : \mathbb{D} \to \mathbb{D}$ such that $\phi(0) = 0$ and $f(z) = g(\phi(z))$ in $\mathbb{D}$. In view of (2.1), (2.2), and Lemma 2.3, for $0 \le |z| = r \le 1/3$, we obtain

$$\begin{aligned} M_r(f) = M_r(g(\phi)) &= M_r\left(\sum_{n=0}^{\infty} B_n(\phi(z))^n\right) \\ &\le \sum_{n=0}^{\infty} \|B_n\| \left(M_r(\phi(z))\right)^n \le \sum_{n=0}^{\infty} \|B_n\| r^n = M_r(g). \end{aligned}$$

This completes the proof. ∎

In particular, for the scalar-valued functions $f, g : \mathbb{D} \to \mathbb{C}$, Theorem 2.1 reduces to the result of Abu Muhanna *et al.* [2], and Bhowmik and Das [10]. In view of Theorem 2.1, we obtain the following interesting result.

Theorem 2.2 *Let $f, g, h : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be holomorphic functions such that $f(z) = h(z)g(\phi(z))$ for some analytic function $\phi : \mathbb{D} \to \mathbb{D}$ with $\phi(0) = 0$. If $\|h(z)\| \le M$ for $|z| < \beta \le 1$ and $h(0) = \alpha I$ with $|\alpha| \le M$, then $M_r(f) \le M\, M_r(g)$ for $0 \le r \le \beta/3$.*

Proof From (2.2), we have

$$M_r(f) \le M_r(h) M_r(g(\phi)). \tag{2.5}$$

The assumption $\|h(z)\| \le M$ in the disk $\mathbb{D}_\beta := \{z \in \mathbb{C} : |z| < \beta\}$ shows that $h_1 : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ defined by $h_1(z) = h(z)/M$ is holomorphic and $\|h_1(z)\| \le 1$ in $\mathbb{D}_\beta$ such that $h_1(0) = (\alpha/M)I$. Since $|\alpha| \le M$, from (1.6), we obtain

$$M_r(h) \le M \quad \text{for } 0 < r \le \frac{\beta}{3}. \tag{2.6}$$

Furthermore, in view of Theorem 2.1, we have

$$M_r(g(\phi)) \le M_r(g) \quad \text{for } 0 < r \le \frac{1}{3}. \tag{2.7}$$

By using (2.6) and (2.7) in (2.5), we obtain

$$M_r(f) \le M\ M_r(g) \quad \text{for } 0 < r \le \frac{\beta}{3}. \tag{2.8}$$

This completes the proof. ∎

https://doi.org/10.4153/S0008439523000541 Published online by Cambridge University Press

Remark 2.1 (1) For a particular case $h(z) \equiv I$, Theorem 2.2 reduces to Theorem 2.1. By taking $f, g, h : \mathbb{D} \to \mathbb{C}$ are analytic functions in Theorem 2.2, we obtain the scalar-valued quasi-subordination result, which has been established in [5].

(2) When $\|h(z)\| \le 1$ in $\mathbb{D}$, we deduce that $M_r(f) \le M_r(g)$ for $|z| = r \le 1/3$.

We now prove the following interesting result, which is an analog of von Neumann inequality (1.2).

Theorem 2.3 *Let $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ such that $f(0) = \alpha I$ for some $\alpha \in \mathbb{C}$ with $|\alpha| < 1$. Then*

$$M_r(f(\phi)) \le \|f\|_\infty \quad \text{for} \quad 0 \le r \le 1/3, \tag{2.9}$$

where $\phi : \mathbb{D} \to \mathbb{D}$ is analytic function with $\phi(0) = 0$.

