11

Quantum kinetic field theory

11.1 The Kadanoff–Baym equations

Quantum kinetic field theory is the theme of this chapter. In this section we get right to the heart of it by showing a derivation of the celebrated Kadanoff–Baym (KB) equations [KadBay62]. The basic idea is that close to equilibrium, propagators are nearly translation invariant. It is possible to define a partial Fourier transform with respect to the relative position of the arguments. The Kadanoff–Baym equations then determine how the partial Fourier transform depends on the average (or "center of mass") of the arguments in the original propagators.

Besides the presentation in Kadanoff and Baym's textbook, there are several derivations of these equations in the literature [Dub67, Dan84a, MroDan90, MroHei94, ZhuHei98]. We shall follow [CalHu88, CaHuRa00]. See also [Hen95, IvKnVo00, KnIvVo01, Nie02, Koi02]. References [BoVeWa00, WBVS00] follow a different path towards quantum kinetic theory, based on the so-called dynamical renormalization group.

11.1.1 The model

To better appreciate the main points in this derivation, we shall consider a simple model, namely, the KB equations for the theory of a single real self-interacting $\lambda\Phi^4$ scalar field, in the absence of background fields. Actually, the key ideas are not sensitive to the particular models, but for concreteness it will be helpful to have a model in mind. The classical action is given by equation (6.106).

A translation-invariant propagator G^{ab} depends on its arguments x and x' only through the so-called "relative" variable $u = x - x'$. The Fourier transform with respect to u yields the momentum representation

$$G^{ab}(x, x') = \hbar \int \frac{d^d k}{(2\pi)^d}\, e^{iku}\, G^{ab}(k) \tag{11.1}$$

We have discussed in Chapter 6 the basic properties of these Fourier transforms.

We say that a G^{ab} is almost translation invariant if, when partially Fourier transformed with respect to u, the Fourier transform is weakly dependent on the "center of mass" variable $X = (x + x')/2$, i.e.

$$G^{ab}(x, x') = \int \frac{d^d k}{(2\pi)^d}\, e^{iku} G^{ab}(X, k) \tag{11.2}$$

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The precise definition of what "weakly dependent" means depends up to a certain point on the problem at hand. For example, in a hard thermal loop scheme such as discussed in Chapter 10, we may find a situation where $\partial_x G^{ab}(x,x') \approx TG^{ab}$, while $\partial_X G^{ab}(X,k) \approx g^n TG^{ab}$ with $n \geq 1$. In such a case, the propagators are almost translation invariant in the weak coupling limit.

On the other hand, beware that in gauge theories the same object may be almost translation invariant in some gauges and not in others (with a corresponding problem in relativistic theories with respect to changes of coordinates). We also mention that in the presence of external gauge background fields, or in curved spacetimes, some care must be taken to define precisely the Fourier transform in (11.2) [Hei83, Win85, CaHaHu88, Fon94]. We shall discuss these issues later in this chapter.

Irrespective of whether the assumption of almost translation-invariance holds, expressions involving $G^{ab}(X,k)$ may be classified according to their "adiabatic order," namely, the number of X derivatives appearing in the expression. We call this the "adiabatic expansion." When almost translation-invariance is satisfied, we may further reject all terms above a given adiabatic order. We call such a truncation of an adiabatic expansion an "adiabatic approximation." In other words, the adiabatic order is used as a tag to bunch together certain terms in the equations of motion in accordance to their derivative orders and the adiabatic approximation determines how many of those terms are kept.

Our aim is to analyze the adiabatic expansion of the 2PI Schwinger–Dyson equations for the propagators. These are deduced from the 2PI CTPEA (cf. Chapter 6)

$$\Gamma = \frac{1}{2Z_B}\int d^dx d^dy\, c_{ab} D(x,y) G^{ab}(x,y) - \frac{i\hbar}{2}\mathrm{Tr}\,\ln G + \Gamma_Q \tag{11.3}$$

where $c_{11} = -c_{22} = 1$, $c_{12} = c_{21} = 0$,

$$D(x,y) = \left[\partial_x^2 - m_b^2\right]\delta(x-y) \tag{11.4}$$

Γ_Q is the sum of all 2PI vacuum bubbles. Taking variations of the 2PI CTPEA we find the equations of motion

$$\frac{1}{Z_B} c_{ab} D(x,y) - i\hbar\left[G^{-1}\right]_{ab}(x,y) - \hbar\Sigma_{ab}(x,y) = 0 \tag{11.5}$$

$$\hbar\Sigma_{ab}(x,y) = -2\frac{\delta\Gamma_Q}{\delta G^{ab}(x,y)} \tag{11.6}$$

These are the exact equations we must solve. We assume there are known relations expressing Σ_{ab} in terms of the propagators. These can be found, for example, by adopting one of the perturbative schemes discussed in Chapter 6.

Observe that to determine $G(X,k)$ as the inverse Fourier transform of $G(x,x')$ we must know the whole evolution of the correlation, both to the past and future of the event X. It is possible to present an alternative formulation where only

equal time correlations are Fourier transformed, thus more in keeping with the spirit of causality [ZhaHei96a, ZhaHei96b, ABZH96, ZhuHei98].

In general, it will be necessary to add nonlocal sources to the classical action to account for nontrivial correlations at the initial time. We consider these sources are included into the $\Sigma_{ab}(x,y)$.

Let us continue with the analysis of (11.5). Our first task is to find an efficient parameterization for the propagators. It is clear that the four basic propagators (Feynman, Dyson, positive and negative frequency) are not independent. As we shall see, there are essentially two (phase space) functions which contain the relevant information from which all propagators may be reconstructed. One of these functions plays the role of a (position-dependent) density of states, and the other one of the nonequilibrium one-particle distribution function. To be able to write all propagators in terms of these two functions we must consider first the so-called Keldysh representation of the propagators, in which the four basic propagators are written in terms of the Hadamard, retarded and advanced propagators as

$$G^{ab}=\begin{pmatrix}G^{11} & G^{12}\\ G^{21} & G^{22}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & -i\hbar G_{\rm adv}\\ -i\hbar G_{\rm ret} & G_1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \tag{11.7}$$

with inverse

$$\begin{pmatrix}0 & -i\hbar G_{\rm adv}\\ -i\hbar G_{\rm ret} & G_1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}G^{11} & G^{12}\\ G^{21} & G^{22}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \tag{11.8}$$

11.1.2 Density of states and distribution function

Let us introduce the density of states $\mathcal{D}(X,k)$ out of the Fourier transform of the Jordan propagator $G=G^{21}-G^{12}$

$$G(X,k)\equiv 2\pi\hbar\mathcal{D}(X,k)\ \text{sign}\left(k^0\right) \tag{11.9}$$

Observe that by symmetry we must have $G(X,(0,\mathbf{k}))=0$. We shall assume G is continuous there, implying $\mathcal{D}(X,(0,\mathbf{k}))=0$.

The Jordan and retarded propagators are related through

$$\begin{aligned}G(X,k)&=2\hbar\,\text{Im}G_{\rm ret}(X,k)\\ &=-2\hbar\left|G_{\rm ret}(X,k)\right|^2\text{Im}\left[G_{\rm ret}(X,k)\right]^{-1}\end{aligned} \tag{11.10}$$

This suggests defining a new kernel $\gamma(X,k)$ such that

$$\mathcal{D}(X,k)=\left|G_{\rm ret}(X,k)\right|^2\gamma(X,k) \tag{11.11}$$

$$\gamma(X,k)=\frac{1}{\pi}\text{Im}\left[-G_{\rm ret}(X,k)\right]^{-1}\text{sign}\left(k^0\right) \tag{11.12}$$

Observe that per earlier assumptions, $\gamma(X,(0,\mathbf{k}))=0$.

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We now define the (dimensionless) distribution function $f(X,k)$ through the partial Fourier transform of the Hadamard propagator

$$G_1(X,k) \equiv 2\pi\hbar\mathcal{D}(X,k)\ F_1(X,k) \tag{11.13}$$

$$F_1(X,k) = 1 + 2f(X,k) \tag{11.14}$$

It follows that

$$G^{21(12)}(X,k) = 2\pi\hbar\, F^{21(12)}(X,k)\,\mathcal{D}(X,k) \tag{11.15}$$

where

$$F^{21(12)}(X,k) = \theta\left(\pm k^0\right) + f(X,k) \tag{11.16}$$

In equilibrium, f is the Bose–Einstein distribution function (KMS theorem). It can be assumed that (11.13) serves as the definition of the function f, valid to all orders in perturbation theory. Observe that, since the relevant Fourier transforms are distributions (e.g. in free theories), this definition may only be applied if both Fourier transforms have the same singularity structure, which amounts to a restriction on allowed quantum states. In what follows, we shall assume these restrictions are met.

While this definition of the one-particle distribution function will prove to be very convenient in practice, and it is guaranteed to give the right result in equilibrium, it is not tied to any fundamental definition of what a particle is. It is also possible to take an alternative route, where one introduces a physically motivated particle destruction operator, builds the corresponding particle number operator, and finally derives an equation of motion for the latter (cf. the discussion of the quantum Vlasov equation in Chapter 4) [GreLeu98].

