# A slowly evolving conical pendulum

SUBHRANIL DE

1. *Introduction*

The idea of this work originally arose from a question pertaining to a laboratory experiment on circular motion in our departmental lab manual. The experiment itself involves rotating a bob along a horizontal circle (Figure 1), where the tension in the string attached to the bob provides the centripetal acceleration of the bob, the string itself passing through a smooth vertical pipe. It is assumed that the rotation is fast enough for the effect of gravity to be neglected and therefore the orientation of the part of the string between the top end of the pipe and the bob can be taken to be horizontal. The above-mentioned question enquires what happens to the speed of the bob in the case that the bottom end of the string is hand-held and pulled slowly so that the radius of the circular orbit decreases. The answer to the question is straightforward. Either a work-energy argument or an argument involving the conservation of angular momentum provides the same correct answer.

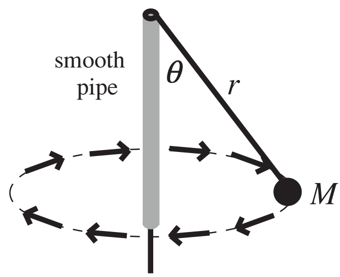


FIGURE 1: The experimental configuration

In reality though, the portion of the string between the bob and the pipe makes an angle with the horizontal, which renders the configuration a classic *conical pendulum* [1, 2], referred to as such because the string traces the surface of a cone with some apex angle $\theta$.

The modified question we pose in this paper is this: we start with some configuration $r$ and $\theta$ (Figure 1), where $r$ is the length of the conical pendulum. If the other end of the string is pulled through the pipe slowly (quasi-statically) with the pipe itself fixed in place, then as $r$ decreases, how will the angle $\theta$ change in the process?

2. *The conical pendulum: basic analysis*

In this section, we define certain quantities of significance as well as present an analysis of the mechanics of the classic conical pendulum. As shown in Figure 2, the length of the portion of the string between the bob (point $B$) and the fixed top end of the pipe (point $A$, referred to as the apex in the rest of the paper) is $r$, while the angle between that portion of the string

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and the vertical is $\theta$. The vertical $y$-axis and the horizontal $x$-axis are chosen as shown with the apex as the origin. The bob undergoes circular motion with speed $v$ along a horizontal circle of radius $x$, with the centre (point $C$) of the circle located on the $y$-axis. Since the bob is at a lower vertical level than the apex, the value of the $y$-coordinate of the bob will be negative.

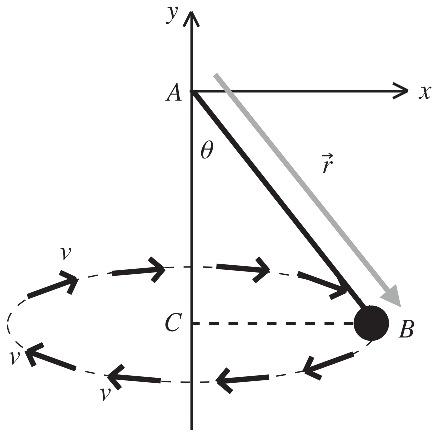


FIGURE 2: The conical pendulum: the geometry and coordinates

From the right-angled triangle $ABC$, we write $x$ and $y$ in terms of $r$ and $\theta$:

$$x = r \sin\theta, \tag{1}$$

and

$$y = -r \cos\theta. \tag{2}$$

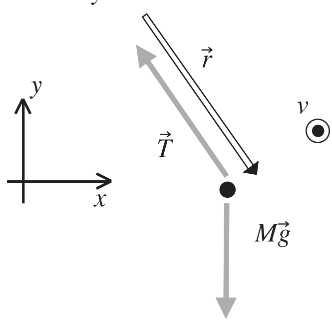


FIGURE 3: The force vectors (the solid arrows) exerted on the bob.

In Figure 3, the two force vectors exerted on the bob are shown with the two solid arrows: the vertically downward force of gravity $M\vec{g}$ and the force of tension $\vec{T}$ pulling on the bob in the direction along the string toward the apex. The position vector $\vec{r}$ of the bob relative to the apex is also shown with a hollow arrow in order to highlight the fact that it is directed exactly opposite to the tension force $\vec{T}$. Also, in this portrayal commensurate with Figure 2, where the plane of the page represents the vertical $xy$-plane

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containing the two force vectors, the horizontal direction represented by the direction normal to the page and out of the page, i.e. the positive $z$-direction, is the direction of the instantaneous velocity of the bob.