Proof Let $f(z) = \sum_{n=0}^{\infty} A_n z^n$ in $\mathbb{D}$, where $A_0 = \alpha I$ and $A_n \in \mathcal{B}(\mathcal{H})$ for $n \in \mathbb{N} \cup \{0\}$. Then, for $r \le 1/3$, Theorem 2.1 gives

$$M_r(f(\phi)) \le \sum_{n=0}^{\infty} \|A_n\| r^n = M_r(f). \tag{2.10}$$

In view of (1.6), for $0 \le r \le 1/3$, we obtain $M_r(f) \le \|f\|_\infty$ which together with (2.10) gives (2.9). ∎

3 Bohr theorem for operator-valued polyanalytic functions

Polyanalytic functions f of order p defined in a simply connected domain $\Omega \subseteq \mathbb{C}$ are complex-valued polynomials in the variable $\overline{z}$ with analytic functions are their coefficients. That is, f has the following form $f(z) = \sum_{l=0}^{p-1} \overline{z}^p f_l(z)$, where f_l's are analytic functions in Ω. Equivalently, polyanalytic functions can also be defined as the $\mathcal{C}^p(\Omega)$-solutions of the generalized Cauchy–Riemann equations $\partial^p f/\partial\overline{z}^p = 0$ in Ω (the Cauchy–Riemann equations of order p). Throughout this paper, we assume that $p \ge 2$.

In 1908, Kolossov [20] first introduced polyanalytic functions in connection with his research in the mathematical theory of elasticity. Polyanalytic function theory has been extensively studied by Balk [9]. In 2011, Agranovsky [3] characterized the polyanalytic functions by meromorphic extensions into chains of circles. It is worth mentioning that the properties of polyanalytic functions can be different from those of analytic functions (see [9]). By considering the polyanalytic function $f(z) = 1 - z\overline{z}$, one can easily see that f vanishes on the boundary of the unit disk $\mathbb{D}$ without vanishing identically in $\mathbb{D}$. Studying polyanalytic functions also reveals some new properties of analytic functions. The study of polyanalytic functions is closely related to numerous research topics of complex analysis, e.g., function theory of several complex variables, the theory of distribution of values of meromorphic functions, the theory of meromorphic curves, and the theory of boundary properties of analytic functions. In 2019, Hachadi and Youssfi [18] have studied several properties of polyanalytic reproducing kernels. In 2021, Abdulhadi and Hajj [1] extensively studied univalency criteria, Landau's theorem, arc-length problem, and the Bohr phenomenon problem for polyanalytic functions in $\mathbb{D}$.

Since complex-valued polyanalytic functions are polynomials in $\overline{z}$ in simply connected domain Ω, this leads to study the operator-valued polyanalytic functions. A operator-valued polyanalytic function F of order p in Ω is a polynomial in $\overline{z}$ with operator-valued analytic functions as its coefficients. That is, F has the following form:

$$F(z) = \sum_{l=0}^{p-1} \overline{z}^p f_l(z),$$

where $f_l : \Omega \to \mathcal{B}(\mathcal{H})$ are analytic functions for $l = 0, 1, \ldots, p-1$ and $f_{p-1} \not\equiv 0$. Now, we consider the simply connected domain Ω containing $\mathbb{D}$.

Although Bohr radius and Bohr phenomenon have been extensively studied, no attempt has been made so far to obtain operator-valued analogs of Bohr phenomenon for polyanalytic functions. Therefore, our main aim of this section is to obtain the Bohr inequality under appropriate considerations and necessary conditions. In the following result, we establish operator-valued analogs of Bohr inequality in simply connected domain Ω containing $\mathbb{D}$.

