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OPERATOR IDEALS AND TENSOR NORMS DEFINED BY A SEQUENCE SPACE

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We study the tensor norm defined by a sequence space λ and its minimal and maximal operator ideals associated in the sense of Defant and Floret. Our results extend the classical theory related to the tensor norms of Saphar [16]. They show the key role played by the finite dimensional structure of the ultrapowers of λ in this kind of problems.

1. INTRODUCTION

In the definition of the interesting tensor norms of Grothendieck, Saphar and Lapresté and its associated operator ideals, the ℓ_p spaces play a central role. It is quite natural to try to replace ℓ_p for another Banach sequence space λ, an idea pointed out in the seventies for De Grande-De Kimpe [2] and Harksen [6]. However until now this idea has not gone far probably because the classical theory is dominated by the special properties of the class of the $\mathcal{L}^p$ spaces of Lindenstrauss and Pelczński, but in more general cases hidden problems emerge.

In this paper, we study the tensor norm defined by a sequence space λ in the sense of De Grande-De Kimpe and Harksen, and also its maximal and minimal associated operator ideals, the relations between them and some metric properties of the involved tensor norms. The main instrument we have to obtain the "expected" results is the so called "local theory" of Banach spaces, that is, the study of Banach spaces (and the operators between them) in terms of finite dimensional subspaces, a tool which has enriched our understanding of Banach spaces in other many aspects.

The notation is standard. All the spaces considered are Banach spaces over the real field in order to more easily use known results in the theory of Banach lattices. If we wish to emphasise the space E where a norm is defined we shall write $\|\cdot\|_E$. The canonical inclusion map of a Banach space E into the bidual E'' will be denoted by J_E. In general if E is a subspace of F, the inclusion of E into F is denoted by $I_{E,F}$. The set of finite dimensional subspaces of a normed space E will be denoted by $FIN(E)$.

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Concerning Banach lattices we refer the reader to [**1**]. A linear map T between Banach lattices E and F is said to be positive if $T(x) \geqslant 0$ in F for every $x \in E, x \geqslant 0$. T is called order bounded if $T(A)$ is order bounded in F for every order bounded set A in E.

Let ω be the vector space of all scalar sequences and φ its subspace of the sequences with finitely many non zero coordinates. A sequence space λ is a linear subspace of ω containing φ provided with a topology finer than the topology of coordinatewise convergence. *A Banach sequence space* will be a sequence space λ provided with a norm which makes it a Banach lattice and an ideal in ω, that is, such that if $|x| \leqslant |y|$ with $x \in \omega$ and $y \in \lambda$, then $x \in \lambda$ and $\|x\|_\lambda \leqslant \|y\|_\lambda$. A *sectional subspace* $S_k(\lambda), k \in \mathbb{N}$, is the topological subspace of λ of those sequences (α_i) such that $\alpha_i = 0$ for every $i \geqslant k$. Clearly $S_k(\lambda)$ is 1-complemented in λ. A Banach sequence space λ will be called *regular* whenever the sequence $\{\mathbf{e}_i\}_{i=1}^\infty$ where $\mathbf{e}_i := (\delta_{ij})_j$ (Kronecker's delta) forms a Schauder basis in λ. Every Banach sequence space λ has a solid and regular subspace $\lambda_r := \overline{\varphi}^\lambda$ such that λ is regular if and only if $\lambda = \lambda_r$ (see [**9**, Lemma 3.3] for example). For technical requisites of the standard theory of tensor norms (see [**3**, Criterion 12.2]), given a Banach sequence space λ with the quoted properties in the introduction, *from now on it will be supposed furthermore that* $\|\mathbf{e}_i\|_\lambda = \|\mathbf{e}_i\|_{\lambda^\times} = 1$ *for every* $i \in \mathbb{N}$. The Köthe dual (or α-dual) $\lambda^\times$ of a sequence space λ is defined as the set of scalar sequences (b_i) such that $\sum_{i=1}^\infty |a_i b_i|$ converges for every $(a_i) \in \lambda$. In general, if λ is a Banach sequence space, the Köthe dual $\lambda^\times$ is a closed subspace of the Banach dual λ'.

We suppose the reader is familiar with the theory of operator ideals and tensor norms. Of course, the fundamental references about tensor norms and operator ideals are the books of Defant and Floret [**3**] and Pietsch [**14**] respectively.

Given a pair of Banach spaces E and F and a tensor norm α, $E \underset{\alpha}{\bigotimes} F$ represents the space $E \otimes F$ endowed with the α-normed topology. The completion of $E \underset{\alpha}{\bigotimes} F$ is denoted by $E \widehat{\underset{\alpha}{\bigotimes}} F$, and the norm of z in $E \widehat{\underset{\alpha}{\bigotimes}} F$ by $\alpha(z; E \otimes F)$. If there is no risk of mistake we write $\alpha(z)$ instead of $\alpha(z; E \otimes F)$.

A sequence $(x_n)_{n=1}^\infty \in E^{\mathbb{N}}$ is said to be strongly λ-summable if

$$\pi_\lambda\big((x_i)\big) := \Big\|\big(\|x_n\|\big)\Big\|_\lambda < \infty$$

and it is said to be weakly λ-summable if

$$\varepsilon_\lambda\big((x_i)\big) := \sup_{\|x'\| \leqslant 1} \Big\|\Big(\big|\langle x_n, x'\rangle\big|\Big)\Big\|_\lambda < \infty.$$

From now on $\lambda[E]$ (respectively $\lambda(E)$) will denote the space of all strongly (respectively weakly) λ-summable sequences in E endowed with the norm $\pi_\lambda(\cdot)$ (respectively $\varepsilon_\lambda(\cdot)$).

Concerning ultraproducts of Banach spaces the standard paper is [**8**] and we refer to it for concrete definitions. We only set the notation we shall use. Let D be a non

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empty index set and $\mathcal{U}$ a non-trivial ultrafilter in D. Given a family $\{X_d,\ d \in D\}$ of Banach spaces, $(X_d)_{\mathcal{U}}$ denotes the corresponding ultraproduct Banach space. If every $X_d,\ d \in D$, coincides with a fixed Banach space X the corresponding ultraproduct is named an ultrapower of X and is denoted by $(X)_{\mathcal{U}}$. Remark that if every $X_d,\ d \in D$ is a Banach lattice, $(X_d)_{\mathcal{U}}$ has a canonical order which makes it a Banach lattice. If we have another family of Banach spaces $\{Y_d,\ d \in D\}$ and a family of operators $\{T_d \in \mathcal{L}(X_d, Y_d),\ d \in D\}$ such that $\sup_{d\in D} \|T_d\| < \infty$, then $(T_d)_{\mathcal{U}} \in \mathcal{L}\big((X_d)_{\mathcal{U}}, (Y_d)_{\mathcal{U}}\big)$ denotes the canonical ultraproduct operator.

The organisation of the paper is the following. Section 2 is devoted to the tensor norm derived from a Banach sequence space λ and a characterisation of the minimal operator ideal associated (the so called λ-nuclear operators) by means of a factorisation theorem. Section 3 describes the class of the *generalised* $\mathcal{L}_c^\lambda$ spaces, denoted $\mathcal{L}_c^{\lambda,g}$ and gives some useful properties which allow us the development of the maximal operator ideal associated to the tensor norm (the ideal of λ-integral operators). Finally in section 4 we apply all the results to the study the coincidence between λ-nuclear and λ-integral spaces, and we obtain some metric properties of the involved tensor norms.

2. The tensor norm associated to a Banach sequence space λ and the ideal of the λ-nuclear operators.

Let E and F be Banach spaces. Inspired by the tensor norm g_p of Saphar [**16**], for all $z \in E \otimes F$ we set

$$g_\lambda(z) := g_\lambda(z; E\otimes F) := \inf\Big\{\pi_\lambda\big((x_n)\big)\, \varepsilon_{\lambda^\times}\big((y_n)\big) : z = \sum_{n=1}^{m} x_n \otimes y_n\Big\}.$$

In general $g_\lambda(\cdot; E\otimes F)$ only is a reasonable *quasi norm* in $E \otimes F$, see [**2, 6**]. We denote $E\widehat{\bigotimes}_{g_\lambda}F$ the corresponding quasi Banach space.