11.1.3 The dissipation and noise kernels

As we have seen, the information content of the almost translation-invariant propagators can be encoded in just two functions $\mathcal{D}(X,k)$ and $f(X,k)$. To proceed, we must perform a similar compression of the self-energies Σ_{ab}. We do this by writing both propagators and self-energies in terms of the dissipation $\mathbf{D}$ and the noise kernel $\mathbf{N}$, which appear in the Hessian of the one-particle irreducible (1PI) effective action. These two kernels are largely independent of each other, and have a distinct physical meaning, with $\mathbf{D}$ carrying the dynamical information and $\mathbf{N}$ the statistical information. This division of labor is most clearly seen in a free theory. Together $\mathbf{D}$ and $\mathbf{N}$ are a more compact description of the theory than the propagators themselves.

We have two different ways of relating $\mathbf{D}$ and $\mathbf{N}$ to the propagators. On one hand, they are constructed from Feynman diagrams which carry propagators in their internal legs. Which diagrams must be considered depends on which

approximation is being used. On the other hand, the Schwinger–Dyson (SD) equations allow us to express the propagators in terms of $\mathbf{D}$ and $\mathbf{N}$. For a true solution, these two paths must be equivalent. This consistency requirement yields the most efficient representation of the dynamics.

The dissipation and noise kernels $\mathbf{D}$ and $\mathbf{N}$ appear in the linearized CTP 1PI EA $\Gamma_{1\mathrm{PI}}$ (cf. Chapter 6)

$$\Gamma_{1\mathrm{PI}} = \frac{1}{2}\int d^d x d^d y \left\{ \left[\varphi^1 - \varphi^2\right](x) \left[\frac{1}{Z_B} D(x,y) + \mathbf{D}(x,y)\right] \left[\varphi^1 + \varphi^2\right](y) + i\left[\varphi^1 - \varphi^2\right](x)\, \mathbf{N}(x,y) \left[\varphi^1 - \varphi^2\right](y) \right\} \tag{11.17}$$

$\mathbf{D}$ is causal and $\mathbf{N}$ is even, and both are real. The causality of $\mathbf{D}$ allows for a more efficient parameterization. Introduce the kernels

$$\mathbf{D}_{\mathrm{even}}(x,y) = \frac{1}{2}\left[\mathbf{D}(x,y) + \mathbf{D}(y,x)\right]; \qquad \mathbf{\Gamma}(x,y) = \frac{1}{2}\left[\mathbf{D}(x,y) - \mathbf{D}(y,x)\right] \tag{11.18}$$

then

$$\mathbf{D} = \mathbf{2\Gamma}\,\theta\left(x^0 - y^0\right) \tag{11.19}$$

$$\mathbf{D}_{\mathrm{even}} = \mathbf{\Gamma}\,\mathrm{sign}\left(x^0 - y^0\right) \tag{11.20}$$

$\mathbf{D}$ and $\mathbf{N}$ are related to the G^{ab} through the identity

$$\frac{\mathcal{D}^2 \Gamma_{1PI}}{\mathcal{D}\varphi^a \mathcal{D}\varphi^b} = i\hbar\left[G^{-1}\right]_{ab} \tag{11.21}$$

The inverse propagators may be read off the Schwinger–Dyson equations, and we get

$$-\hbar\Sigma_{11} = \mathbf{D}_{\mathrm{even}} + i\mathbf{N} \tag{11.22}$$

$$-\hbar\Sigma_{12} = \mathbf{\Gamma} - i\mathbf{N} \tag{11.23}$$

$$-\hbar\Sigma_{21} = -\mathbf{\Gamma} - i\mathbf{N} \tag{11.24}$$

$$-\hbar\Sigma_{22} = -\mathbf{D}_{\mathrm{even}} + i\mathbf{N} \tag{11.25}$$

Since $\mathbf{D}(x,y)$ is real, we know the real part of $\mathbf{D}(X,k)$ is even and the imaginary part is odd: $\mathbf{D}(X,k) = \mathbf{D}(X,-k)^*$, so

$$\mathbf{D}_{\mathrm{even}}(X,k) = \mathrm{Re}\,\mathbf{D}(X,k) \tag{11.26}$$

$$\mathbf{\Gamma}(X,k) = i\,\mathrm{Im}\,\mathbf{D}(X,k) \tag{11.27}$$

and (11.20) transforms into the Kramers–Kronig relations for the causal kernel $\mathbf{D}$.

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11.1.4 The retarded and advanced propagators

We have seen how to relate the dissipation and noise kernels $\mathbf{D}$ and $\mathbf{N}$ to the inverse propagators. To relate them to the propagators themselves, let us first investigate the SD equations in the Keldysh representation.

Write

$$\hbar \left[G^{-1}\right]_{ab} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \hbar^{-1}G_{\mathrm{ret}}^{-1}G_1 G_{\mathrm{adv}}^{-1} & iG_{\mathrm{ret}}^{-1} \\ iG_{\mathrm{adv}}^{-1} & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{11.28}$$

The equations of motion now read

$$\begin{pmatrix} \hbar^{-1}G_{\mathrm{ret}}^{-1}G_1 G_{\mathrm{adv}}^{-1} & iG_{\mathrm{ret}}^{-1} \\ iG_{\mathrm{adv}}^{-1} & 0 \end{pmatrix} = (-i)\begin{pmatrix} 2i\mathbf{N} & \frac{D}{Z_B} + \mathbf{D} \\ \frac{D}{Z_{\mathrm{zB}}} + \mathbf{D}_{\mathrm{adv}} & 0 \end{pmatrix} \tag{11.29}$$

where

$$\mathbf{D}_{\mathrm{adv}}(x,y) = \mathbf{D}_{\mathrm{even}} - \mathbf{\Gamma} = \mathbf{D}(y,x) \tag{11.30}$$

These equations show that G_1 and G_{ret} may be considered functionals of $\mathbf{D}$ and $\mathbf{N}$. The formulae above upon partial Fourier transform become

$$\mathbf{D}(X,k) = \left\{ -G_{\mathrm{ret}}^{-1}(X,k) + \frac{1}{Z_B}\left(k^2 + m_b^2\right) \right\} \tag{11.31}$$

To relate $G_{\mathrm{ret}}^{-1}(X,k)$ to $\left[G_{\mathrm{ret}}(X,k)\right]^{-1}$ we recall the formula for the partial Fourier transform of a convolution

$$\begin{aligned}[f * g](X,k) = f(X,k)\, g(X,k) - \frac{i}{2}\{f,g\}\frac{-1}{8}\left\{ \frac{\partial^2 f}{\partial X^\mu \partial X^\nu}\frac{\partial^2 g}{\partial k_\mu \partial k_\nu} \right. \\ \left. + \frac{\partial^2 g}{\partial X^\mu \partial X^\nu}\frac{\partial^2 f}{\partial k_\mu \partial k_\nu} - 2\frac{\partial^2 f}{\partial X^\mu \partial k_\nu}\frac{\partial^2 g}{\partial k_\mu \partial X^\nu} \right\} + \ldots \end{aligned} \tag{11.32}$$

where we use the Poisson bracket (cf. Chapter 2)

$$\{f,g\} = \frac{\partial f}{\partial k}\frac{\partial g}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial g}{\partial k} \tag{11.33}$$

We obtain the adiabatic expansion of $G_{\mathrm{ret}}^{-1}(X,k)$ by applying equation (11.32) to the obvious statement that the convolution of G_{ret}^{-1} and G_{ret} is the identity operator. To simplify the resulting expression, we assume that in the second-order terms we may approximate $G_{\mathrm{ret}}^{-1}(X,k)$ by its quasi-particle approximation form $\left[(k+i\epsilon)^2 + M^2\right]$.

More generally, the so-called quasi-particle approximation consists in replacing the actual propagators for those of a free field (see Chapter 5) with a yet-to-be-determined mass M^2. The physical basis of this approximation is that one expects the most interesting dynamics may be described in terms of localized excitations which, in between collisions, propagate as free particles with a well-defined mass. This leads to propagators concentrated on a sharp mass shell, which can be well approximated by free propagators. The quasi-particle

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approximation is expected to hold when the mean free path for quasi-particles is long compared with the Debye length M^{-1}. We warn the reader beforehand that important processes, such as thermalization, are not well described within this approximation (see the next chapter).

Computing the required derivatives and rearranging, we obtain

$$1 = \left\{ \frac{1}{Z_B} \left(k^2 + m_b^2\right) - \mathbf{D}\left(X,k\right) \right\} G_{\rm ret}\left(X,k\right) - \frac{1}{4}\nabla_X^2 G_{\rm ret}\left(X,k\right) - \frac{1}{8}\frac{\partial^2 M^2}{\partial X^\mu \partial X^\nu}\frac{\partial^2 G_{\rm ret}\left(X,k\right)}{\partial k_\mu \partial k_\nu} \tag{11.34}$$

Over and above the need to renormalize (11.34), observe that this is a full-fledged evolution equation for the Fourier transform of the retarded propagator. To make the approach more definite, we may request that the quasi-particle approximation for $G_{\rm ret}$ actually becomes exact as $k \to 0$. We shall discuss mass renormalization in more detail below.

Ideally one would seek simultaneous solutions for (11.34) and the transport equation to be derived below, but these are hard (and may be impossible) to find [Mro97]. In such a case, one simply regards (11.34) as a way to generate the adiabatic expansion of $G_{\rm ret}$.

In the approximation where only terms linear in the gradients of the Fourier transforms of the propagators are retained, it is possible to write down a non-perturbative (in the coupling constant) expression for the retarded and Jordan propagators. The advantage of this approach is that it goes beyond the quasi-particle approximation. In particular, it is sufficient for the discussion of the transition to hydrodynamics and the computation of transport functions.