Theorem 3.1 *Let F be a polyanalytic function of order p in Ω with $F(z) = \sum_{l=0}^{p-1} \overline{z}^l f_l(z)$, where each $f_l : \Omega \to \mathcal{B}(\mathcal{H})$ is an analytic function such that $f_l(z) = \sum_{n=0}^{\infty} A_{n,l} z^n$ in $\mathbb{D}$ and $A_{n,l} \in \mathcal{B}(\mathcal{H})$ for $n \in \mathbb{N} \cup \{0\}$. Also, assume that:*

(a) *$\|f_0(z)\| \le 1$ in Ω such that $f_0(0) = \mathbf{0}$ and $f_l'(0) = \alpha_l f_0'(0)$ with $|\alpha_l| < k$ for $k \in [0,1]$ and each $l = 1, \ldots, p-1$.*

(b) *$\omega_l : \Omega \to \mathcal{B}(\mathcal{H})$ is analytic with $\|\omega_l(z)\| \le k$ in Ω for $k \in [0,1]$, where $\omega_l(z) = f_l'(z)(f_0'(z))^{-1}$ in Ω such that $\omega_l(z) = \sum_{n=0}^{\infty} \omega_{n,l} z^n$ in $\mathbb{D}$, provided $(f_0'(z))^{-1}$ exists for all $z \in \Omega$.*

Then $M_r(F) \le 1$ for $|z| = r \le R_f = \min\{r_f(p), 1/(1+2\lambda_{\mathcal{H}})\}$, where $r_f(p)$ is the smallest root in $(0,1)$ of

$$(1-r)^2 - k\lambda_{\mathcal{H}} r + k\lambda_{\mathcal{H}} r^{p+1} = 0. \tag{3.1}$$

Here, $\lambda_{\mathcal{H}}$ is given by (1.3).

Proof Let $F(z) = \sum_{l=0}^{p-1} \overline{z}^l f_l(z)$ with $f_l(z) = \sum_{n=0}^{\infty} A_{n,l} z^n$ in $\mathbb{D}$. Then

$$M_r(F) = M_r\left(\sum_{l=0}^{p-1} \overline{z}^l f_l(z)\right) \le \sum_{l=0}^{p-1} r^l M_r(f_l) \quad \text{for} \quad |z| = r < 1. \tag{3.2}$$

Since $\omega_l(z) = f_l'(z)(f_0'(z))^{-1}$ in Ω with $\|\omega_l(z)\| \le k$ in Ω for each l such that $f_l'(0) = \alpha_l f_0'(0)$, it follows that $f_l'(z) = \omega_l(z) f_0'(z)$ in Ω with $\omega_l(0) = \alpha_l I$, where $|\alpha_l| < k$ for each $l = 1, \ldots, p-1$. Let $\lambda_{\mathcal{H}}$ be given by (1.3). In view of Theorem 1.2, for $|z| = r \le 1/(1+2\lambda_{\mathcal{H}})$, we have $M_r(\omega_l) \le k$, which together with (2.2) gives

$$M_r(f_l) = \int_0^r M_r(f_l')dt = \int_0^r M_r(\omega_l f_0')dt \le k \int_0^r M_r(f_0')dt = k\, M_r(f_0). \tag{3.3}$$

https://doi.org/10.4153/S0008439523000541 Published online by Cambridge University Press

Using (3.2) and (3.3), for $|z| = r \le 1/(1+2\lambda_{\mathcal{H}})$, we obtain

$$M_r(F) \le k \sum_{l=0}^{p-1} r^l M_r(f_0) = k M_r(f_0) \left(\frac{1-r^p}{1-r} \right). \tag{3.4}$$

We now wish to find the upper bound for $M_r(f_0)$. We observe that $f_0 : \Omega \to \mathcal{B}(\mathcal{H})$ is analytic function with $\|f_0(z)\| \le 1$ in Ω such that $f_0(z) = \sum_{n=0}^{\infty} A_{n,0} z^n$ in $\mathbb{D}$, where $f_0(0) = A_{0,0} = \mathbf{0}$. Then, in view of (1.3), we have $\|A_{n,0}\| \le \lambda_{\mathcal{H}}$ for $n \ge 1$, and hence

$$M_r(f_0) = \sum_{n=0}^{\infty} \|A_{n,0}\| \, r^n \le \lambda_{\mathcal{H}} \left(\frac{r}{1-r} \right). \tag{3.5}$$