Then we consider the Minkowski functional $g_\lambda^c(\cdot; E\otimes F)$ of the absolutely convex hull of the unit ball $B_{g_\lambda} := \{z \in E\otimes F : g_\lambda(z) \leqslant 1\}$ and it is straightforward that $g_\lambda^c(z; E\otimes F)$ can be evaluated as

$$g_\lambda^c(z) := g_\lambda^c(z; E\otimes F) := \inf\Big\{\sum_{i=1}^{n} \pi_\lambda\big((x_{ij})\big)\, \varepsilon_{\lambda^\times}\big((y_{ij})\big) :\ \ z = \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}\otimes y_{ij}\Big\}$$

Moreover g_λ^c is *a tensor norm* in the class of Banach spaces less than or equal to g_λ.

If it is possible as in this case, the series representation of the elements of the completed tensor products $E\widehat{\bigotimes}_{g_\lambda}F$ and $E\widehat{\bigotimes}_{g_\lambda^c}F$ is a basic tool in the study of the involved operator ideals. In particular (see for instance [**2, 16**]) if $z \in E\widehat{\bigotimes}_{g_\lambda}F$, there are $(x_i)_{i=1}^\infty$,

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$\in \lambda_r[E]$ and $(y_i)_{i=1}^{\infty} \in \lambda^{\times}(F)$ such that $\pi_\lambda\big((x_i)\big)\, \varepsilon_{\lambda^\times}\big((y_i)\big) < \infty$ and

$$z = \sum_{i=1}^{\infty} x_i \otimes y_i.$$

Moreover the quasi norm is given by

$$g_\lambda(z) = \inf \pi_\lambda\big((x_i)\big)\, \varepsilon_{\lambda^\times}\big((y_i)\big)$$

taking the infimum over all such representations of z. In the same way it is easy to see that every $z \in E\widehat{\bigotimes}_{g_\lambda^c} F$ can be represented as

$$z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} \otimes y_{ij} \tag{1}$$

where $\big\{(x_{ij})_{i=1}^{\infty};\ j \in \mathbb{N}\big\} \subset \lambda_r[E], \big\{(y_{ij})_{i=1}^{\infty} : j \in \mathbb{N}\big\} \subset \lambda^{\times}(F)$ and

$$\sum_{j=1}^{\infty} \pi_\lambda\big((x_{ij})\big)\, \varepsilon_{\lambda^\times}\big((y_{ij})\big) < \infty. \tag{2}$$

Moreover, the norm of z in $E\widehat{\bigotimes}_{g_\lambda^c} F$ is the infimum of the numbers in (2) over all representations of type (1).

But g_λ and g_λ^c are equivalent. In fact if we consider the bilinear and continuous onto map

$$R : \lambda_r[E] \times \lambda^{\times}(F) \to E\widehat{\bigotimes}_{g_\lambda} F$$

such that $R\big((x_i),(y_i)\big) = \sum_{i=1}^{\infty} x_i \otimes y_i$ with quasi norm less or equal than one, by [**17**] there exists a unique linear and continuous map $\lambda_r[E] \bigotimes_{\pi} \lambda^{\times}(F) \to E\widehat{\bigotimes}_{g_\lambda} F$. This map can be extended to a continuous linear and onto map $\lambda_r[E]\widehat{\bigotimes}_{\pi}\lambda^{\times}(F) \to E\widehat{\bigotimes}_{g_\lambda} F$, and by the open mapping theorem, $E\widehat{\bigotimes}_{g_\lambda} F$ is isomorphic to a quotient of a Banach space, hence it is a Banach space. Then the topology defined by g_λ in $E \otimes F$ is always normable. Now it is easy to see that $g_\lambda(\cdot; E \otimes F)$ and $g_\lambda^c(\cdot; E \otimes F)$ are equivalent *In view of this equivalence, one is tempted to use the easier g_λ quasinorm instead of the norm g_λ^c, but in the Sections 4 and 5 the g_λ^c norm is necessary.*

Definition 1: Let $T \in \mathcal{L}(E,F)$ be, we say that T is λ-absolutely summing if exist a real number $C > 0$, such that for all sequence (x_i) in E, with $\varepsilon_\lambda\big((x_i)\big) < \infty$,

$$\pi_\lambda\Big(\big(T(x_i)\big)\Big) \leqslant C\varepsilon_\lambda\big((x_i)\big) \tag{3}$$

For $\mathcal{P}_\lambda(E,F)$ we denote the Banach ideal of the λ-absolutely summing operators $T : E \to F$ endowed with the topology of the norm $\Pi_\lambda(T) := \inf\big\{C \geqslant 0 : C \text{ satisfies } (3)\big\}$.

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Theorem 2. *For E, F Banach spaces, $\left(E \underset{g_\lambda^c}{\bigotimes} F\right)' = \mathcal{P}_{\lambda^\times}(F, E')$ isometrically.*

Proof: For $T \in \mathcal{P}_{\lambda^\times}(F, E')$, for every $z = \sum_{i=1}^{n}\sum_{j=1}^{l_i} x_{ij} \otimes y_{ij} \in E \underset{g_\lambda^c}{\bigotimes} F$ we define $\varphi_T : E \underset{g_\lambda^c}{\bigotimes} F \to \mathbb{R}$ by

$$\langle \varphi_T, z\rangle = \sum_{i=1}^{n}\sum_{j=1}^{l_i} \langle x_{ij}, T(y_{ij})\rangle.$$

This definition for φ_T is not dependent on representation of z and it can be seen that $\varphi_T \in \left(E \underset{g_\lambda^c}{\bigotimes} F\right)'$ with $\left|\langle \varphi_T, z\rangle\right| \leqslant \Pi_{\lambda^\times}(T) g_\lambda^c(z; E, F)$ and therefore $\|\varphi_T\| \leqslant \Pi_{\lambda^\times}(T)$.

On the other hand, for $\varphi \in \left(E \underset{g_\lambda}{\bigotimes} F\right)'$ we define $T_\varphi : F \longrightarrow E'$ by

$$\langle T_\varphi(y), x\rangle = \langle \varphi, x \otimes y\rangle \quad \forall\, y \in F,\ x \in E$$

Then, if $(y_i) \in F^{\mathbf{N}}$ such as $\varepsilon_{\lambda^\times}\left((y_i)\right) < \infty$, as B_E is weakly dense in $B_{E''}$, given $\varepsilon > 0$ and $(\delta_i) \in \lambda^\times$ with $\left\|(\delta_i)\right\|_{\lambda^\times} \leqslant 1$, for all $i \in \mathbb{N}$ there is $x_i \in E$ such as $\|x_i\| \leqslant 1$ and $\|T_\varphi(y_i)\| \leqslant \left|\langle \varphi, x_i \otimes y_i\rangle\right| + \varepsilon\delta_i$. Hence

$$\left\|\left(\|T_\varphi(y_i)\|\right)\right\|_{\lambda^\times} \leqslant \sup_{\|(\eta_i)\|_\lambda \leqslant 1} \left|\sum_{i=1}^{\infty} \eta_i \langle \varphi, x_i \otimes y_i\rangle\right| + \varepsilon$$

but $\pi_{h_\lambda}\left((\eta_i x_i)\right) = \left\|\left(\|\eta_i x_i\|\right)\right\|_{h_\lambda} \leqslant \left\|(\eta_i)\right\|_{h_\lambda} \leqslant 1$ and $\varepsilon_{\lambda^\times}\left((y_i)\right) < \infty$ so that $\sum_{i=1}^{\infty} \eta_i x_i \otimes y_i \in E \underset{g_\lambda^c}{\widehat{\bigotimes}} F$, hence

$$\left\|\left(\|T_\varphi(y_i)\|\right)\right\|_{\lambda^\times} \leqslant \sup_{\|(\eta_i)\|_\lambda \leqslant 1} \|\varphi\| g_\lambda^c\left(\sum_{i=1}^{\infty} \eta_i x_i \otimes y_i\right) + \varepsilon \leqslant \|\varphi\| \varepsilon_{\lambda^\times}\left((y_i)\right) + \varepsilon$$

and since ε is arbitrary, it follows that $\left\|\left(\|T_\varphi(y_i)\|\right)\right\|_{\lambda^\times} \leqslant \|\varphi\| \varepsilon_{\lambda^\times}\left((y_i)\right)$ and $\Pi_{\lambda^\times}(T_\varphi) \leqslant \|\varphi\|$. ∎