Now the horizontal component of the tension force provides the centripetal acceleration $\left(v^2/x\right)$ [1] of the bob. Hence, from Newton's second law,

$$T \sin\theta = M\frac{v^2}{x}. \tag{3}$$

At the same time, the vertical component of the tension force exactly balances the vertically downward force of gravity. Hence we have

$$T \cos\theta = Mg. \tag{4}$$

Using (1), (3) and (4), we derive for the speed of the bob:

$$v^2 = \frac{gr \sin^2\theta}{\cos\theta}. \tag{5}$$

3. *Quasi-static evolution: the work-energy approach*

Now we consider the case where the other end of the string is pulled quasi-statically through the frictionless pipe so that the length $r$ of the conical pendulum decreases accordingly. Since the process is quasi-static, i.e. $\left|\frac{dr}{dt}\right| << v$, the system can be treated as a perfect conical pendulum at any point during the process.

According to the work-energy theorem, the change in mechanical energy of the bob equals the external work done on it [1, 3]. Here the mechanical energy is the sum of its kinetic energy $\frac{1}{2}Mv^2$ and its gravitational potential energy $Mgy$. As for the work done on the bob, the two forces that do the work are the tension force and the gravitational force. However, the work done by gravity is already accounted for through the aforementioned gravitational potential energy term.

Hence, at the end, for a differential change $dr$ in the length of the pendulum, we write using the work-energy theorem:

$$d\left(\tfrac{1}{2}Mv^2 + Mgy\right) = -T\,dr, \tag{6}$$

where the right-hand side of the above relation is the work done by the tension. It is worth noting that $dr < 0$ during the pulling of the string, since $r$ decreases in the process. However, the analysis we present below holds perfectly for $dr > 0$ as well, i.e. when the string is quasi-statically released instead of pulled and $r$ increases as a result.

Dividing both sides of (6) by $dr$, we write:

$$\frac{1}{2}M\frac{dv^2}{dr} + Mg\frac{dy}{dr} = -T. \tag{7}$$

Differentiating both sides of (5) with respect to $r$, we obtain

$$\frac{dv^2}{dr} = \frac{d}{dr}\left(\frac{gr \sin^2\theta}{\cos\theta}\right) = g\frac{\sin^2\theta}{\cos\theta} + gr\frac{d\theta}{dr}\frac{d}{d\theta}\left(\frac{\sin^2\theta}{\cos\theta}\right).$$

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Completing the derivative with respect to $\theta$ in the last term on the right-hand side of the above relation, we obtain after simplifying:

$$\frac{dv^2}{dr} = g\frac{\sin^2\theta}{\cos\theta} + gr\frac{d\theta}{dr}\sin\theta\left(2 + \tan^2\theta\right). \tag{8}$$

At the same time, differentiating both sides of (2) with respect to $r$, we obtain

$$\frac{dy}{dr} = -\cos\theta + r\frac{d\theta}{dr}\sin\theta. \tag{9}$$

Lastly, we rearrange (4) to give:

$$T = \frac{Mg}{\cos\theta}. \tag{10}$$

Putting (8), (9) and (10) in (7), we obtain after simplification:

$$r\frac{d\theta}{dr}\left(3\cos\theta + \sec\theta\right) + 3\sin\theta = 0.$$

Rearranging this gives

$$3\cot\theta\, d\theta + \frac{d\theta}{\sin\theta\cos\theta} + \frac{3\,dr}{r} = 0.$$

Then integrating, we get:

$$3\ln(\sin\theta) + \ln(\tan\theta) + 3\ln\left(\frac{r}{r_0}\right) = 0, \tag{11}$$

where $r_0$ is a constant.

We can now rearrange (11) to write

$$\left(\sin^3\theta\right)\left(\tan\theta\right)r^3 = c_1,$$

or

$$r^3 = c_1\frac{\cos\theta}{\sin^4\theta}. \tag{12}$$

The above relation relates $r$ to $\theta$ for the conical pendulum in the course of its quasi-static evolution, where the positive constant $c_1$ depends on initial conditions.

## 4. *The torque-angular momentum approach*

In this section, we attempt to apply the principle that the net torque on a particle is equal to the rate of change of its angular momentum [1, 3]. In particular, we consider the angular momentum of the bob about the fixed apex $A$, for which we refer to Figure 3. Since the position vector of the bob about the apex is $\vec{r}$, the net torque on the bob is given by $\vec{\tau}_{\text{net}} = \vec{r} \times \vec{T} + \vec{r} \times M\vec{g}$.