In view of (3.4) and (3.5), for $r \le 1/(1+2\lambda_{\mathcal{H}})$, we obtain

$$M_r(F) \le k\, \lambda_{\mathcal{H}} \left(\frac{r}{1-r} \right) \left(\frac{1-r^p}{1-r} \right). \tag{3.6}$$

Therefore, $M_r(F) \le 1$ for $r \le \min\{1/(1+2\lambda_{\mathcal{H}}), r_f(p)\}$, where $r_f(p)$ is the smallest root in $(0,1)$ of

$$k\, \lambda_{\mathcal{H}} \left(\frac{r}{1-r} \right) \left(\frac{1-r^p}{1-r} \right) = 1,$$

which is equivalent to $(1-r)^2 - k\, \lambda_{\mathcal{H}}\, r + k\, \lambda_{\mathcal{H}}\, r^{p+1} = 0$. This completes the proof. ∎

As a consequence of Theorem 3.1, we obtain Bohr-type inequality for bi-analytic functions in a domain Ω.

Corollary 3.7 *Let F be a bi-analytic function in a domain Ω with the series expansion as in Theorem 3.1. Also, assume all the hypotheses as in Theorem 3.1. Then $M_r(F) \le 1$ for $|z| = r \le \min\{r_f(2), 1/(1+2\lambda_{\mathcal{H}})\}$, where $r_f(2)$ is the smallest root in $(0,1)$ of*

$$(1-r)^2 - k\lambda_{\mathcal{H}}\, r + k\lambda_{\mathcal{H}}\, r^3 = 0, \tag{3.8}$$

where $\lambda_{\mathcal{H}}$ is given by (1.3).

For $\Omega = \Omega_\gamma$, we have $\lambda_{\mathcal{H}} = \lambda_{\mathcal{H}}(\Omega_\gamma) \le 1/(1+\gamma)$ (see [6]). In view of Theorem 3.1, we obtain the following corollaries.

Corollary 3.9 *Let F be a polyanalytic function in Ω_γ with the series expansion as in Theorem 3.1. Also, assume all the hypotheses as in Theorem 3.1. Then $M_r(F) \le 1$ for $|z| = r \le \min\{r_f(p,\gamma), (1+\gamma)/(3+\gamma)\}$, where $r_f(p,\gamma)$ is the smallest root in $(0,1)$ of*

$$(1+\gamma)(1-r)^2 - kr + kr^{p+1} = 0. \tag{3.10}$$

The following result is the limiting case of Corollary 3.9. Consider the domain $\tilde{\Omega} = \{z : \operatorname{Re} z < 1\}$, which can be obtained as the limiting case of the domain Ω_γ by considering $\gamma \to 1^-$.

Corollary 3.11 *Let F be a polyanalytic function in $\tilde{\Omega}$ with the series expansion as in Theorem 3.1. Also, assume all the hypotheses as in Theorem 3.1. Then $M_r(F) \le 1$ for $|z| = r \le \min\{r_f(p,1), 1/2\}$, where $r_f(p,1)$ is the smallest root in $(0,1)$ of*

$$2(1-r)^2 - kr + kr^{p+1} = 0. \tag{3.12}$$

In the next result, we obtain Bohr radius for the polyanalytic function $F(z) = \sum_{l=0}^{p-1} \overline{z}^l f_l(z)$, where f_0 is a subordinate to a convex biholomorphic function in the unit disk $\mathbb{D}$.