We shall now consider the ideal $\mathcal{N}_\lambda$ of the λ-nuclear operators in the sense of Dubinsky and Ramanujan introduced previously in [5] in order to deal with a different kind of problem. Every representation of $z \in E' \underset{g_\lambda^c}{\widehat{\bigotimes}} F$ of the type (1) defines a map $T_z \in \mathcal{L}(E, F)$ such that $\forall x \in E$,

$$T_z(x) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle x'_{ij}, x\rangle\, y_{ij}.$$

We remark that all these representations of the same z define the same map T_z. Let $\Phi_{EF} : E' \underset{g_\lambda^c}{\widehat{\bigotimes}} F \to \mathcal{L}(E, F)$ be defined by $\Phi_{EF}(z) := T_z$.

Definition 3: Let E, F be Banach spaces. An operator $T : E \to F$ is said to be λ- nuclear if $T = \Phi_{EF}(z)$, for some $z \in E' \underset{g_\lambda^c}{\widehat{\bigotimes}} F$.

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$\mathcal{N}_\lambda(E,F)$ denotes the space of the λ-nuclear operators $T : E \to F$ endowed with the topology of the norm

$$\mathrm{N}_\lambda^c(T) := \inf\{g_\lambda^c(z) \ / \ \Phi_{EF}(z) = T\}$$

or with the equivalent quasi-norm

$$\mathrm{N}_\lambda(T) := \inf\{g_\lambda(z) \ / \ \Phi_{EF}(z) = T\}.$$

For every pair of Banach spaces E and F, $\left(\mathcal{N}_\lambda(E,F), \mathrm{N}_\lambda^c\right)$ is a component of the minimal Banach operator ideal $(\mathcal{N}_\lambda, \mathrm{N}_\lambda^c)$ associated to the tensor norm g_λ^c.

We have the following characterisation of λ-nuclear operators:

Theorem 4. *Let E and F be Banach spaces and let T be an operator in $\mathcal{L}(E,F)$. Then the following statements are equivalent:*

(1) *T is λ-nuclear.*

(2) *T factors continuously in the following way:*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "A"'] & F \\ \ell_\infty \arrow[r, "B"'] & \lambda_r \arrow[u, "C"'] \end{tikzcd}$$

where B is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \lambda_r$.

Furthermore $\mathrm{N}_\lambda(T) = \inf\left\{\|C\|\,\|B\|\,\|A\|\right\}$, taken over all such factorisations.

(3) *T factors continuously in the following way:*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "A"'] & F \\ \ell_\infty[\ell_\infty] \arrow[r, "B"'] & \ell_1[\lambda_r] \arrow[u, "C"'] \end{tikzcd}$$

where B is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \ell_1[\lambda_r]$.

Furthermore $\mathrm{N}_\lambda^c(T) = \inf\left\{\|C\|\,\|B\|\,\|A\|\right\}$, taken over all such factorisations.

Proof: (2) $\Longrightarrow$ (1). Assume we have a factorisation $T = C\ B\ A$ as in the diagram. Put $x_i' := A'(\mathrm{e}_i)$. Then for every $i \in \mathbb{N}$ $A(x) = \left(\langle x_i', x\rangle\right)_{i=1}^\infty$. Suppose that $B\left((u_i)\right)$

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$= (b_i u_i)_{i=1}^{\infty}$. Then $\|B\| \leqslant \|(b_i)\|_\lambda$. Let $C(\mathbf{e}_i) = y_i$ for every $i \in \mathbb{N}$. Then

$$\forall\, (\beta_i) \in \lambda_r \quad \left\|C\big((\beta_i)\big)\right\|_F = \sup_{\|y'\|_{F'} \leqslant 1} \sum_{i=1}^{\infty} |\beta_i \langle y_i, y'\rangle| \leqslant \|C\|\, \|(\beta_i)\|_\lambda.$$

We obtain that $T = \Phi_{EF}(z)$ with $z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_i x_i' \otimes y_i \in E' \widehat{\bigotimes_{g_\lambda}} F$, hence $T \in \mathcal{N}(E,F)$ and $\mathrm{N}_\lambda(T) \leqslant \|A\|\, \|B\|\, \|C\|$.

(1)$\Longrightarrow$(2). Assume T is λ-nuclear. Given $\varepsilon > 0$, there is a representation $T = \sum_{i=1}^{\infty} x_i' \otimes y_i$ such that $(x_i')_{i=1}^{\infty} \in \lambda_r[E']$, $(y_i)_{i=1}^{\infty} \in \lambda^\times(F)$ and

$$\mathrm{N}_\lambda(T) + \varepsilon \geqslant \pi_\lambda\big((x_i')\big)\, \varepsilon_{\lambda^\times}\big((y_i)\big).$$

Indeed we can suppose that $\varepsilon_{\lambda^\times}\big((y_i)\big) = 1$ and $\mathrm{N}_\lambda(T) + \varepsilon \geqslant \sum_{i=1}^{\infty} \pi_\lambda\big((x_{ij}')\big)$.

Let $A : E \to \ell_\infty$ be given by $A(x) := \big(\langle x_{ij}', x\rangle / \|x_i'\|\big)_{i=1}^{\infty}$ which is linear and continuous with $\|A\| \leqslant 1$.

Let $B : \ell_\infty \longrightarrow \lambda_r$ be given by $B\big((\lambda_i)\big) := \big(\lambda_{ij}\ \|x_i'\|\big)_{i=1}^{\infty}$. Then $\left\|\big(B(\lambda_i)\big)\right\|_\lambda \leqslant \|(\lambda_i)\|_{\ell_\infty} \|(\|x_i'\|)\|_\lambda = \|(\lambda_i)\|_{\ell_\infty} \pi_\lambda\big((x_i')\big)$ and hence B is linear and continuous with $\|B\| \leqslant \pi_\lambda\big((x_i')\big)$.

Finally let $C : \lambda \longrightarrow F$ be given by $C\big((\beta_i)\big) := \sum_{i=1}^{\infty} \beta_i\, y_i$. C is linear and continuous with $\left\|C\big((\beta_i)\big)\right\| = \sup_{\|y'\| \leqslant 1} \sum_{i=1}^{\infty} \langle \beta_i\, y_i, y'\rangle \leqslant \|(\beta_{ij})\|_\lambda\, \varepsilon_{\lambda^\times}\big((y_i)\big) = \|(\beta_i)\|_\lambda$ and then $\|C\| \leqslant 1$.

Clearly we have $T = C\ B\ A$ and

$$\mathrm{N}_\lambda(T) + \varepsilon \geqslant \pi_\lambda\big((x_i')\big)\, \varepsilon_{\lambda^\times}\big((y_i)\big) \geqslant \|A\|\, \|B\|\, \|C\|.$$

Since $\varepsilon > 0$ is arbitrary the implication is proved.

The proof of (1)$\Longleftrightarrow$(3) is similar using the norm $\mathrm{N}_\lambda^{\mathrm{c}}$. ∎

3. λ-INTEGRAL OPERATORS.

The Banach ideal of λ-integral operators $(\mathcal{I}_\lambda, \mathrm{I}_\lambda)$ is the maximal operator ideal associated to the tensor norm g_λ^{c} in the sense of Defant and Floret [**3**], or equivalently the maximal Banach operator ideal associated to the ideal of the λ-nuclear operators in the sense of Pietsch [**14**]. From [**3**], for every pair of Banach spaces E and F, an operator $T : E \to F$ is λ-integral if and only if $J_F T \in \Big(E \bigotimes_{(g_\lambda^{\mathrm{c}})'} F'\Big)'$.