It is convenient to introduce a real kernel

$$R\left(X,k\right) = \frac{1}{Z_B}\left(k^2 + m_b^2\right) - \mathbf{D}_{\rm even}\left(X,k\right) \tag{11.35}$$

The required expression is

$$\left[G_{\rm ret}\left(X,k\right)\right]^{-1} = R - \mathbf{\Gamma} \tag{11.36}$$

From (11.27)

$$\mathbf{\Gamma}\left(X,k\right) = (-i)\ \mathrm{Im}\ \left[G_{\rm ret}\left(X,k\right)\right]^{-1} = i\pi\,\gamma\left(X,k\right)\mathrm{sign}\left(k^0\right) \tag{11.37}$$

and finally, from (11.11),

$$\mathcal{D}\left(X,k\right) = \frac{\gamma\left(X,k\right)}{R^2 - \mathbf{\Gamma}^2} \tag{11.38}$$

Recall that since we assume we know how to express R and $\mathbf{\Gamma}$ (and therefore also γ) in terms of propagators, this is really a consistency condition linking the density of states and the distribution function. Also recall that in deriving it we have neglected terms of second adiabatic order and higher.

11.1.5 The off-shell kinetic equation

To obtain the dynamics of the distribution function f, we make use of the remaining equation involving the noise kernel

$$\mathbf{N}(x,y) = \frac{1}{2\hbar}\int d^d z_1 d^d z_2\, G_{\rm ret}^{-1}(x,z_1)\, G_1(z_1,z_2)\, G_{\rm adv}^{-1}(z_2,y) \tag{11.39}$$

Iterating the formula for the partial Fourier transform of a convolution (11.32), and dropping second-order terms and higher, we get

$$\begin{aligned} 2\hbar\mathbf{N} = &\left[G_{\rm ret}^{-1}G_{\rm adv}^{-1} - \frac{i}{2}\left\{G_{\rm ret}^{-1}, G_{\rm adv}^{-1}\right\}\right] G_1 \\ &- \frac{i}{2}\left[G_{\rm adv}^{-1}\left\{G_{\rm ret}^{-1}, G_1\right\} - G_{\rm ret}^{-1}\left\{G_{\rm adv}^{-1}, G_1\right\}\right] \end{aligned} \tag{11.40}$$

Observe that by performing the adiabatic expansion on this form of the SD equations we avoid the appearance of $\mathbf{N}$ within Poisson brackets. This choice is related to the so-called Botermans and Malfliet approach [BotMal90, IvKnVo00, KnIvVo01, IvKnVo03]. Next, write

$$G_1 = 2\pi\hbar\gamma G_{\rm ret} G_{\rm adv} F_1 \tag{11.41}$$

$$\begin{aligned} \mathbf{N} = &\left[\pi\gamma\left(1 - \frac{i}{2}G_{\rm ret}G_{\rm adv}\left\{G_{\rm ret}^{-1}, G_{\rm adv}^{-1}\right\}\right)\right. \\ &\left. - \frac{i\pi}{2}\left(G_{\rm ret}\left\{G_{\rm ret}^{-1}, \gamma\right\} - G_{\rm adv}\left\{G_{\rm adv}^{-1}, \gamma\right\}\right)\right] F_1 \\ &- \frac{i\pi\gamma}{2}\left[G_{\rm ret}\left\{G_{\rm ret}^{-1}, F_1\right\} - G_{\rm adv}\left\{G_{\rm adv}^{-1}, F_1\right\}\right] \end{aligned} \tag{11.42}$$

Recall that

$$\{\mathbf{\Gamma}, \gamma\} = i\pi\,\gamma(X,p)\left\{\mathrm{sign}\left(p^0\right), \gamma\right\} = i\pi\,\delta\left(p^0\right)\frac{\partial}{\partial X^0}\gamma^2(X,p) = 0 \tag{11.43}$$

because $\gamma(X,(0,\mathbf{p})) = 0$. Therefore

$$\left\{G_{\rm ret}^{-1}, G_{\rm adv}^{-1}\right\} = \{R - \mathbf{\Gamma}, R + \mathbf{\Gamma}\} = 2\{R, \mathbf{\Gamma}\} \tag{11.44}$$

$$\left\{G_{\rm ret}^{-1}, \gamma\right\} = \left\{G_{\rm adv}^{-1}, \gamma\right\} = \{R, \gamma\} \tag{11.45}$$

$$\begin{aligned} \mathbf{N} = &\left[\pi\gamma\left(1 - 2iG_{\rm ret}G_{\rm adv}\{R, \mathbf{\Gamma}\}\right)\right] F_1 \\ &- \frac{i\pi\gamma}{2}\left[G_{\rm ret}\left\{G_{\rm ret}^{-1}, F_1\right\} - G_{\rm adv}\left\{G_{\rm adv}^{-1}, F_1\right\}\right] \end{aligned} \tag{11.46}$$

Introduce the *collision integral*

$$\begin{aligned} I_{\rm col} &\equiv \left[\mathbf{N} - \pi\gamma F_1\right]\mathrm{sign}\left(k^0\right) \\ &= \frac{-i\hbar}{2}\left[\left(\Sigma_{12} + \Sigma_{21}\right)\left(F^{21} - F^{12}\right) + \left(\Sigma_{12} - \Sigma_{21}\right)\left(F^{21} + F^{12}\right)\right] \\ &= -i\hbar\left[\Sigma_{12}F^{21} - \Sigma_{21}F^{12}\right] \end{aligned} \tag{11.47}$$

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Then we obtain the kinetic equation

$$A\{R, F_1\} + B\{\mathbf{\Gamma}, F_1\} + CF_1 = I_{\text{col}} \text{ sign}\left(k^0\right) \tag{11.48}$$

where

$$A = -\frac{\mathbf{\Gamma}^2}{R^2 - \mathbf{\Gamma}^2} \tag{11.49}$$

$$B = \frac{R\mathbf{\Gamma}}{R^2 - \mathbf{\Gamma}^2} \tag{11.50}$$

$$C = -2\frac{\mathbf{\Gamma}}{R^2 - \mathbf{\Gamma}^2}\{R, \mathbf{\Gamma}\} \tag{11.51}$$

Equation (11.48) is the key result of this chapter.

11.1.6 Weakly coupled theories and the Boltzmann equation

For weakly coupled theories, a series of approximations allow us to reduce (11.48) to the more familiar Boltzmann kinetic equation (cf. Chapter 2).

We observe that in terms of the coupling constant λ we have, for a generic momentum p, $R \sim O(1)$ while $\mathbf{\Gamma} \sim O\left(\lambda^2\right)$. The C term in (11.48) combines both space derivatives (assumed small) and radiative corrections. It is therefore expected to be smaller than the other terms in the equation, and thus neglected (approximations of this kind are further discussed in Section 11.1.9). An alternative, which we shall not follow, is to consider these terms as parts of the collision integral, in which case we could regard them as a first-order approximation to a more general, non-Markovian kinetic equation [KBKS97, Ike04].

A second observation is that in general $\mathbf{\Gamma}$, which involves the coupling constants, will be much smaller than R for a generic choice of p. When the coupling constants go to zero $\mathbf{\Gamma} \to 0$, but the retarded propagator has a well-defined asymptotic value, and (11.36) becomes

$$G_{\text{ret}} \sim \text{PV}\frac{1}{R} + i\pi \text{ sign}(k^0)\delta(R) \tag{11.52}$$

From equations (11.9) and (11.10), the density of states

$$\mathcal{D} = \delta(R) \tag{11.53}$$

In this limit the propagators are insensitive to the behavior of the distribution function "off shell" (i.e. when $R \neq 0$), because the distribution function is always multiplied by the density of states, and this is very small there. Therefore, only "on shell" modes (i.e. those for which $R = 0$) really contribute to the field correlation functions. If our only concern is to follow the evolution of the distribution function on shell, we are allowed to replace the A and B coefficients in (11.48) by their "on shell" values, namely $A = 1$ and $B = 0$. We thus obtain the Kadanoff–Baym equations

$$\{R, F_1\} = -i\hbar \text{ sign}\left(k^0\right) \left[\Sigma_{12}F^{21} - \Sigma_{21}F^{12}\right] \tag{11.54}$$

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Observe that in this argument we first took the weakly coupled limit, and then went on-shell. Also we assumed that somehow the adiabatic expansion and the expansion in powers of the coupling constant were linked; otherwise the C coefficient would be found to be comparable to A and B. A way to put this on a systematic basis is the hard thermal loop expansion discussed in Chapter 10.

After these approximations (in keeping with the weak coupling assumption, we are entitled to keep only $O(\lambda)$ terms in R as well), the nontrivial content of (11.54) is given by the form of the collision integral, namely, which Feynman graphs contribute to the self-energies.

The Kadanoff–Baym equations are formally valid to all orders in the coupling constant. It is convenient to consider the loop expansion of the self-energies to reduce this equation to a more familiar form. However, even now we recognize the structure of the collision term as the difference between a gain and a loss term for particles moving in or out of a phase space cell around the point (X, p) per unit time. Taking $p^0 > 0$ for simplicity, we see that $\Sigma_{12} F^{21}$ is the gain term, with $F^{21} = 1 + f$ accounting for stimulated emission of particles into the cell, while the other term is the loss term, which is proportional to the number of particles $F^{12} = f$ already there.

Let us consider the expansion of the self-energies in terms of Feynman graphs of increasing loop order, as a means of obtaining a definite expression for the collision term in the kinetic equation. Since we have the relationship $\Sigma_{21}(p) = \Sigma_{12}(-p)$ it is enough to analyze only the expansion of Σ_{12}. Physically this means considering only the gain processes, which produce a particle within a given phase space cell. The collision term is then obtained by subtracting the loss processes, which remove a particle therein.