Theorem 3.2 *Let F be a polyanalytic function of order p in $\mathbb{D}$ with $F(z) = \sum_{l=0}^{p-1} \overline{z}^l f_l(z)$, where each $f_l : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is an analytic function such that $f_l(z) = \sum_{n=0}^{\infty} A_{n,l} z^n$ in $\mathbb{D}$ and $A_{n,l} \in \mathcal{B}(\mathcal{H})$ for $n \in \mathbb{N} \cup \{0\}$. Also, assume that:*

(1) *$f_0 \in S(g)$ such that $f_0(0) = \mathbf{0}$ and $f_l'(0) = \alpha_l f_0'(0)$ with $|\alpha_l| < k$ for $k \in [0,1]$ and each $l = 1, \dots, p-1$, where $g : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is a convex biholomorphic function with $g(z) = \sum_{n=0}^{\infty} g_n z^n$ in $\mathbb{D}$ and $g_n \in \mathcal{B}(\mathcal{H})$ for $n \in \mathbb{N} \cup \{0\}$.*

(2) *$\omega_l : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is an analytic function with $\|\omega_l(z)\| \le k$ in $\mathbb{D}$ for $k \in [0,1]$, where $\omega_l(z) = f_l'(z)(f_0'(z))^{-1}$ in $\mathbb{D}$ such that $\omega_l(z) = \sum_{n=0}^{\infty} \omega_{n,l} z^n$ in $\mathbb{D}$, provided $(f_0'(z))^{-1}$ exists for all $z \in \mathbb{D}$.*

Then $M_r(F) \le 1$ for $|z| = r \le R_C = \min\{r_C(p,k,\beta), 1/3\}$, where $r_C(p,k,\beta)$ is the smallest root in $(0,1)$ of

$$(1-r)^2 - k\beta r + k\beta r^{p+1} = 0, \tag{3.13}$$

where $\|g'(0)\| = \beta$.

Proof From (3.4), it is enough to estimate the upper bound of $M_r(f_0)$. Let $g : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be univalent and convex biholomorphic function in $\mathbb{D}$ such that $g(z) = \sum_{n=0}^{\infty} g_n z^n$, where $g_n \in \mathcal{B}(\mathcal{H})$. Set $\xi = e^{2\pi i/n}$. Since g is convex, then by the similar argument used in proving [26, Theorem X], we obtain

$$\Psi(z^n) = \frac{f_0(\xi z) + f_0(\xi^2 z) + \cdots + f_0(\xi^n z)}{n} = A_{n,0} z^n + A_{2n,0} z^{2n} + \cdots \prec g(z),$$

and hence $\Psi(z) = A_{n,0} z + A_{2n,0} z^2 + \cdots \prec g(z)$ for $z \in \mathbb{D}$. Hence, there exits a holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $\Psi(z) = g(\omega(z))$, which implies that $\Psi'(0) = \omega'(0) g'(0)$. That is, $A_{n,0} = \omega'(0) g'(0)$, which leads to

$$\|A_{n,0}\| \le \|g'(0)\|. \tag{3.14}$$

Since $f_l(z) = \sum_{n=0}^{\infty} A_{n,l} z^n$ in $\mathbb{D}$, using (3.14) and the fact $f_0(0) = \mathbf{0}$, we obtain

$$M_r(f_0) = \sum_{n=0}^{\infty} \|A_{n,0}\| r^n \le \left(\frac{r}{1-r}\right) \|g'(0)\| = \beta \left(\frac{r}{1-r}\right). \tag{3.15}$$

Since $F(z) = \sum_{l=0}^{p-1} \overline{z}^l f_l(z)$ with $f_l(z) = \sum_{n=0}^{\infty} A_{n,l} z^n$ in $\mathbb{D}$, we have

$$M_r(F) = M_r\left(\sum_{l=0}^{p-1} \overline{z}^l f_l(z)\right) \le \sum_{l=0}^{p-1} r^l M_r(f_l) \quad \text{for} \quad |z| = r < 1. \tag{3.16}$$

https://doi.org/10.4153/S0008439523000541 Published online by Cambridge University Press