We remark that if E, F are Banach spaces, we can define the finitely generated tensor norm g_λ' such that $g_\lambda'(z; M \otimes N) := \sup\Big\{\big|\langle z, w\rangle\big| \ :\ g_\lambda(w; M' \otimes N') \leqslant 1\Big\}$, $M \in FIN(E), N \in FIN(F)$. Clearly $g_\lambda' = (g_\lambda^{\mathrm{c}})'$, but we also remark that $E' \bigotimes_{g_\lambda^{\mathrm{c}}} F'$ (and not $E' \bigotimes_{g_\lambda} F'$) is an isometric subspace of $\Big(E \bigotimes_{g_\lambda'} F\Big)'$, see [**3**, 15.3].

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Define $\mathbf{I}_\lambda(T)$ to be the norm of $J_F\, T$ considered as an element of the topological dual of $E \underset{g'_\lambda}{\bigotimes} F'$. We remark that $\mathbf{I}_\lambda(T) = \mathbf{I}_\lambda(J_F\, T)$ because F' is complemented in F'''.

The following example of an λ-integral operator is essential for the next purpose of the paper which is to characterise λ-integral operator by a factorisation theorem.

Theorem 5. *Let (Ω, Σ, μ) be a measure space and let λ be a Banach sequence space with a regular predual Banach sequence space Δ. Then every order bounded operator $S : L_\infty(\mu) \to \ell_1[\lambda]$ is λ-integral with $\mathbf{I}_\lambda(S) = \|S\|$.*

Proof: By hypothesis, the linear span $\mathcal{T}$ of the set $\{\mathbf{e}_{ij},\ i, j \in \mathbb{N}\}$ is dense in $c_0[\Delta]$. Then by the representation theorem of maximal operator ideals (see [**3**, 17.5]) and the density lemma ([**3**, Theorem 13.4]) we only have to show that $S \in \Big(L_\infty(\mu) \underset{g'_\lambda}{\bigotimes} \mathcal{T}\Big)'$.

Given $z \in L_\infty(\mu) \underset{g'_\lambda}{\bigotimes} \mathcal{T}$ and $\varepsilon > 0$, let X and Y be finite dimensional subspaces of $L_\infty(\mu)$ and $\mathcal{T}$ respectively such that $z \in X \otimes Y$ and

$$g'_\lambda(z; X \otimes Y) \leqslant g'_\lambda(z; L_\infty(\mu) \otimes \mathcal{T}) + \varepsilon. \tag{4}$$

Let $\{\mathbf{g}_s\}_{s=1}^m$ be a basis for Y and let $k, t \in \mathbb{N}$ be such that

$$\forall\, 1 \leqslant s \leqslant m\ \ \mathbf{g}_s = \sum_{i=1}^{k} \sum_{j=1}^{t} c_{sij} \mathbf{e}_{ij}.$$

Then

$$\forall\, f \in X,\ \ \forall\, 1 \leqslant s \leqslant m\ \langle S, f \otimes \mathbf{g}_s\rangle = \langle f, S'(\mathbf{g}_s)\rangle$$
$$= \Big\langle f, \Big(\sum_{i=1}^{k}\sum_{j=1}^{t} c_{sij}\Big) S'(\mathbf{e}_{ij})\Big\rangle = \Big\langle f \otimes \sum_{a=1}^{k}\sum_{b=1}^{t} c_{sab}\, \mathbf{e}_{ab}, \sum_{i=1}^{k}\sum_{j=1}^{t} S'(\mathbf{e}_{ij}) \otimes \mathbf{e}_{ij}\Big\rangle.$$

Then if U denotes the tensor

$$U := \sum_{i=1}^{k}\sum_{j=1}^{t} S'(\mathbf{e}_{ij}) \otimes \mathbf{e}_{ij} \in L_\infty(\mu)' \otimes \ell_1[\lambda],$$

by bilinearity we get

$$\forall\, z \in X \otimes Y,\ \ \langle z, S\rangle = \langle U, z\rangle.$$

Given $\nu > 0$, for every $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant t$ there is $f_{ij} \in L_\infty(\mu)$ such that $\|f_{ij}\| \leqslant 1$ and $\|S'(\mathbf{e}_{ij})\| \leqslant \Big|\langle S'(\mathbf{e}_{ij}), f_{ij}\rangle\Big| + \nu$. Then $f := \sup\limits_{1\leqslant i\leqslant k, 1\leqslant j\leqslant t} f_{ij}$ lies in the closed unit ball of $L_\infty(\mu)$. On the other hand, $\ell_1[\lambda]$ is a dual lattice and hence it is order complete. By the Riesz-Kantorovich theorem (see [**1**, Theorem 1.13] for instance), the modulus $|S|$ of

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the operator S exists in $\mathcal{L}(L_\infty(\mu), \ell_1[\lambda])$. By the lattice properties of $\ell_1[\lambda]$ we have that

$$\begin{aligned}
\sum_{i=1}^{k} \pi_\lambda(S'(e_{ij})) &= \sum_{i=1}^{k} \Big\| \sum_{j=1}^{t} \|S'(\mathrm{e}_{ij})\| \mathrm{e}_{ij} \Big\|_\lambda = \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \|S'(\mathrm{e}_{ij})\| \, \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} \\
&\leqslant \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \big| \langle S'(\mathrm{e}_{ij}), f_{ij} \rangle \big| \, \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} + \nu \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} \\
&\leqslant \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \big| \langle S(f_{ij}), \mathrm{e}_{ij} \rangle \big| \, \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} + \nu \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} \\
&\leqslant \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \Big\langle |S(f_{ij})|, \mathrm{e}_{ij} \Big\rangle \Big\|_{\ell_1[\lambda]} + \nu \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} \\
&\leqslant \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \Big\langle |S|(|f_{ij}|), \mathrm{e}_{ij} \Big\rangle \, \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} + \nu \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} \\
&\leqslant \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \Big\langle |S|(|f|), \mathrm{e}_{ij} \Big\rangle \, \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} + \nu \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} \\
&= \Big\| |S| \, (|f|) \Big\|_{\ell_1[\lambda]} + \nu \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} \leqslant \| |S| \| + \nu \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]}.
\end{aligned}$$

Moreover for every $1 \leqslant i \leqslant k$

$$\varepsilon_{\lambda'}\big((\mathrm{e}_{ij})\big) = \sup_{\|(\beta_a)\|_\Delta \leqslant 1} \Big\| \sum_{j=1}^{t} \langle \mathrm{e}_{ij}, (\beta_a) \rangle \, \mathrm{e}_{ij} \Big\|_\Delta = \Big\| \sum_{j=1}^{t} \beta_j \, \mathrm{e}_{ij} \Big\|_\Delta \leqslant 1.$$

Hence, denoting by I_X and I_Y the corresponding inclusion maps into $L_\infty(\mu)$ and $\ell_1[\lambda]$ respectively, we have

$$\begin{aligned}
|\langle S, z \rangle| &= |\langle U, z \rangle| = \Big| \Big\langle U, \big((I_X)' \otimes (I_Y)'\big)(z) \Big\rangle \Big| \\
&\leqslant g_\lambda^c(U; X \otimes Y) \, g'_\lambda\Big(\big((I_X)' \otimes (I_Y)'\big)(z); X' \otimes Y'\Big) \\
&\leqslant \Big(g_\lambda^c(U; L_\infty \otimes \ell_1[\lambda]) + \varepsilon\Big) \, g'_\lambda\Big(z; L_\infty(\mu) \otimes (\ell_1[\lambda])'\Big) \\
&\leqslant g'_\lambda\Big(z; L_\infty(\mu) \otimes (\ell_1[\lambda])'\Big) \Big(\sum_{i=1}^{k} \pi_\lambda(S'(\mathrm{e}_{ij})) \, \varepsilon_{\lambda'}((\mathrm{e}_{ij})) + \varepsilon \Big) \\
&\leqslant g'_\lambda\Big(z; L_\infty(\mu) \otimes (\ell_1[\lambda])'\Big) \Big(\| |S| \| + \nu \Big\| \sum_{i=1}^{k} \sum_{j=1}^{t} \mathrm{e}_{ij} \Big\|_{\ell_1[\lambda]} + \varepsilon \Big)
\end{aligned}$$

and ν being arbitrary

$$|\langle S, z \rangle| \leqslant g'_\lambda\Big(z; L_\infty(\mu) \otimes (\ell_1[\lambda])'\Big) \Big(\| |S| \| + \varepsilon \Big).$$