Since the vectors $\vec{r}$ and $\vec{T}$ are in opposite directions, the cross product $\vec{r} \times \vec{T}$ is zero. As for the torque of the force of gravity, since both $\vec{r}$ and $\vec{T}$ lie in the $xy$ (vertical) plane, the cross product $\vec{r} \times M\vec{g}$ will be perpendicular

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to that plane, and therefore directed horizontally (specifically, in the negative $z$-direction, as per Figure 3). What is worth emphasising is that this fact, namely that the net torque on the bob will be directed purely horizontally and therefore will have *no* vertical component, will be true at each and every point in time. This implies that the vertical component of the angular momentum of the bob about the apex $A$ will remain unaltered during the quasi-static evolution of the conical pendulum.

Now, if the unit vectors in the positive $x$, $y$ and $z$ directions are $\hat{x}$, $\hat{y}$ and $\hat{z}$ respectively, then, referring back to Figure 3, we can write the following vector expressions:

$$\vec{r} = x\hat{x} + y\hat{y}, \tag{13}$$

and

$$\vec{v} = v\hat{z}. \tag{14}$$

Remembering that the angular momentum is defined as $\vec{L} = \vec{r} \times M\vec{v}$, we obtain using (13) and (14),

$$\vec{L} = \left(x\hat{x} + y\hat{y}\right) \times Mv\hat{z} = -Mvx\hat{y} - Mvy\hat{x}.$$

The above expression elucidates the fact that the absolute value of the vertical ($y$) component of the angular momentum is $Mvx$. However, as we inferred above already, this vertical component of the angular momentum must remain unaltered. Hence we have

$$Mvx = \text{const.},$$

or

$$vx = \text{const.},$$

or

$$v^2x^2 = \text{const.} \tag{15}$$

Using (5) and (1) in (15), we obtain

$$\left(r\frac{\sin^2\theta}{\cos\theta}\right)r^2\sin^2\theta = \text{const.},$$

or

$$r^3\frac{\sin^4\theta}{\cos\theta} = \text{const.},$$

or

$$r^3 = \text{const.}\frac{\cos\theta}{\sin^4\theta},$$

thus corroborating (12).

5. *A quartic surface emerges*

It turns out that we can go even further with (12).

We rewrite (12) as

$$r^3\sin^3\theta = c_1\frac{r\cos\theta}{r\sin\theta}.$$

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From which, using (1) and (2), we obtain

$$x^3 = c_1 \frac{(-y)}{x},$$

or

$$y = c_2 x^4, \tag{16}$$

where $c_2$ is a negative constant.

Equation (16) is a neat, quartic equation, which, when rotated about the $y$-axis, produces the surface of revolution which contains the bob of the conical pendulum during its quasi-static evolution.

Figure 4 presents the plot of (16) with the example value of the constant $c_2$ chosen to be $-1$. Several configurations of the conical pendulum during its quasi-static evolution, along with the corresponding positions of the bob on the quartic contour, are shown.

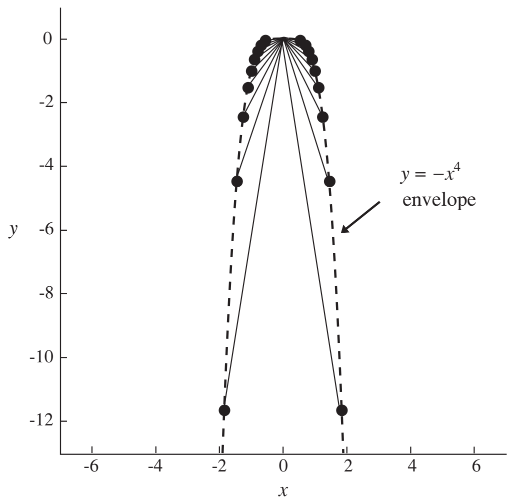


FIGURE 4: The vertical projection of the surface of revolution containing the bob described by the quartic equation (16), with the value of the constant $c_2$ taken to be $-1$

*References*

1. Resnick, Halliday and Walker, *Fundamentals of physics,* Wiley (2021).
2. https://demonstrations.wolfram.com/MotionOfAConicalPendulum/
3. Taylor, *Classical mechanics*, University Science Books (2005).

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SUBHRANIL DE
*School of Natural Sciences,*
*Indiana University Southeast,*
*New Albany, IN 47150 USA*
e-mail: *subde@ius.edu*