The first term in the expansion is the setting-sun graph. To this order,

$$\Sigma_{12}(x, y) = \frac{i}{6} \lambda^2 \hbar \mathcal{G}(x, y) \tag{11.55}$$

$$\mathcal{G} = \frac{1}{\hbar^3} \left[G^{12}(x, y) \right]^3 \tag{11.56}$$

In momentum space, dealing with the propagators as if they were translation invariant, and using the definition of F^{12}, we get

$$\mathcal{G}(p) = (2\pi)^4 \int \frac{d^4 r \mathcal{D}(r)}{(2\pi)^3} \frac{d^4 s \mathcal{D}(s)}{(2\pi)^3} \frac{d^4 t \mathcal{D}(t)}{(2\pi)^3} \delta(p - r - s - t) F^{12}(r) F^{12}(s) F^{12}(t) \tag{11.57}$$

We also replace $\mathcal{D}$ by its quasi-particle form $\mathcal{D}_0 = \delta\left(p^2 + M^2\right)$. We must then find sets of four on-shell momenta adding up to zero. If $p^0 > 0$, this means that two of the r, s, t momenta must be future oriented, and the third past oriented. Using the symmetries of this expression, we obtain

$$\mathcal{G}(p) = 3(2\pi)^4 \int DrDsDt\, \delta(p + r - s - t) [1 + f(r)] f(s) f(t) \tag{11.58}$$

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where

$$Dp = \frac{d^4 p \mathcal{D}(p) \theta\left(p^0\right)}{(2\pi)^3} \tag{11.59}$$

It is fairly obvious that the resulting kinetic equation is just Boltzmann's.

An important if simple consequence of this fact is that the usual arguments showing that the only stationary solutions of the Boltzmann equation are thermal distributions carry over to the Kadanoff–Baym equations. In other words, the only translation-invariant propagators which solve the Kadanoff–Baym equations, or, for that matter, the 2PI Schwinger–Dyson equations, to this order in perturbation theory are thermal propagators. This fact is relevant to the discussion of thermalization in quantum field theory [JuCaGr04].

The basic formalism we presented here can be extended in several ways, such as including higher terms in the derivative expansion [Mro97, Jak02], higher correlations [WanHei02] or non-Markovian effects [MorRop99, SeKrBo00]. Another important generalization consists of explicitly incorporating the effects of quantum fluctuations in higher composite operators by including a stochastic source besides the collision integral [ReiToe94, AARS96, CalHu00]. We may regard this so-called Boltzmann–Langevin equation (see Chapter 2) as the quantum kinetic analog of the Langevin approach we discussed in Chapter 8.

The classical limit

It is interesting to consider the classical limit of the Boltzmann equation. Naively, we have, in powers of $\hbar$, that $\{R, F_1\} \sim O(1)$ and $I_{\rm col} \sim O\left(\hbar^2\right)$. However, in the classical limit we must have that the Jordan propagator $G \to 0$ but the Hadamard propagator G_1 remains finite. To allow for a nonzero limit we must have

$$f = \hbar^{-1} f_{\rm cl} \tag{11.60}$$

and counting powers of $\hbar$ we get

$$\{R, f_{\rm cl}\} = I^{(3)}_{\rm col}\left[f_{\rm cl}\right] \tag{11.61}$$

where $I^{(3)}_{\rm col}\left[f_{\rm cl}\right]$ contains all terms in the collision integral that are cubic in $f_{\rm cl}$.

The conclusion is that the correlation functions for a weakly interacting classical *field*, in the nearly translation-invariant limit, may be captured by a kinetic equation describing two by two scattering of on-shell excitations [MueSon04].

The classical Boltzmann equation describing interacting *particles* has a collision term quadratic in the distribution function. To obtain equation (11.61) instead, we must include the Bose enhancement factors, although of course this is a classical theory [Ein17].

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11.1.7 The Vlasov equation

To lowest order in λ, our theory reduces to the Vlasov equation, namely, a transport theory for collisionless particles interacting with a self-consistent field.

This theory is obtained by neglecting the $O\left(\lambda^2\right)$ terms in our equations. The unperturbed equations are

$$R = \Omega_0 = p^2 + M^2 \tag{11.62}$$

and

$$\mathcal{D}(p) = \delta\left(p^2 + M^2\right) + O\left(\lambda^2\right) \tag{11.63}$$

The kinetic equation reduces to

$$0 = \mathcal{D}\left[p\frac{\partial}{\partial X} - \frac{1}{2}\partial_X M^2 \partial_p\right] f \tag{11.64}$$

which is indeed in the form of a Vlasov equation. The mass is defined through the self-consistent gap equation

$$M^2 = m_b^2 + m_V^2 + \frac{\lambda_b \hbar}{2} M_T^2 \tag{11.65}$$

where

$$M_T^2 = \int \frac{d^4p}{(2\pi)^3} \delta\left(\Omega_0\right) f\left(X, p\right) \tag{11.66}$$

$$m_V^2 = \frac{\lambda_b \hbar}{4} \int \frac{d^4p}{(2\pi)^3} \delta\left(\Omega_0\right) \tag{11.67}$$

This second quantity is actually divergent, so to evaluate it we need to regularize it first. We shall use dimensional regularization, writing (cf. Chapter 5)

$$m_V^2 = -\frac{\lambda_b \hbar M^2}{16\pi^2}\left[z - \frac{1}{2}\ln\left(\frac{M^2}{4\pi\mu^2}\right)\right] \tag{11.68}$$

$$z \equiv \frac{\Gamma\left[1 + \frac{\varepsilon}{2}\right]}{\varepsilon\left[1 - \frac{\varepsilon}{2}\right]} = \frac{1}{\varepsilon} + \frac{1}{2}\left(1 - \gamma\right) + \ldots \tag{11.69}$$

($\gamma = 0.5772\ldots$).

We go back to the gap equation and write it as

$$M^2\left\{1 + \frac{\lambda_b \hbar}{16\pi^2}\left[z - \frac{1}{2}\ln\left(\frac{M^2}{4\pi\mu^2}\right)\right]\right\} = m_b^2 + \frac{\lambda_b \hbar}{2} M_T^2 \tag{11.70}$$

which implies

$$\frac{1}{2}\frac{dM_T^2}{dM^2} = \frac{1}{\lambda_b \hbar} + \frac{1}{16\pi^2}\left[z - \frac{1}{2} - \frac{1}{2}\ln\left(\frac{M^2}{4\pi\mu^2}\right)\right] \tag{11.71}$$

Since the left-hand side is finite, the expression

$$\frac{1}{\lambda_b \hbar} + \frac{1}{16\pi^2}\left[z - \frac{1}{2}\right] \equiv \frac{1}{\lambda \hbar} \tag{11.72}$$

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must also be finite, and the differential gap equation becomes

$$\frac{1}{2}\frac{dM_T^2}{dM^2} = \frac{1}{\lambda\hbar} - \frac{1}{32\pi^2}\ln\left(\frac{M^2}{4\pi\mu^2}\right) \tag{11.73}$$

Mass renormalization entails defining an initial condition for this differential equation, such as $M_T^2(M^2 = 0) = T^2/6\hbar^2$.

What is the small parameter?

Since reducing the theory to just the Vlasov equation means dropping terms of order λ^2, it might appear that one ought to replace the physical mass by the solution of the gap equation to the same order, namely

$$M^2 = M_0^2 + \frac{\lambda\hbar}{32\pi^2}M_0^2\ln\left(\frac{M_0^2}{4\pi\mu^2}\right) \tag{11.74}$$

where

$$M_0^2 = \frac{\lambda\hbar}{2}\left(M_T^2 - \frac{T_c^2}{6}\right) \tag{11.75}$$

(we gloss over the fact that M_T^2 itself depends on M^2; at high enough temperature M_T^2 stabilizes at a value of $T^2/6$, as in the massless theory). However, a moment's thought shows that, at least in the high-temperature limit, this is not a good idea. For high enough temperature, the second term in our expansion is of the order of the first term, meaning the breakdown of naive perturbation theory.

However, we can also proceed differently. In the regime where the derivation of the gap equation is valid, we can also write it as

$$M^2 = \frac{M_0^2}{1 - \frac{\lambda\hbar}{32\pi^2}\ln\left(\frac{M_0^2}{4\pi\mu^2}\right)} \tag{11.76}$$

(of course, this expression also blows up when the denominator vanishes, but that is a pathology of the $\lambda\phi^4$ theory, which is not asymptotically free). If we replace back equation (11.74) into the gap equation, we see that there is an error term of order

$$2\left[\frac{\lambda\hbar}{32\pi^2}\ln\left(\frac{M_0^2}{4\pi\mu^2}\right)\right]^2 M_0^2 \tag{11.77}$$

If we repeat the same with the expression (11.76), we see that the error has been reduced to

$$\left[\frac{\lambda\hbar}{32\pi^2}\right]^2 \ln\left(\frac{M_0^2}{4\pi\mu^2}\right) M_0^2 \tag{11.78}$$

that is, an improvement by a factor of $\left(\ln\left[M_0^2/4\pi\mu^2\right]\right)^{-1}$. An analysis of the perturbative expansion shows that the terms from higher order Feynman graphs are also of this order (see [CaJaPA86] and references therein).