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Finally, by the arbitrariness of ε we get

$$\left|\langle S, z\rangle\right| \leqslant g'_\lambda\Big(z; L_\infty(\mu) \otimes \left(\ell_1[\lambda]\right)'\Big)\, |||S|||\,.$$

But from [**1**, Theorem 1.10],

$$|S|(\chi_\Omega) = \sup\Big\{\left|S(f)\right|, |f| \leqslant \chi_\Omega\Big\}$$

and as λ is order continuous

$$|||S||| = \left\|\,|S|\,(\chi_\Omega)\right\| = \sup\Big\{\Big\|\left|S(f)\right|\Big\|, \|f\| \leqslant 1\Big\} = \|S\|.$$

Then S is $\lambda-$integral with $\mathbf{I}_\lambda(S) \leqslant \|S\|$. But as $(\mathcal{I}_\lambda, \mathbf{I}_\lambda)$ is a Banach operator ideal, $\|S\| \leqslant \mathbf{I}_\lambda(S)$, hence $\mathbf{I}_\lambda(S) = \|S\|$. ∎

Corollary 6. *Let (Ω, Σ, μ) be a measure space and $n, k \in \mathbb{N}$. Then every operator $T : L_\infty(\mu) \to S_n(\ell_1)\left[S_k(\lambda)\right]$ satisfies that $\mathbf{I}_\lambda(T) = \|T\|$.*

Proof: The results follows easily from the Theorem 5, because every operator $T : L_\infty(\mu) \to S_n(\ell_1)\left[S_k(\lambda)\right]$ is order bounded and $S_n(\ell_1)\left[S_k(\lambda)\right]$ is reflexive hence order continuous. ∎

To get our best results the structure of finite dimensional subspaces of the relevant Banach sequence space λ and so its behaviour under ultraproducts will be crucial. For this goal we introduce the following definition:

Definition 7: Given a Banach sequence space λ, we say that a Banach space X is $\mathcal{L}_c^{\lambda,g}$ space if there exists a real constant $c > 0$ such that for every finite dimensional subspace F of X, there is a section $S_n(\lambda)$ of λ and linear operators $u : F \to S_n(\lambda)$ and $v : S_n(\lambda) \to X$ such that $\|u\|\,\|v\| \leqslant c$ and $v\,u = I_{F,X}$.

The following definition was introduced by Pelczyǹnski and Rosenthal [**13**] in 1975.

Definition 8: A Banàch space X has the *uniform projection property* if there is a $b > 0$ such that for each natural number n there is a natural number $m(n)$ such that for every n-dimensional subspace $M \subset X$ there exists a k-dimensional and b-complemented subspace Z of X containing M with $k \leqslant m(n)$.

Remark 9.

(1) The constant b of the above definition is called a uniform projection property of X, and in this case we also say that X has the b-uniform projection property. If X has the b-uniform projection property for every $b > B$ we say that X has the B^+-uniform projection property

(2) The class of Banach spaces with the uniform projection property is quite large and includes, for example the reflexive Orlicz spaces, see [**10**]. In particular they have the 1^+-uniform projection property,

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(3) The Bochner space $L_p(\mu, E)$ has the b-uniform projection property if E does, see $1 \leqslant p \leqslant \infty$, [**8**].

(4) It is known that the uniform projection property is stable under ultrapowers, see [**8**].

The next proposition is more or less obvious, see [**15**].

PROPOSITION 10. *Let λ be a regular Banach sequence space. Then*

(a) *λ is a $\mathcal{L}_c^{\lambda,g}$ space, for every $c > 1$.*

(b) *Complemented subspaces of $\mathcal{L}_c^{\lambda,g}$ spaces are $\mathcal{L}_c^{\lambda,g}$ spaces.*

And the results of the following also are proved in [**15**].

PROPOSITION 11. *If λ is a regular Banach sequence space satisfying the b-uniform projection property, then every ultrapower of λ_r is a $\mathcal{L}_\beta^{\lambda,g}$ space for every $\beta > b$.*

For our next theorem we need a very deep technical result of Lindenstrauss and Tzafriri [**10**] which gives us a kind of "uniform approximation" of finite dimensional subspaces by finite dimensional sublattices in Banach lattices.

LEMMA 12. *Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be fixed. There is a natural number $h(n, \varepsilon)$ such that for every Banach lattice X and every subspace $F \subset X$ of dimension $\dim(F) = n$ there are $h(n, \varepsilon)$ disjoints elements $\{z_i,\ 1 \leqslant i \leqslant h(n,\varepsilon)\}$ and an operator A from F into the linear span G of $\{z_i,\ 1 \leqslant i \leqslant h(n,\varepsilon)\}$ such that*

$$\forall\, x \in F\ \left\|A(x) - x\right\| \leqslant \varepsilon \left\|x\right\|.$$

THEOREM 13. *Let λ be a regular Banach sequence space with the uniform projection property and let E and F be Banach spaces. The following statements are equivalent:*

(1) *$T \in \mathcal{I}_\lambda(E, F)$.*

(2) *$J_F T$ factors continuously in the following way:*

$$\begin{tikzcd} E \arrow[r, "J_F\, T"] \arrow[d, "A"'] & F'' \\ L_\infty(\mu) \arrow[r, "B"'] & X \arrow[u, "C"'] \end{tikzcd}$$

where X is an $\mathcal{L}^{\ell_1[\lambda],g}$ space. Furthermore the norm $\mathrm{I}_\lambda(T)$ is equivalent to $\inf\{\|C\|\,\|B\|\,\|A\|\}$, taken over all such factorisations.

(3) *$J_F T$ factors continuously in the following way:*

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$$\begin{tikzcd} E \arrow[rr, "J_F T"] \arrow[d, "A"'] & & F'' \\ L_\infty(\mu) \arrow[rr, "B"'] & & X \arrow[u, "C"'] \end{tikzcd}$$

where X is an ultrapower of $\ell_1[\lambda]$ and B is a lattice homomorphism. Furthermore $\mathbf{I}_\lambda(T)$ is equivalent to $\inf\{\|C\|\,\|B\|\,\|A\|\}$, taken over all such factorisations.

PROOF: (1)$\Longrightarrow$(3) We define the set

$$D := \{(M,N) : M \in FIN(E), N \in FIN(F')\}$$

where $FIN(Y)$ is the set of finite dimensional subspace of a Banach space Y, endowed with the natural inclusion order

$$(M_1,N_1) \leqslant (M_2,N_2) \Longleftrightarrow M_1 \subset M_2,\ N_1 \subset N_2.$$

For every $(M_0,N_0) \in D$, $R(M_0,N_0) := \{(M,N) \in D : (M_0,N_0) \subset (M,N)\}$ and $\mathcal{R} = \{R(M,N), (M,N) \in D\}$. $\mathcal{R}$ is filter basis in D, and according to Zorn's lemma, let $\mathcal{D}$ be an ultrafilter on D containing $\mathcal{R}$. If $d \in D$, M_d and N_d denote the finite dimensional subspaces of E and F' respectively so that $d = (M_d, N_d)$. For every $d \in D$, if $z \in M_d \otimes N_d$, $J_F T_{|M_d \otimes N_d} \in \left(M_d \bigotimes_{g'_\lambda} N_d\right)' = M'_d \bigotimes_{g_\lambda} N'_d = \mathcal{N}_\lambda(M_d, N'_d)$. Then from Theorem 4 of characterisation of λ-nuclear operators, $J_F T_{|M_d \otimes N_d}$ factors as

$$\begin{tikzcd} M_d \arrow[rr, "J_F T_{|M_d \otimes N_d}"] \arrow[d, "A_d"'] & & N'_d \\ \ell_\infty[\ell_\infty] \arrow[rr, "B_d"'] & & \ell_1[\lambda_r] \arrow[u, "C_d"'] \end{tikzcd}$$

where B_d is a positive diagonal operator and $\|A_d\|\,\|B_d\|\,\|C_d\| \leqslant \mathbf{N}_\lambda^c(T_{|M_d \otimes N_d}) + \varepsilon = \mathbf{I}_\lambda(T_{|M_d \otimes N_d}) + \varepsilon$. Then

$$\|A_d\|\,\|B_d\|\,\|C_d\| \leqslant \mathbf{I}_\lambda(T_{|M_d \otimes N_d}) + \varepsilon \leqslant \mathbf{I}_\lambda(T) + \varepsilon$$