By the given assumption $\omega_l(z) = f_l'(z)(f_0'(z))^{-1}$ in Ω with $\|\omega_l(z)\| \le k$ in Ω for each l such that $f_l'(0) = \alpha_l f_0'(0)$, it follows that $f_l'(z) = \omega_l(z) f_0'(z)$ in Ω with $\omega_l(0) = \alpha_l I$, where $|\alpha_l| < k$ for each $l = 1, \ldots, p-1$. Let $\lambda_{\mathcal{H}}$ be given by (1.3). For $\Omega_\gamma = \mathbb{D}$, i.e., $\gamma = 0$, we have $\lambda_{\mathcal{H}} \le 1$ (see [6]). In view of Theorem 1.2, for $|z| = r \le 1/3$, we have $M_r(\omega_l) \le k$, which together with (2.2) gives

$$M_r(f_l) = \int_0^r M_r(f_l')dt = \int_0^r M_r(\omega_l f_0')dt \le k \int_0^r M_r(f_0')dt = k\, M_r(f_0). \tag{3.17}$$

Using (3.15)–(3.17), for $|z| = r \le 1/3$, we obtain

$$M_r(F) \le k \sum_{l=0}^{p-1} r^l M_r(f_0) = k M_r(f_0) \left(\frac{1-r^p}{1-r}\right) \le k\beta \left(\frac{r}{1-r}\right)\left(\frac{1-r^p}{1-r}\right). \tag{3.18}$$

Hence, $M_r(F) \le 1$ for $r \le R_C = \min\{r_C(p,k,\beta), 1/3\}$, where $r_C(p,k,\beta)$ is the smallest root in $(0,1)$ of (3.13). This completes the proof. ■

Let $h : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be holomorphic and $g \in S(h)$ with the expansions $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$, respectively, in $\mathbb{D}$, where $h_n, g_n \in \mathcal{B}(\mathcal{H})$ for $n \in \mathbb{N} \cup \{0\}$. Then, in view of [11, Lemma 2], for $|z| = r \le 1/3$, it is known that

$$\sum_{n=1}^{\infty} \|g_n\|\, r^n \le \sum_{n=1}^{\infty} \|h_n\|\, r^n. \tag{3.19}$$

In the following result, we obtain Bohr radius for the polyanalytic function $F(z) = \sum_{l=0}^{p-1} \overline{z}^l f_l(z)$, where f_0 is subordinate to a starlike biholomorphic function in the unit disk $\mathbb{D}$.

Theorem 3.3 *Let F be a polyanalytic function of order p in $\mathbb{D}$ with $F(z) = \sum_{l=0}^{p-1} \overline{z}^l f_l(z)$, where each $f_l : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ are analytic functions such that $f_l(z) = \sum_{n=0}^{\infty} A_{n,l} z^n$ in $\mathbb{D}$ and $A_{n,l} \in \mathcal{B}(\mathcal{H})$ for $n \in \mathbb{N} \cup \{0\}$. Also, assume that:*

(1) *$f_0 \in S(g)$ such that $f_0(0) = \mathbf{0}$ and $f_l'(0) = \alpha_l f_0'(0)$ with $|\alpha_l| < k$ for $k \in [0,1]$ and each $l = 1, \ldots, p-1$, where $g : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is a normalized starlike biholomorphic function with $g(z) = zI + \sum_{n=2}^{\infty} g_n z^n$ in $\mathbb{D}$ and $g_n \in \mathcal{B}(\mathcal{H})$ for $n \in \mathbb{N} \cup \{0\}$.*

(2) *$\omega_l : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is an analytic function with $\|\omega_l(z)\| \le k$ in $\mathbb{D}$ for $k \in [0,1]$, where $\omega_l(z) = f_l'(z)(f_0'(z))^{-1}$ in $\mathbb{D}$ such that $\omega_l(z) = \sum_{n=0}^{\infty} \omega_{n,l} z^n$ in $\mathbb{D}$.*