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In other words, by adopting expression (11.76) we obtain a nonperturbative (in powers of the coupling constant) approximation to the physical mass, which is equivalent to summing all terms of the form $\left(\lambda\hbar \ln\left[M_0^2/4\pi\mu^2\right]\right)^p$ in the perturbative expansion (the so-called *leading logs*) while leaving aside terms of the form $(\lambda\hbar)^p \left(\ln\left[M_0^2/4\pi\mu^2\right]\right)^q$ with $q < p$. In this sense, the true small parameter in our expansion is not the coupling constant, but rather $\left(\ln\left[M_0^2/4\pi\mu^2\right]\right)^{-1}$ [ArSoYa99a, ArSoYa99b].

11.1.8 Time reversal invariance

Time reversal invariance means that for *any* solution $G^{ab}(x,y)$ of the equations of motion the time-reversed expression $G^{ab}_{\text{rev}}(x,y)$ is also a solution. The form of G^{ab}_{rev} is determined by the time reversal operation appropriate to the underlying field theory. In our case, time reversal transforms an expectation value $\left\langle \Phi\left(\mathbf{x}, x^0\right) \Phi\left(\mathbf{y}, y^0\right)\right\rangle$ into $\left\langle \Phi\left(\mathbf{y}, -y^0\right) \Phi\left(\mathbf{x}, -x^0\right)\right\rangle$ (see Streater and Wightman [StrWig80] and T.D. Lee [Lee81]). If $x = \left(\mathbf{x}, x^0\right)$, write $\bar{x} = \left(\mathbf{x}, -x^0\right)$ (observe that $\bar{x}^\mu = x_\mu = \eta_{\mu\nu}x^\nu$, where $\eta_{\mu\nu}$ is the Minkowski metric); therefore

$$G^{21}_{\text{rev}}(x,y) = G^{21}(\bar{y},\bar{x}) \tag{11.79}$$

$$G^{12}_{\text{rev}}(x,y) = G^{12}(\bar{y},\bar{x}) \tag{11.80}$$

For the Feynman propagator, we have

$$\begin{aligned} G^{11}_{\text{rev}}(x,y) &= \theta\left(x^0 - y^0\right) G^{21}_{\text{rev}}(x,y) + \theta\left(y^0 - x^0\right) G^{12}_{\text{rev}}(x,y) \\ &= \theta\left(-y^0 - \left(-x^0\right)\right) G^{21}(\bar{y},\bar{x}) + \theta\left(-x^0 - \left(-y^0\right)\right) G^{12}(\bar{y},\bar{x}) \\ &= G^{11}(\bar{y},\bar{x}) \end{aligned} \tag{11.81}$$

Similarly,

$$G^{22}_{\text{rev}}(x,y) = G^{22}(\bar{y},\bar{x}) \tag{11.82}$$

These formulae are summarized by

$$G^{ab}_{\text{rev}}(x,y) = G^{ab}(\bar{y},\bar{x}) \tag{11.83}$$

It is convenient to introduce the notation: for any kernel $A(x,y)$, we define the kernel $\bar{A}(x,y) = A(\bar{y},\bar{x})$. Therefore the time reversal operation means changing G^{ab} into $\bar{G}^{ab}$. Observe that a spherically symmetric translation invariant solution is automatically a fixed point under time reversal.

In terms of the partial Fourier transform we get for any kernel A that

$$\bar{A}(X,p) = A\left(\bar{X}, -\bar{p}\right) \tag{11.84}$$

and as a consequence the first-order terms on the left-hand side of the kinetic equations (11.48) change sign (there is always one derivative that does), while the right-hand side $I_{\text{col}}\text{sign}\left(k^0\right)$ does not. So equations (11.48) are *not* time reversal

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invariant, unless $I^{(0)}_{\rm col}$ vanishes (which implies a thermal solution). On the other hand, a local thermal solution cannot be a solution to first order, because then $I^{(0)}_{\rm col}$ vanishes, but just four degrees of freedom $\beta_\mu(X)$ are not enough to kill the left-hand side terms identically in p.

Since (11.48) is the result of a systematic adiabatic expansion of the original 2PI SD equations, and the expansion itself would not break time reversal invariance, we conclude that the equations derived from the 2PIEA, unlike the Heisenberg equations themselves, break time reversal symmetry. This is to be expected, since these equations result from the slaving of higher correlations to the two-point functions [IvKnVo99, CalHu00].

However, we must not conclude that the observed thermalization in solutions of the evolution equations derived from the 2PIEA [Ber02, Ber04b, JuCaGr04, ArSmTr05] is an artifact of the approach. Thermalization is also observed in classical field theories, where the wave equation is directly solved [BoDeVe04]. We shall discuss this important issue in Chapter 12.

11.1.9 The limits of the kinetic approach

The derivation of the kinetic equations in this chapter is important for both practical and fundamental reasons. The fact that it can be done, as we have seen, already shows that the equations derived from the 2PIEA are not time reversal invariant. We will see in the next chapter that the Kadanoff–Baym equations play an important role in the derivation of the transport coefficients for a quantum field, and that kinetic equations may be used to describe an important stage in the thermalization process.

However, whether the kinetic equations, and more generally the adiabatic approximation, are quantitatively accurate, is a difficult issue and should not be taken lightly. We present an example, taken from [Mro97], which clearly displays the dangers at hand.

Consider a simple free Klein–Gordon field. The Heisenberg equations may be solved exactly, and the field decomposed in creation and destruction operators:

$$\Phi = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{e^{i\mathbf{kx}}}{\sqrt{2\omega_k}} \left\{ a_{\mathbf{k}} e^{-i\omega_k t} + a^\dagger_{\mathbf{k}} e^{i\omega_k t} \right\} \tag{11.85}$$

where $\omega_k^2 = k^2 + M^2$. Thus we may write any correlation function in terms of the expectation values of products of $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$. Let us assume for simplicity a spatially homogeneous and isotropic state, so that

$$\left\langle a^\dagger_{\mathbf{p}} a_{\mathbf{q}} \right\rangle = f_p \delta(\mathbf{p} - \mathbf{q}) \tag{11.86}$$

$$\left\langle a_{\mathbf{p}} a_{\mathbf{q}} \right\rangle = \left\langle a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} \right\rangle^* = g_p \delta(\mathbf{p} + \mathbf{q}) \tag{11.87}$$

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The Jordan propagator is of course state independent and translation invariant. The Hadamard propagator is

$$G_1(x,x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{\omega_k} \left\{ [1+2f_k] \cos\omega_k (t-t') + g_k e^{-2i\omega_k T} + g_k^* e^{2i\omega_k T} \right\} \tag{11.88}$$

where $T = (t+t')/2$. Observe that no nontrivial choice of the g_k makes this almost translation invariant. The propagator is either *exactly* translation invariant (if all $g_k = 0$) or else strongly T dependent; in particular, no modification of the distribution function f in a neighborhood of the mass-shell may account for the g_k terms.

We find the kinetic theory formalism is of no help in this problem, except in the case where it is unnecessary, since the state is time independent.

11.2 Quantum kinetic field theory on nontrivial backgrounds

11.2.1 The scalar Wigner function in scalar quantum electrodynamics (SQED)

The application of quantum kinetic field theory methods to fields defined on nontrivial backgrounds (both abelian and non-abelian gauge fields and gravitational backgrounds) presents special features which are not found in the general formulation presented above. We shall now discuss some of these characteristics.

For simplicity we shall concentrate on the basic issues of how to define a Wigner transform on a nontrivial background, the nature of the object so introduced and the "transport" part of the kinetic equation. Once these difficulties are overcome, the construction of the "collision" term of the kinetic equation follows the general guidelines presented above. For the remainder of this Chapter we set $\hbar = 1$.

The first difficulty encountered in applying the formalism of quantum kinetic field theory to a scalar field on an electromagnetic background is also the most obvious. Quantum kinetic theory assumes the two-point functions of the theory are nearly translation invariant. But this is not a gauge-invariant statement. For example, the Green function for a charged scalar field

$$G_1(x,x') = \left\langle \left\{ \hat{\phi}(x), \hat{\phi}^\dagger(x') \right\} \right\rangle \tag{11.89}$$

becomes

$$G_1(x,x') \to e^{i\left\{\varepsilon(x)-\varepsilon(x')\right\}} G_1(x,x') \tag{11.90}$$

under a gauge transformation. It is clear that a nearly translation-invariant kernel in one gauge may seem to be arbitrarily far from translation invariance in some other gauge.

As we shall show, it is possible nevertheless to associate a "Wigner function" with the propagators under a well-defined gauge transformation law. For

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an abelian theory, the Wigner function is actually gauge invariant; in the non-abelian case, it transforms as an element of the adjoint representation.

The price to be paid is to relinquish the identification of the Wigner function as the partial Fourier transform of a propagator. That relationship will hold only if both the propagator and the Wigner function are expressed in the Fock–Schwinger gauge to be introduced momentarily [Foc37, Sch70, Jac02]. Observe that this is not a very original procedure: it is the same logic by which one specifies the temperature of a fluid by identifying the frame in which it should be measured, namely the rest frame.