Without loss of generality we can suppose that $\|A_d\| = \|C_d\| = 1$. We define $W_E : E \to (M_d)_\mathcal{D}$ such that $W_E(x) = (x_d)_\mathcal{D}$ so that $x_d = x$ if $x \in M_d$ and $x_d = 0$ if $x \notin M_d$. In the same way we define $W_{F'} : F' \to (N_d)_\mathcal{D}$ such that $W_{F'}(a) = (a_d)_\mathcal{D}$ so that $a_d = a$ if $a \in N_d$ and $a_d = 0$ if $a \notin N_d$. Then we have the following commutative diagram:

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$$\begin{tikzcd}
E \arrow[rr, "J_F T"] \arrow[d, "W_E"'] & & F'' \\
(M_d)_{\mathcal{D}} \arrow[r, "(J_F T_{|M_d \otimes N_d})_{\mathcal{D}}"'] \arrow[d, "(A_d)_{\mathcal{D}}"'] & (N'_d)_{\mathcal{D}} \arrow[r, "I"'] & ((N_d)_{\mathcal{D}})' \arrow[u, "W'_{F'}"'] \\
(\ell_\infty[\ell_\infty])_{\mathcal{D}} \arrow[r, "(B_d)_{\mathcal{D}}"'] & (\ell_1[\lambda_r])_{\mathcal{D}} \arrow[u, "(C_d)_{\mathcal{D}}"'] &
\end{tikzcd}$$

where I is the canonical inclusion map. As from [**10**] $\left((\ell_1[\lambda_r])_{\mathcal{D}}\right)''$ is a 1-complemented subspace of some ultrapower $\left((\ell_1[\lambda_r])_{\mathcal{D}}\right)_{\mathcal{U}}$ which is another ultrapower $(\ell_1[\lambda_r])_{\mathcal{U}_1}$ with projection Q, the result follows with $A = (A_d)_{\mathcal{D}}$, $B = ((B_d)_{\mathcal{D}})''$ which is a lattice homomorphism, $C = P_{F''''}(W'_{F'}I(C_d)_{\mathcal{D}})''Q$, where $P_{F''''}$ is the projection of F'''' in F'', and $X = (\ell_1[\lambda])_{\mathcal{U}_1}$, having in mind that as $(\ell_\infty[\ell_\infty])_{\mathcal{D}}$ is an abstract M-space, there is a measure space such that $L_\infty(\mu) = \left((\ell_\infty[\ell_\infty])_{\mathcal{D}}\right)''$, where the equality means that the spaces are lattice isometric.

(3)$\Longrightarrow$(2): It is straightforward.

(2)$\Longrightarrow$(1) We only have to see that B is a λ-integral operator, or a little bit more, that every operator $B : G \longrightarrow X$, where G is an abstract M-space and X is an $\mathcal{L}^{\ell_1[\lambda],g}$ space, is λ-integral and $\mathbf{I}_\lambda(B) \leqslant c\,\|B\|$ for some $c > 0$.

By the representation theorem of maximal operator ideals (see [**3**, 17.5]), we only need to show that $J_X\, B \in (G \otimes_{g'_\lambda} X')'$.

Given $z \in G \otimes X'$ and $\varepsilon > 0$, let $M \subset G$ and $N \subset X'$ be finite dimensional subspaces and let $z = \sum_{i=1}^{n} f_i \otimes x'_i$ be a *fixed* representation of z with $f_i \in M$ and $x'_i \in N$, $i = 1, 2, \ldots, s, n$ such that

$$g'_\lambda(z; G \otimes X') \leqslant g'_\lambda(z; M \otimes N) \leqslant g'_\lambda(z; G \otimes X') + \varepsilon.$$

By Lemma 12 we find a finite dimensional sublattice M_1 of G and an operator $A_1 : M \to M_1$ so that

$$\forall\, f \in M,\ \|A_1(f) - f\| \leqslant \varepsilon \|f\|.$$

Then, if id_G denotes the identity map on G we have

$$\begin{aligned}
|\langle J_X B, z\rangle| = \left|\sum_{i=1}^{n} \langle B(f_i), x'_i\rangle\right| &\leqslant \left|\sum_{i=1}^{n} \langle T(id_G - A_1)(f_i), x'_i\rangle\right| + \left\|\sum_{i=1}^{n} \langle B\, A_1(f_i), x'_i\rangle\right\| \\
&\leqslant \varepsilon \|B\| \sum_{i=1}^{n} \|f_i\|\, \|x'_i\| + \left|\sum_{i=1}^{n} \langle B\, A_1(f_i), x'_i\rangle\right|.
\end{aligned}$$

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Put $X_1 := B(M_1)$. By hypothesis X is an $\mathcal{L}^{\ell_1[\lambda],g}$ space, hence there are $c > 0$, $n, k \in \mathbb{N}$, $u : X_1 \to S_n(\ell_1)[S_k(\lambda)]$ and $v : S_n(\ell_1)[S_k(\lambda)] \to X$ such that $I_{X_1,X} = v\ u$ and $\|u\|\,\|v\| \leqslant c$. We denote $X_2 = v\ u(X_1)$ which is a finite dimensional subspace of X containing X_1 and $I_{X_1,X_2} = v\ u$. Let $K_2 : X' \longrightarrow X_2' = X'/X_2^\circ$ be the canonical quotient map. Then

$$\begin{aligned}\sum_{i=1}^n \Big\langle B\big(A_1(f_i)\big), x_i'\Big\rangle &= \sum_{i=1}^n \Big\langle I_{X_1,X_2} B\big(A_1(f_i)\big), K_2(x_i')\Big\rangle \\ &= \sum_{i=1}^n \Big\langle v\ u\ B\ \big(A_1(f_i)\big), K_2(x_i')\Big\rangle \\ &= \sum_{i=1}^n \Big\langle u\ B\big(A_1(f_i)\big), v'\ K_2(x_i')\Big\rangle \\ &= \Big\langle u\ B, \sum_{i=1}^n A_1(f_i) \otimes v'\ K_2(x_i')\Big\rangle\end{aligned}$$

with $\sum_{i=1}^n A_1(f_i) \otimes v'\ K_2(x_i') \in M_1 \otimes \Big(S_n(\ell_1)\big[S_k(\lambda)\big]\Big)'$ and $u\ B : M_1 \to S_n(\ell_1)\big[S_k(\lambda)\big]$.