Then $M_r(F) \le 1$ for $|z| = r \le R_S = \min\{r_S(p,k), 1/3\}$, where $r_s(p,k)$ is the smallest root in $(0,1)$ of

$$(1-r)^3 - kr + kr^{p+1} = 0. \tag{3.20}$$

Proof Let $g : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be a normalized starlike biholomorphic function. Then, in view of [17, Theorem 6.2.6], g satisfies

$$z\,g'(z) = q(z) g(z) \quad \text{for} \quad z \in \mathbb{D}, \tag{3.21}$$

where $q : \mathbb{D} \to \mathbb{C}$ is an analytic function with $\operatorname{Re} q(z) > 0$ in $\mathbb{D}$ and $q(0) = 1$. By comparing the coefficients in the power series of both the sides of (3.21), we obtain

$$(n-1)g_n = g_{n-1}q_1 + g_{n-2}q_2 + \cdots + q_{n-1} \quad \text{for} \quad n \geq 2. \tag{3.22}$$

By using induction and (3.22), we obtain

$$(n-1)\|g_n\| \leq 2(n-1+n-2+\cdots+1)I = n(n-1)I,$$

which turns out that $\|g_n\| \leq n$ for all $n \geq 2$. Since, $f_0 \in S(g)$, by using (3.19), for $r \leq 1/3$, we obtain

$$M_r(f_0) = \sum_{n=1}^{\infty} \|A_{n,0}\| r^n \leq \sum_{n=1}^{\infty} \|g_n\| r^n \leq r + \sum_{n=2}^{\infty} nr^n = \frac{r}{(1-r)^2}. \tag{3.23}$$

It is known that, for $\Omega_\gamma = \mathbb{D}$, i.e., $\gamma = 0$, if $h : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is holomorphic, then we have $\lambda_{\mathcal{H}} \leq 1$ (see [6]). From (3.17), we have

$$M_r(f_l) \leq k\, M_r(f_0). \tag{3.24}$$

Then, from (3.16), (3.23), and (3.24), we obtain

$$M_r(F) \leq k\, \frac{r}{(1-r)^2} \left(\frac{1-r^p}{1-r} \right) \quad \text{for} \quad r \leq \frac{1}{3}.$$

Hence, $M_r(F) \leq 1$ for $r \leq R_S = \min\{r_S(p,k), 1/3\}$, where $r_C(p,k,\beta)$ is the smallest root in $(0,1)$ of (3.20). This completes the proof. ∎

Acknowledgment The authors would like to express their sincerest gratitude to the referees for careful reading of the manuscript and many valuable suggestions, which greatly helped to improve the clarity of the exposition in this manuscript.

Competing interests The authors declare none.

References

[1] Z. Abdulhadi and L. E. Hajj, *On the univalence of poly-analytic functions*. Comput. Methods Funct. Theory 22(2021), 169–181. https://doi.org/10.1007/s40315-021-00378-5.
[2] Y. Abu Muhanna, R. M. Ali, Z. C. Ng, and S. K. Lee, *The Bohr operator on analytic functions and sections*. J. Math. Anal. Appl. **496**(2021), 124837.
[3] M. L. Agranovsky, *Characterization of polyanalytic functions by meromorphic extensions from chains of circles*. J. Anal. Math. **113** (2011), 305–329.
[4] M. B. Ahamed, V. Allu, and H. Halder, *The Bohr phenomenon for analytic functions on shifted disks*. Ann. Fenn. Math. 47(2022), 103–120.
[5] S. A. Alkhaleefah, I. R. Kayumov, and S. Ponnusamy, *On the Bohr inequality with a fixed zero coefficient*. Proc. Amer. Math. Soc. 147(2019), 5263–5274.
[6] V. Allu and H. Halder, *Bohr radius for Banach spaces on simply connected domains*. Preprint, 2021. https://arxiv.org/pdf/2111.10880.pdf.
[7] C. BÉnÉteau, A. Dahlner, and D. Khavinson, *Remarks on the Bohr phenomenon*. Comput. Methods Funct. Theory 4(2004), 1–19.
[8] A. Aytuna and P. Djakov, *Bohr property of bases in the space of entire functions and its generalizations*. Bull. Lond. Math. Soc. 45(2013), no. 2, 411–420.