Concretely, let X be a spacetime event at which we want to define the Wigner function $F(X,p)$ for the charged scalar field. We want to specify a gauge in which the gauge aspects of the background are suppressed as much as possible. As a start, we demand $A_\mu^{(X)}(X)=0$, where $A_\mu^{(X)}$ is the background abelian gauge field in the special gauge around X. We cannot remove all the derivatives of the abelian field by gauge transformations (unless the field is trivial to begin with), but we can and will set to zero the symmetric combination

$$A^{(X)}_{(\mu,\nu)}=\frac{1}{2}\left[A^{(X)}_{\mu,\nu}+A^{(X)}_{\nu,\mu}\right] \tag{11.91}$$

In general, we define

$$A_{(\mu,\nu_1\ldots\nu_n)}=\frac{1}{n+1}\left[A_{\mu,\nu_1\ldots\nu_n}+\sum_{i=1}^{n}A_{\nu_i,\mu\overline{\nu_1\ldots\nu_n}}\right] \tag{11.92}$$

where the overbar means that ν_i is omitted. Then the Fock–Schwinger gauge is defined by the conditions

$$A^{(X)}_{(\mu,\nu_1\ldots\nu_n)}(X)=0 \tag{11.93}$$

Without loss of generality we may take $X=0$. The above equation (11.93) reduces to

$$u^\mu A^{(0)}_\mu(tu)=0 \tag{11.94}$$

where t is just a parameter, unrelated to time. We may now define the Wigner function as

$$F(0,p)=\int d^4u\, e^{-ipu}G_1^{(0)}(u/2,-u/2) \tag{11.95}$$

where

$$G_1^{(0)}(x,x')=\left\langle\left\{\hat\phi^{(0)}(x),\hat\phi^{\dagger(0)}(x')\right\}\right\rangle \tag{11.96}$$

is the Hadamard propagator in the Fock–Schwinger gauge. On the other hand, suppose the background field in the gauge we happen to be working in (which we shall refer to as "the gauge," for short) is A_μ. There must exist a gauge parameter $\epsilon^{(0)}(x)$ such that

$$A^{(0)}_\mu(u)=A_\mu(u)+\frac{\partial}{\partial u^\mu}\varepsilon^{(0)} \tag{11.97}$$

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The Fock–Schwinger gauge condition (11.94) becomes an equation for $\epsilon^{(0)}$

$$\frac{d}{dt}\varepsilon^{(0)}(tu) = -u^{\mu}A_{\mu}(tu) \tag{11.98}$$

with solution

$$\varepsilon^{(0)}(u) = \varepsilon^{(0)}(0) - \int_0^1 dt\, u^{\mu}A_{\mu}(tu) \tag{11.99}$$

By the same token

$$G_1^{(0)}(u/2,-u/2) = G_1(u/2,-u/2)\exp\left\{\left(-\frac{i}{2}\right)\int_{-1}^{1} dt\, u^{\mu}A_{\mu}\left(\frac{tu}{2}\right)\right\} \tag{11.100}$$

Performing a simultaneous gauge transformation of G_1 and A_μ in (11.100), we see that $G_1^{(0)}$, and therefore also F, are gauge invariant. Also, observe that the constant of integration in the gauge parameter (11.99) drops out. In the non-abelian case, the constant of integration matters, and the Wigner function will be merely gauge covariant, rather than invariant.

The next step is to invert (11.95), that is, to express the Hadamard propagator in terms of the Wigner function. Let x and x' be the points at which we want to evaluate the Hadamard propagator. Let $X(x,x')$ be the midpoint and $u(x,x')$ the relative variable

$$X^{\mu} = \frac{1}{2}(x^{\mu}+x'^{\mu}); \quad u^{\mu} = x^{\mu}-x'^{\mu} \tag{11.101}$$

Then

$$G_1(x,x') = \exp\left\{\left(\frac{i}{2}\right)\int_{-1}^{1} dt\, u^{\mu}A_{\mu}\left(X+\frac{tu}{2}\right)\right\}\int \frac{d^4p}{(2\pi)^4}\, e^{ipu}\, F(X,p) \tag{11.102}$$

The transport equation

We shall use (11.102) to obtain the transport equation for the Wigner function F. Observe that since we already know F is a gauge-invariant object, the transport equation we are looking for must be gauge invariant. This observation will be useful in our search.

If we disregard scalar field self-interactions, the field operators obey the Heisenberg equations

$$\left[D^{\mu}D_{\mu} - m^2\right]\hat{\phi} = 0 \tag{11.103}$$

where D is the covariant derivative

$$D_{\mu} = \partial_{\mu} - iA_{\mu} \tag{11.104}$$

For a non-self-interacting theory the Hadamard propagator obeys the same equation. From (11.102) we find

$$D_{\mu}G_1 = e^{\left(\frac{i}{2}\right)\int_{-1}^{1} dt\, u^{\mu}A_{\mu}\left(X+\frac{tu}{2}\right)}\mathbf{D}_{\mu} \tag{11.105}$$

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where

$$\mathbf{D}_\mu = \int \frac{d^4p}{(2\pi)^4} e^{ipu} \left\{ ip_\mu + \frac{1}{2}\frac{\partial}{\partial X^\mu} + i\mathcal{P}_\mu \right\} F(X,p) \tag{11.106}$$

$$\mathcal{P}_\mu = \frac{1}{2}\int_{-1}^{1} dt \left[A_\mu\left(X + \frac{tu}{2}\right) + \frac{1+t}{2} u^\lambda A_{\lambda,\mu}\left(X + \frac{tu}{2}\right)\right] - A_\mu\left(X + \frac{u}{2}\right) \tag{11.107}$$

We now make a crucial observation. The object $\mathbf{D}_\mu$ defined in (11.106) is gauge invariant, so we may evaluate it in any gauge, and in particular in the Fock–Schwinger gauge around X. Similarly, now that all derivatives have been made explicit, there is no harm done if we set $X = 0$. We add the assumption that the background field tensor is slowly varying, so we may approximate the background field (in the special gauge) by its Taylor expansion. Up to two derivatives, we get

$$A_\nu^{(0)}\left(\frac{u}{2}\right) = \frac{u^\lambda}{4} F_{\lambda\nu}(0) + \frac{u^\lambda u^\rho}{24}\left[F_{\lambda\nu,\rho} + F_{\rho\nu,\lambda}\right](0) + \dots \tag{11.108}$$

$$\mathbf{D}_\mu = \int \frac{d^4p}{(2\pi)^4} e^{ipu} \left\{ ip_\mu + \frac{1}{2}\left[F_{\lambda\mu}\frac{\partial}{\partial p_\lambda} + \frac{\partial}{\partial X^\mu}\right]\right\} F(X,p) \tag{11.109}$$

Since this expression is gauge invariant, it holds in any gauge.

We now observe that $D_\mu G_1$ has the same structure as G_1 itself, namely, with the factor

$$e^{\left(\frac{i}{2}\right)\int_{-1}^{1} dt\, u^\mu A_\mu\left(X + \frac{tu}{2}\right)} \tag{11.110}$$

multiplying a Fourier integral over momentum space of a gauge-invariant quantity. Therefore, we immediately find

$$D^\mu D_\mu G_1 = e^{\left(\frac{i}{2}\right)\int_{-1}^{1} dt\, u^\mu A_\mu\left(X + \frac{tu}{2}\right)} \mathbf{D}^2 \tag{11.111}$$

for some operator $\mathbf{D}^2$. This operator, like $\mathbf{D}_\mu$, has both real and imaginary parts. The former, together with the mass term in the Klein–Gordon equation, give rise to the mass-shell constraint of the theory, while the latter yields the transport equation

$$p^\mu\left[\frac{\partial}{\partial X^\mu} + F_{\lambda\mu}\frac{\partial}{\partial p_\lambda}\right] F(X,p) = 0 \tag{11.112}$$

The transport equation describes the evolution of a swarm of particles acted upon by the Lorentz force. A similar calculation yields the conserved current

$$j^\mu(x) = \left(\frac{-i}{2}\right)\left\{\left[D_x^\mu - D_{x'}^\mu\right] G_1(x,x')\right\}_{x=x'} \tag{11.113}$$

Using our previous results for the covariant derivatives we get

$$j^\mu(X) = \int \frac{d^4p}{(2\pi)^4} p^\mu F(X,p) \tag{11.114}$$

whose conservation follows from the transport equation.

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We see that both the transport equation and the conserved current agree (to this order) with the corresponding expressions in classical kinetic theory, while the mass-shell condition begins to show traces of nonlocality.

11.2.2 Scalar Wigner functions on non-abelian backgrounds

We now consider the case in which the scalar field forms a multiplet minimally coupled to a non-abelian gauge field. Observe that now the Hadamard propagator carries group indices, transforming as $\hat{\phi}$ at x and as $\hat{\phi}^{+}$ at x'.

The first issue we must confront is whether the Fock–Schwinger gauge condition (11.94) can be realized [Cro80]. We now have

$$A_{\mu}^{(0)}(u) = g\left[A_{\mu}(u) - ig^{-1}\frac{\partial g}{\partial u^{\mu}}\right]g^{-1} \tag{11.115}$$

where $g = \exp\{i\epsilon\}$ is a group element and $\epsilon = \epsilon^{A}T_{A}$ belongs to the group algebra. If we impose the condition (11.94) we get

$$\frac{d}{dt}g(tu) = -ig\, u^{\mu}A_{\mu}(tu) \tag{11.116}$$

whose solution is

$$g(u) = g(0)\,\tilde{T}\left[e^{-i\int_0^1 dt\, u^{\mu}A_{\mu}(tu)}\right] \tag{11.117}$$

where the operator $\tilde{T}$ anti-orders with respect to the parameter t. Recall that $A_{\mu} = A_{\mu}^{A}T_{A}$ are matrices, so they may not commute at different values of t.