Since M_1 is a reflexive abstract M-space it is lattice isometric to some $L_\infty(\mu)$ space, hence from corollary 6 this map is λ-integral with $\mathbf{I}_\lambda(u\ B) \leqslant \|u\|\,\|B\|$. Then

$$\begin{aligned}\left|\sum_{i=1}^n \Big\langle B\big(A_1(f_i)\big), x_i'\Big\rangle\right| &= \left|\Big\langle u\ B, \sum_{i=1}^n A_1(f_i) \otimes v'\ K_2(x_i')\Big\rangle\right| \\ &\leqslant \mathbf{I}_\lambda(u\ B)\ g_\lambda'\Big(\sum_{i=1}^n A_1(f_i) \otimes v'\ K_2(x_i'); M_1 \otimes S_k(\lambda)\Big) \\ &\leqslant \|u\|\|B\| g_\lambda'\big((A_1 \otimes v'K_2)(z); M_1 \otimes S_k(\lambda)\big) \\ &\leqslant \|u\|\,\|B\|\,\|A_1\|\,\|v'\|\,\|K_2\|\ \ g_\lambda'(z; M \otimes N) \\ &\leqslant (1+\varepsilon)c\,\|B\|\,g_\lambda'(z; M \otimes N) \\ &\leqslant (1+\varepsilon)c\,\|B\|\,\big(g_\lambda'(z; G \otimes X') + \varepsilon\big)\end{aligned}$$

and by the arbitrariness of $\varepsilon > 0$ we obtain

$$\big|\langle J_X B, z\rangle\big| \leqslant c\,\|B\|\,g_\lambda'(z; G \otimes X')$$

∎

4. On the coincidence of λ-nuclear and λ-integral operators.

The following new formulation of the Theorem 13 for Banach sequence spaces having finite cotype and satisfying the uniform projection property is needed in our setting:

Theorem 14. *Let λ be a Banach sequence space having the uniform projection property and finite cotype. For every pair of Banach spaces E and F, the following statements are equivalent:*

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(1) $T \in \mathcal{I}_\lambda(E,F)$.

(2) *There exists a σ-finite measure space $(\mathcal{O},\mathcal{S},\nu)$ and an order continuous Köthe function space $\mathcal{K}(\nu)$ which is an $\mathcal{L}^{\ell_1[\lambda],g}$ space, such that $J_F T$ factors continuously in the following way:*

$$\begin{tikzcd} E \arrow[r, "J_F T"] \arrow[d, "A"'] & F'' \\ L_\infty(\nu) \arrow[r, "B"'] & \mathcal{K}(\nu) \arrow[u, "C"'] \end{tikzcd}$$

where B is a multiplication operator for a positive function of $\mathcal{K}(\nu)$. Furthermore $\mathbf{I}_\lambda(T)$ is equivalent to $\inf\{\|C\|\,\|B\|\,\|A\|\}$, taking the infimum over all such factors.

PROOF: We start from the Theorem 13. As λ has finite cotype from [**12**] it implies that $\ell_1[\lambda_r]$ has the same property, hence every ultrapower of $\ell_1[\lambda_r]$ is order continuous (Henson and Moore, [**7**, 4.6]). From [**11**, Theorem 1.a.9] every ultrapower of $\ell_1[\lambda_r]$ can be decomposed into an unconditional direct sum of a family of mutually disjoint ideals $\{X_h, h \in H\}$ having a positive weak unit, and then from [**11**, 1.b.14], as every X_h is order isometric to a Köthe space of functions defined on a probability space $(\mathcal{O}_h,\mathcal{S}_h,\nu_h)$, $(\ell_1[\lambda_r])_{\mathcal{U}}$ is order isometric to a Köthe function space $\mathcal{K}(\nu^1)$ over a measure space $(\mathcal{O}^1,\mathcal{S}^1,\nu^1)$. Hence we can substitute $(\ell_1[\lambda_r])_{\mathcal{U}}$ for $\mathcal{K}(\nu^1)$ in the Theorem 13. But if we denote $z := B(\chi_\Omega)$ where $z = \sum\limits_{i=1}^{\infty} y_{h_i}$ with $y_{h_i} \in X_{h_i}$ for every $i \in \mathbb{N}$, then $B(L_\infty(\mu))$ is contained in the unconditional direct sum of $\{X_{h_i}, i \in \mathbb{N}\}$ which is order isometric to a Köthe function space $\mathcal{K}(\nu)$ over a σ-finite measure space $(\mathcal{O},\mathcal{S},\nu)$ which is 1-complemented in $\mathcal{K}(\nu^1)$, and hence it is an $\mathcal{L}^{\ell_1[\lambda],g}$ space.

Finally, as $\mathcal{K}(\nu)$ is order complete, $g := \sup\limits_{\|f\|_{L_\infty(\mu)}} B(f)$ exists in $\mathcal{K}(\nu)$. Then the operators $B_1 : L_\infty(\mu) \to L_\infty(\nu)$ and $B_2 : L_\infty(\nu) \to \mathcal{K}(\nu)$, such that $B_1(f)(\omega) := B(f)(\omega)/g(\omega)$, for all $f \in L_\infty(\mu)$, $\omega \in \mathcal{O}$ with $g(\omega) \neq 0$ and $B_1(f)(\omega) = 0$ otherwise, and $B_2(h)(\omega) := g(\omega)h(\omega)$ for all $h \in L_\infty(\nu)$, $\omega \in \mathcal{O}$, satisfy that $B = B_2 B_1$ and B_2 is a multiplication operator for a positive element $g \in \mathcal{K}(\nu)$. ∎

We introduce a new operator ideal, which is contained in the ideal of the λ-integral operators.

DEFINITION 15: Let λ be a Banach sequence space having finite cotype and satisfying the uniform projection property. We say that $T \in \mathcal{L}(E,F)$ is *strictly λ-integral* if exist a σ-finite measure space $(\mathcal{O},\mathcal{S},\nu)$ and an order continuous Köthe function space $\mathcal{K}(\nu)$ which is an $\mathcal{L}^{\ell_1[\lambda],g}$ space, such that T factors continuously in the following way:

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$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[d, "A"'] & & F \\ L_\infty(\nu) \arrow[rr, "B"'] & & \mathcal{K}(\nu) \arrow[u, "C"'] \end{tikzcd}$$

where B is a multiplication operator for a positive function of $\mathcal{K}(\nu)$.

We denote by $\mathcal{SI}_\lambda(E,F)$ the set of the strictly λ-integral operators between E and F which is a closed subspace of $\mathcal{I}_\lambda(E,F)$ and $\mathbf{SI}^c_\lambda(T) = \mathbf{I}_\lambda(T)$ for every $T \in \mathcal{SI}_\lambda(E,F)$. It is clear that if F is a dual space, or it is complemented in its bidual space, then $\mathcal{SI}_\lambda(E,F) = \mathcal{I}_\lambda(E,F)$.

Theorem 16. *Let λ be a Banach sequence space having the uniform projection property and finite cotype, and let E and F be Banach spaces such that E' satisfies the Radon-Nikodým property. Then $\mathcal{N}_\lambda(E,F) = \mathcal{SI}_\lambda(E,F)$.*

Proof: Let $T \in \mathcal{SI}_\lambda(E,F)$.

(a) We suppose that B is an multiplication operator for a function $g \in \mathcal{K}(\nu)$ with support the set D, and D has finite measure. We denote ν_D the restriction of ν to D.

As $(\chi_D A) : E \to L_\infty(\nu_D)$, then $(\chi_D A)' : \big(L_\infty(\nu_D)\big)' \to E'$ and the restriction $(\chi_D A)' \restriction_{L_1(\nu_D)}: L_1(\nu_D) \to E'$. Thus, for every $x \in E$ and $f \in L_1(\nu_D)$

$$\big\langle x, (\chi_D A)'(f)\big\rangle = \big\langle \chi_D A(x), f\big\rangle = \int_D \chi_D A(x) f d(\nu_D).$$

As E' has the Radon-Nikodým property, applying [4, Theorem III(5)], we have that $(\chi_D A)'$ has a Riesz representation, so there exists a function $\phi \in L_\infty(\nu_D, E')$ such that for every $f \in L_1(\nu_D)$

$$(\chi_D A)'(f) = \int_D f\phi d(\nu_D).$$

Then, for every $x \in E$, we have that $\chi_D A(x)(t) = \langle \phi(t), x\rangle$, ν_D-almost everywhere in D, and then $B(\chi_D A)(x) =< g\phi(\cdot), x >$, ν_D-almost everywhere in D. We denote $g\phi$ this last operator, and we can consider it as an element in $\mathcal{K}(\nu_D, E')$.

As the simple functions are dense in $\mathcal{K}(\nu_D, E')$, $g\phi$ can be approximated by a sequence of simple functions $(S_k)_{k=1}^\infty$.