[9] M. B. Balk, Polyanalytic functions and their generalizations. In: A. A. Gonchar, V. P. Havin, and N. K. Nikolski (eds.) Complex analysis I, Encyclopedia of Mathematical Sciences, 85, Springer, Berlin, 1997, pp. 195–253.
[10] B. Bhowmik and N. Das, *Bohr phenomenon for subordinating families of certain univalent functions*. J. Math. Anal. Appl. **462**(2018), 1087–1098.
[11] B. Bhowmik and N. Das, *Bohr phenomenon for operator-valued functions*. Proc. Edinburgh Math. Soc. **64**(2021), no. 1, 72–86. https://doi.org/10.1017/S0013091520000395.
[12] H. P. Boas and D. Khavinson, *Bohr's power series theorem in several variables*. Proc. Amer. Math. Soc. **125**(1997), 2975–2979.
[13] H. Bohr, *A theorem concerning power series*. Proc. Lond. Math. Soc. **s2**–13(1914), 1–5.
[14] A. Defant, D. García, M. Maestre, and D. Pérez-García, *Bohr's strip for vector valued Dirichlet series*. Math. Ann. **342**(2008), 533–555.
[15] P. G. Dixon, *Banach algebras satisfying the non-unital von Neumann inequality*. Bull. Lond. Math. Soc. **27**(1995), no. 4, 359–362.
[16] R. Fournier and S. Ruscheweyh, *On the Bohr radius for simply connected domains*. In: Hilbert spaces of analytic functions, CRM Proceedings and Lecture Notes, 51, American Mathematical Society, Providence, RI, 2010, 165–171.
[17] I. Graham and G. Kohr, *Geometric function theory in one and higher dimensions*, Monographs and Textbooks in Pure and Applied Mathematics, 255, Marcel Dekker, Inc., New York, 2003.
[18] H. Hachadi and E. H. Youssfi, *The polyanalytic reproducing kernels*. Complex Anal. Oper. Theory **13**(2019), 3457—3478.
[19] H. Hamada, T. Honda, and G. Kohr, *Bohr's theorem for holomorphic mappings with values in homogeneous balls*. Israel. J. Math. **173**(2009), 177–187.
[20] G. V. Kolossov, *Sur les problèmes d'élasticité à deux dimensions*. C. R. Acad. Sci. **146** (1908), 522—525.
[21] V. I. Paulsen, G. Popescu, and D. Singh, *On Bohr's inequality*. Proc. Lond. Math. Soc. **s3**–85(2002), 493–512.
[22] V. I. Paulsen and D. Singh, *Bohr's inequality for uniform algebras*. Proc. Amer. Math. Soc. **132**(2002), 3577–3579.
[23] V. I. Paulsen and D. Singh, *A simple proof of Bohr's inequality*. https://www.math.uh.edu/vern/bohrconf.pdf.
[24] G. Popescu, *Bohr inequalities for free holomorphic functions on polyballs*. Adv. Math. **347**(2019), 1002–1053.
[25] J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*. Math. Nachr. **4**(1951), 258–281.
[26] W. Rogosinski, *On the coefficients of subordinate functions*. Proc. Lond. Math. Soc. **48**(1943), 48–82.

School of Basic Sciences, Indian Institute of Technology Bhubaneswar, Bhubaneswar 752050, India
e-mail: avrao@iitbbs.ac.in

Mathematics Department, Indian Institute of Technology Bombay, Mumbai 400076, India
e-mail: himadrihalder119@gmail.com, himadri@math.iitb.ac.in