This shows that the Fock–Schwinger gauge exists. However, the constant of integration is no longer irrelevant. When we express the Hadamard propagator in terms of the propagator in the Fock–Schwinger gauge we find

$$G_1 = T\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\, u^{\mu}A_{\mu}\left(X+\frac{tu}{2}\right)}\right]g^{-1}(0)\,G_1^{(0,g)}g(0)\,\tilde{T}\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\, u^{\mu}A_{\mu}\left(X-\frac{tu}{2}\right)}\right] \tag{11.118}$$

To obtain the right gauge properties for G_1 we must assume $g^{-1}(0)\,G_1^{(0,g)}g(0) = G_1^{(0)}$ is independent of g. This means that $G_1^{(0)}$ transforms as an element of the adjoint representation at X.

We adopt the same definition (11.95) for the Wigner function as in the abelian case; now F is an element of the adjoint representation. The inverse relationship reads

$$G_1 = T\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\, u^{\mu}A_{\mu}\left(X+\frac{tu}{2}\right)}\right]\int\frac{d^4p}{(2\pi)^4}\,e^{ipu}\,F(X,p)\,\tilde{T}\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\, u^{\mu}A_{\mu}\left(X-\frac{tu}{2}\right)}\right] \tag{11.119}$$

To compute the transport equation, observe that

$$\begin{aligned}&\frac{\partial}{\partial X^{\lambda}}T\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\, u^{\mu}A_{\mu}\left(X+\frac{tu}{2}\right)}\right]\\ &\quad= T\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\, u^{\mu}A_{\mu}\left(X+\frac{tu}{2}\right)}\right]\left(\frac{i}{2}\right)\int_0^1 dt\, u^{\mu}\hat{A}_{\mu,\lambda}\left(X+\frac{tu}{2}\right)\end{aligned} \tag{11.120}$$

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where

$$\hat{A}_{\mu,\lambda}\left(X+\frac{tu}{2}\right)=\tilde{T}\left[e^{\left(\frac{-i}{2}\right)\int_0^t dr\,u^\mu A_\mu\left(X+\frac{ru}{2}\right)}\right]\times A_{\mu,\lambda}\left(X+\frac{tu}{2}\right)T\left[e^{\left(\frac{i}{2}\right)\int_0^t dr\,u^\mu A_\mu\left(X+\frac{ru}{2}\right)}\right] \tag{11.121}$$

also

$$A_\lambda\left(X+\frac{u}{2}\right)T\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\,u^\mu A_\mu\left(X+\frac{tu}{2}\right)}\right]=T\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\,u^\mu A_\mu\left(X+\frac{tu}{2}\right)}\right]\hat{A}_\lambda\left(X+\frac{u}{2}\right) \tag{11.122}$$

and so

$$D_\mu G_1=T\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\,u^\mu A_\mu\left(X+\frac{tu}{2}\right)}\right]\mathcal{D}_\mu\tilde{T}\left[e^{\left(\frac{i}{2}\right)\int_0^1 dt\,u^\mu A_\mu\left(X-\frac{tu}{2}\right)}\right] \tag{11.123}$$

where

$$\begin{aligned}\mathcal{D}_\mu=\int\frac{d^4p}{(2\pi)^4}\,e^{ipu}\,\Bigg\{&ip^\mu F+\frac{1}{2}\frac{\partial}{\partial X^\mu}F(X,p)-i\hat{A}_\mu\left(X+\frac{u}{2}\right)F(X,p)\\&+\left(\frac{i}{2}\right)\int_0^1 dt\left[\hat{A}_\mu\left(X+\frac{tu}{2}\right)+\frac{u^\rho}{2}(1+t)\hat{A}_{\rho,\mu}\left(X+\frac{tu}{2}\right)\right]F(X,p)\\&+\left(\frac{i}{2}\right)\int_0^1 dt\left[\hat{A}_\mu\left(X-\frac{tu}{2}\right)+\frac{u^\rho}{2}(1-t)\,\hat{A}_{\rho,\mu}\left(X-\frac{tu}{2}\right)\right]F(X,p)\Bigg\}\end{aligned} \tag{11.124}$$

Since $\mathcal{D}_\mu$ has definite gauge transformation properties (it belongs to the adjoint representation) it is enough to evaluate it in the Fock–Schwinger gauge, where $\hat{A}=A$. Moreover, we replace the background fields by their Taylor expansion around $X=0$, which, since $A^{(0)}(0)=0$, is formally identical to the expansion (11.108), taking into account that now the field tensor is a matrix. In the same way that in the abelian case the covariant derivative of the propagator decomposes into real and imaginary parts, here the covariant derivative is the sum of Hermitian and anti-Hermitian terms

$$\mathcal{D}^{(0)}_\mu=\int\frac{d^4p}{(2\pi)^4}\,e^{ipu}\,\left\{ip^\mu F+\frac{1}{2}\frac{\partial}{\partial X^\mu}\,F+\frac{1}{4}\left\{F_{\lambda\mu},\frac{\partial F}{\partial p_\lambda}\right\}+\frac{1}{8}\left[F_{\lambda\mu},\frac{\partial F}{\partial p_\lambda}\right]\right\} \tag{11.125}$$

This expression is valid in the Fock–Schwinger gauge. To obtain the corresponding expression in an arbitrary gauge we must replace $\partial F/\partial X^\mu$ by the covariant derivative for an element of the adjoint representation

$$D_\mu F=\frac{\partial F}{\partial X^\mu}-i\,[A_\mu,F] \tag{11.126}$$

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The double covariant derivative $D_\mu D^\mu$ in the wave equation may be analyzed in the same terms. Its anti-Hermitian part gives rise to the transport equation

$$0 = p^\mu \left(D_\mu F + \frac{1}{2} \left\{ F_{\lambda\mu}, \frac{\partial F}{\partial p_\lambda} \right\} \right) - \frac{i}{8} \left(\left[F_{\lambda\mu}, \frac{\partial^2 F}{\partial p_\lambda \partial X^\mu} \right] + \frac{1}{2} \left[F^\mu_{\lambda,\mu}, \frac{\partial F}{\partial p_\lambda} \right] + \frac{1}{4} \left[\left\{ F_{\lambda\mu}, F^\mu_\rho \right\}, \frac{\partial^2 F}{\partial p_\lambda \partial p_\rho} \right] \right) \tag{11.127}$$

The conserved current is $-i$ times the momentum integral of the anti-Hermitian part of the covariant derivative of the propagator. Discarding total derivatives it is given formally by the same expression (11.114) as in the abelian case.

Classical limit and the Wong equations

The issue of the classical limit in the kinetic theory of particles on a non-abelian background is subtler than in the abelian case, because at first sight the objects involved are of a quite different nature. In the quantum case, as we have seen, the distribution function is a Hermitian matrix $F(X,p)$ belonging to the adjoint representation of the group; in the classical case, particles carry a non-abelian charge q^A which may rotate within the group manifold, and the distribution function $f\left(X,p,q^A\right)$ is then an ordinary function with extra arguments.

One simple way of connecting these two objects is by demanding that the sequence of moments of both distributions are the same. The moments are defined as

$$M^Q_{A_1 \ldots A_n} = \mathrm{Tr}\left\{ T_{A_1} \ldots T_{A_n} F \right\} \tag{11.128}$$

in the quantum case, and as

$$m^c_{A_1 \ldots A_n} = \int dq \; q_{A_1} \ldots q_{A_n} \; f \tag{11.129}$$

where dq is the invariant measure on the group manifold. Observe that because of the group algebra only a few quantum moments are truly independent. We find no such restriction in the classical case, which underlines the difference between both approaches.

We shall carry the comparison in the "near-equilibrium" case where F is close to a diagonal matrix in color space

$$F = f^0(X,p)\, 1 + f^A(X,p)\, T_A \tag{11.130}$$

Let us assume the trace relations

$$\mathrm{Tr}\; T_A = 0; \quad \mathrm{Tr}\; T_A T_B = \frac{1}{2}\delta_{AB} \tag{11.131}$$

The first few moments are then

$$M^Q_0 = N f^0(X,p); \quad M^Q_A = \frac{1}{2} f_A(X,p) \tag{11.132}$$

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where N is the dimension of the representation. If we have the corresponding moments

$$\int dq = N; \quad \int dq\, q^A = 0; \quad \int dq\, q^A q^B = \frac{1}{2}\delta^{AB} \tag{11.133}$$

then we are led to suggest

$$f = f^0(X,p) + f^A(X,p)\, q_A \tag{11.134}$$

Our recipe meant the replacement of the identity matrix 1 by the number 1, and the group generators T_A by q_A. Assume further the multiplication table

$$T_A T_B = \frac{1}{2N} 1\delta_{AB} + \frac{1}{2}\left(K^C_{AB} + iC^C_{AB}\right) T_C \tag{11.135}$$

The C^C_{AB} are the structure constants; the K^C_{AB} vanish for $SU(2)$, but not for $SU(3)$. We insert the quantum distribution function (11.130) into the transport equation (11.127), with the added assumption that the f^A, being small, can be neglected in terms involving the field tensor. Applying our recipe of replacing generators by classical charges we obtain the classical transport equation

$$p^\mu \frac{\partial f}{\partial X^\mu} + q_C C^C_{AB}\left(p^\mu A^A_\mu\right)\frac{\partial f}{\partial q_B} + p^\mu\left(q_A F^A_{\lambda\mu}\right)\frac{\partial f}{\partial p_\lambda} = 0 \tag{11.136}$$

If we wish to interpret this as a conservation equation for the number of particles in a phase-space volume, then we must conclude that these particles move along worldlines whose tangent is proportional to p^μ, and whose momenta and charge evolve according to

$$p^\mu \frac{\partial p_\lambda}{\partial X^\mu} = \left(q_A F^A_{\lambda\mu}\right) p^\mu \tag{11.137}$$

$$p^\mu \frac{\partial q_B}{\partial X^\mu} = \left(p^\mu A^A_\mu\right) C^C_{AB} q_C \tag{11.138}$$

These are the so-called Wong equations [Won70, LitMan02], which form the basis for a classical theory of non-abelian plasmas.