We suppose $S_k = \sum\limits_{j=1}^{m_k} x'_{kj}\chi_{A_{kj}}$, where $\{A_{ki} : i = 1, \dots, m\}$ is a family of ν-measurable sets of Ω pairwise disjoint. For each $k \in \mathbb{N}$, we can interpret S_k as a map $S_k : E \to \mathcal{K}(\nu)$ such that $S_k(x) = \sum\limits_{j=1}^{m_k} < x'_{kj}, x > \chi_{A_{kj}}$ with norm less or equal than the norm of S_k in $\mathcal{K}(\nu, E')$.

Obviously for all $k \in \mathbb{N}$, S_k is λ-nuclear because it has finite range, but we need to evaluate its λ-nuclear norm $\mathbf{N}^c_\lambda(S_k)$ which coincides with it λ-integral norm $\mathbf{I}_\lambda(S_k)$.

Let $S^1_k : E \to L_\infty(\nu)$ be such that

$$S^1_k(x) = \sum_{j=1}^{m_k} \frac{\langle x'_{kj}, x\rangle}{\|x'_{kj}\|}\chi_{A_{kj}}$$

and let $S^2_k : L_\infty(\nu) \to \mathcal{K}(\nu)$ be such that

$$S^2_k(f) = \sum_{j=1}^{m_k} \|x'_{kj}\| f \chi_{A_{kj}}.$$

Then it is easy to see that $\|S^1_k\| \leqslant 1$, $\|S^2_k\| \leqslant \|S_k\|_{\mathcal{K}(\nu,E')}$ and $S_k = S^2_k\, S^1_k$. But as $\mathcal{K}(\nu)$ is an $\mathcal{L}^{\ell_1[\lambda],g}$ space, from the Theorem 13, there is $K > 0$ such that $\mathbf{N}^c_\lambda(S^2_k) = \mathbf{I}_\lambda(S^2_k)$ $\leqslant K\ \|S^2_k\| \leqslant K\ \|S_k\|_{\mathcal{K}(\nu,E')}$, hence $\mathbf{N}^c_\lambda(S_k) \leqslant K\ \|S_k\|_{\mathcal{K}(\nu,E')}$.

Then, as $(S_k)_{k=1}^\infty$ converges in $\mathcal{K}(\nu_D, E')$, it can be also considered as a Cauchy sequence in $\mathcal{N}_\lambda\big(E, \mathcal{K}(\nu_D)\big)$, and as this space is complete, $(S_k)_{k=1}^\infty$ converges to $g\phi$, that is to say, $g\phi \in \mathcal{N}_\lambda\big(E, \mathcal{K}(\nu_D)\big)$. Therefore, $g\phi = B\chi_D A$ is λ-nuclear and so T is also λ-nuclear with.

(b) If g is any element of $\mathcal{K}(\nu)$, g can be approximated in norm by means of a sequence $(t_n)_{n=1}^\infty$ of simple functions with support having finite measure, and therefore by a), the sequence $T_n = CB_{t_n}A$ is a Cauchy sequence in $\mathcal{N}_\lambda(E, F)$ converging to T in $\mathcal{L}(E, F)$, and then $T \in \mathcal{N}_\lambda(E, F)$. ∎

As consequence of the former result and of the factorisation Theorems 14 and 4, we obtain the following metric properties of g^c_λ and g'_λ.

Theorem 17. *If λ has finite cotype and satisfies the uniform projection property, then g'_λ is a totally accessible tensor norm.*

Proof: As g'_λ is finitely generated, it is sufficient to prove that the map $F \otimes_{g'_\lambda} E \hookrightarrow \mathcal{P}_{\lambda^\times}(E', F'')$, is an isometry.

In fact, let $z = \sum_{i=1}^{n} \sum_{j=1}^{l_i} y_{ij} \otimes x_{ij} \in F \otimes_{g'_\lambda} E$, and let $H_z \in \mathcal{P}_{\lambda^\times}(E', F'')$ be the canonical map associated to z, that is to say,

$$H_z(x') = \sum_{i=1}^{n} \sum_{j=1}^{l_i} \langle x_{ij}, x'\rangle y_{ij}$$

for all $x' \in E'$, con $H_z \in \mathcal{L}(E', F) \subset \mathcal{L}(E', F'')$.

Applying the [**3**, Theorem 15.5] for $\alpha = g'_\lambda$, the Theorem 2, and the equality $(g_\lambda)''$ $= g^c_\lambda$ since g^c_λ finitely generated, we have that the inclusion

$$F \bigotimes_{\overleftarrow{g'_\lambda}} E \hookrightarrow \Big(F' \bigotimes_{g^c_\lambda} E'\Big)' \to \mathcal{P}_{\lambda^\times}(E', F'').$$

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is an isometry, and therefore by the in [**3**, Proposition 12.4] we obtain

$$\Pi_{\lambda^\times}(H_z) = \overleftarrow{g'_\lambda}(z; F \otimes E) \leqslant g'_\lambda(z; F \otimes E).$$

On the other hand, given N, a finite dimensional subspace of F such that $z \in N \bigotimes_{g'_\lambda} E$, so there exists a function $V \in (N \bigotimes_{g'_\lambda} E)' = \mathcal{I}_\lambda(N, E')$ such that $\mathbf{I}_\lambda(V) \leqslant 1$ and $g'_\lambda(z; N \otimes E) = \langle z, V \rangle$. Clearly enough $V \in \mathcal{SI}_\lambda(N, E') = \mathcal{I}_\lambda(N, E')$ because E' is a dual space, and N', being finite dimensional, has the Radon-Nikodým property. Therefore by Theorem 16, $V \in \mathcal{N}_\lambda(N, E')$ and by Theorem 4, given $\varepsilon > 0$, there is a factorisation V in the way

$$\begin{tikzcd} N \arrow[r, "V"] \arrow[d, "A"'] & E' \\ \ell_\infty[\ell_\infty] \arrow[r, "B"'] & \ell_1[\lambda_r] \arrow[u, "C"'] \end{tikzcd}$$

such that $\|C\|\,\|B\|\,\|A\| \leqslant \mathbf{N}^c_\lambda(V) + \varepsilon = \mathbf{I}_\lambda(V) + \varepsilon \leqslant 1 + \varepsilon$.

As $\ell_\infty[\ell_\infty]$ has the extension metric property, (to see [**14**, Proposition 1, C.3.2]), A can be extended to a continuous map $\overline{A} \in \mathcal{L}(F, \ell_\infty[\ell_\infty])$ such that $\|\overline{A}\| = \|A\|$. By Theorem 4 again, $W := CB\overline{A}$ is in $\mathcal{N}_\lambda(F, E')$, so there is a representation $w =: \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} y'_{ij} \otimes x'_{ij} \in F' \widehat{\bigotimes}_{g^c_\lambda} E'$ of W verifying

$$\sum_{i=1}^{\infty} \pi_{h_\lambda}\big((y'_{ij})\big)\varepsilon_{\lambda^\times}\big((x'_{ij})\big) \leqslant \mathbf{N}^c_\lambda(W) + \varepsilon \leqslant \|C\|\,\|B\|\,\|\overline{A}\| + \varepsilon \leqslant 1 + 2\varepsilon.$$

Then, $g'_\lambda(z; F \otimes E) \leqslant g'_\lambda(z; N \otimes E) = \langle z, V \rangle = \langle z, W \rangle$ it follows that

$$g'_\lambda(z; F \otimes E) \leqslant g^c_\lambda(w)\Pi_{\lambda^\times}(H_z) \leqslant (1 + 2\varepsilon)\Pi_{\lambda^\times}(H_z)$$

thus $g'_\lambda(z; F \otimes E) \leqslant \Pi_{\lambda^\times}(H_z)$, and the equality is obvious. ∎

Corollary 18. *If λ has finite cotype and satisfies the uniform projection property, then g^c_λ is an accesible tensor norm.*

Proof: It is a direct consequence of the former theorem and of [**3**, Proposition 15.6]. ∎

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