In most problems of interest the back-reaction of the particles described by the distribution function on the background fields is not negligible and one must seek a self-consistent dynamical framework. One possibility is to couple the transport equation for the particle distribution function to the Yang–Mills equations for the soft part of the background fields. On general grounds [LitMan02] one expects that such an approach is reliable when the plasma parameter ϵ is small. The plasma parameter is the inverse to the number of particles within a sphere whose radius is the screening length (see Chapter 10). In a gluon plasma, for example (see below for the application of quantum kinetic theory to the gauge fields themselves), the density scales as T^3 and the screening length as $(gT)^{-1}$, so $\epsilon \approx g^3$. In this case, this scheme works for theories with weak coupling.

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11.2.3 Quantum kinetic theory in curved spacetimes

Quantum kinetic field theory in curved spacetimes has both similarities with and important differences from the transport theory in non-abelian field backgrounds. To begin with, there is one more layer of structure, because besides the Riemann tensor (to be defined momentarily), which is the natural analog to the field tensor, and the Christoffel symbols, which are the analogs to the field 4-vector, there is the metric tensor itself, which has no analog in non-abelian gauge theory. In particular, we shall carry the derivation of the transport equation up to two derivatives of the metric, which means only one derivative of the connection and no derivative of the Riemann tensor.

Let us begin by summarizing the useful definitions and conventions. The metric tensor appears in the expression of Pythagoras' theorem appropriate to the spacetime in question: the geodesic distance between two events whose coordinates differ by infinitesimal amounts dx^μ is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (we adopt the MTW conventions [MiThWh72] throughout this book). The connection appears in the covariant derivative for a contravariant vector field A^μ

$$\nabla_\nu A^\mu = A^\mu_{;\nu} = A^\mu_{,\nu} + \Gamma^\mu_{\nu\lambda} A^\lambda \tag{11.139}$$

We shall adopt the so-called Levi-Civita connection, whose components are the Christoffel symbols

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left\{ g_{\nu\rho,\lambda} + g_{\lambda\rho,\nu} - g_{\nu\lambda,\rho} \right\} \tag{11.140}$$

The Riemann tensor is the commutator of two covariant derivatives

$$\left[\nabla_\nu, \nabla_\rho \right] A^\mu = R^\mu_{\lambda\nu\rho} A^\lambda \tag{11.141}$$

It is related to the connection through

$$R^\mu_{\lambda\nu\rho} = \partial_\nu \Gamma^\mu_{\lambda\rho} - \partial_\rho \Gamma^\mu_{\lambda\nu} + \Gamma^\mu_{\nu\sigma} \Gamma^\sigma_{\lambda\rho} - \Gamma^\mu_{\rho\sigma} \Gamma^\sigma_{\lambda\nu} \tag{11.142}$$

As in our earlier discussions, start from an event P on the spacetime manifold at which we wish to define the Wigner function. We will build a special coordinate system in a neighborhood of P: the so-called Riemann normal coordinates (RNC) centered at P [Pet69] comes in handy. In this system, the coordinates of P are $X^\mu = 0$. We also perform a linear change of variables such that the metric tensor at P becomes $g_{\mu\nu} = \eta_{\mu\nu}$. We now consider a second point P' and assume there is a unique geodesic joining P and P' (we say P' belongs to a normal neighborhood of P). Moreover we parameterize this geodesic as $P'(t)$, such that $P'(0) = P$ and $P'(1) = P'$. We define the RNC of P' as the components u^μ of the tangent vector to this geodesic at $t = 0$. Observe that $\eta_{\mu\nu} u^\mu u^\nu$ gives the geodesic distance $\sigma(P', P)$ between P and P'.

In RNC the line tu^μ is by definition a geodesic. Substituting it into the geodesic equation we obtain the identity

$$u^\nu u^\rho \Gamma^\mu_{\nu\rho}(tu) = 0 \tag{11.143}$$

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This allows us to express the connection in terms of the Riemann tensor

$$\Gamma^{\mu}_{\nu\rho}(u) = \frac{1}{3}\left[R^{\mu}_{\nu\lambda\rho} + R^{\mu}_{\rho\lambda\nu}\right] u^{\lambda} + \dots \tag{11.144}$$

and similarly for the metric

$$g_{\mu\lambda} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\nu\lambda\rho} u^{\nu} u^{\rho} + \dots \tag{11.145}$$

Now we define the Wigner function $F(P,p)$ by demanding that in RNC the Hadamard propagator evaluated at opposite points may be represented as

$$G_1\left(\frac{u}{2}, -\frac{u}{2}\right) = \frac{\Delta^{1/2}_{\mathrm{VM}}\left(\frac{u}{2}, -\frac{u}{2}\right)}{\sqrt{-g(P)}} K(P,u) \tag{11.146}$$

where

$$K(P,u) = \int \frac{d^4p}{(2\pi)^4}\, e^{ipu}\, F(P,p) \tag{11.147}$$

and Δ_{VM} is the Van Vleck–Morette determinant [Vle28, Mor51]

$$\Delta_{\mathrm{VM}}(x,x') = \frac{1}{16\sqrt{-g(x)}\sqrt{-g(x')}} \det\left[\frac{\partial^2 \sigma(x,x')}{\partial x \partial x'}\right] \tag{11.148}$$

Δ_{VM} is a biscalar, that is, a scalar both at x and x'. It is included so that the lowest order adiabatic expansion of the propagator agrees with its WKB approximation.

The factor $\sqrt{-g(P)} = 1$, but we have made it explicit for the following reason. The Hadamard propagator is a biscalar. If we make a coordinate transformation from coordinates x to coordinates x', then u transforms as a contravariant vector at P. To make the product pu a scalar, p must transform as a covariant vector at P, and in this case, $d^4p/\sqrt{-g(P)}$ is the invariant measure. So we get the right transformation properties, provided $F(P,p_\mu)$ transforms into

$$F'\left(P, p'_{\mu}\right) = F\left(P, \frac{\partial x'^{\lambda}}{\partial x^{\mu}} p'_{\lambda}\right) \tag{11.149}$$

The representation (11.146) may be generalized to the case when the propagator is evaluated at two arbitrary points. Consider three points x, y and z in a normal neighborhood of P, and let $x(s)$ be the geodesic going from $x(0) = z$ to $x(1) = x$. Then in an adiabatic expansion we have $x(s) = xs + z(1-s) + \xi(s)$. Plug this into the geodesic equation to get

$$\begin{aligned}\frac{d^2\xi^{\mu}}{ds^2} &= \frac{-1}{3}\left[R^{\mu}_{\nu\lambda\rho} + R^{\mu}_{\rho\lambda\nu}\right](xs + z(1-s))^{\lambda}(x-z)^{\nu}(x-z)^{\rho} + \dots \\ &= \frac{-z^{\lambda}}{3}\left[R^{\mu}_{\nu\lambda\rho} + R^{\mu}_{\rho\lambda\nu}\right](x-z)^{\nu}(x-z)^{\rho}\end{aligned} \tag{11.150}$$

$$x^{\mu}(s) = x^{\mu}s + z^{\mu}(1-s) + \frac{z^{\lambda}}{6}\left[R^{\mu}_{\nu\lambda\rho} + R^{\mu}_{\rho\lambda\nu}\right](x-z)^{\nu}(x-z)^{\rho}\, s(1-s) \tag{11.151}$$

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The tangent at $s = 0$ is

$$t^{\mu}(x,z) = (x-z)^{\mu} + \frac{z^{\lambda}}{6}\left[R^{\mu}_{\nu\lambda\rho} + R^{\mu}_{\rho\lambda\nu}\right](x-z)^{\nu}(x-z)^{\rho} \tag{11.152}$$

To obtain the RNC of x around z we should make a linear coordinate transform so that the metric tensor at z assumes its Minkowski value. However, this last step is nonessential for obtaining the representation of the propagator, because it is compensated by a change of variables in the momentum integral and the $\sqrt{-g(P)}$ factor.

If the point $\bar{z}$ is the geodesic midpoint between x and y, then $t^{\mu}(x,\bar{z}) = -t^{\mu}(y,\bar{z})$. We get

$$\bar{z}^{\mu} = \frac{(x+y)^{\mu}}{2} + \frac{(x+y)^{\lambda}}{48}\left[R^{\mu}_{\nu\lambda\rho} + R^{\mu}_{\rho\lambda\nu}\right](x-y)^{\nu}(x-y)^{\rho} + \ldots \tag{11.153}$$

$$t^{\mu}(x,\bar{z}) = \frac{(x-y)^{\mu}}{2} + \ldots \tag{11.154}$$

and the representation of the propagator is

$$G_1(x,y) = \frac{\Delta^{1/2}_{\mathrm{VM}}(x,y)}{\sqrt{-g(\bar{z})}} K(\bar{z}, 2t^{\mu}(x,\bar{z})) \tag{11.155}$$

where K was defined in (11.147). To this adiabatic order we may approximate $\bar{z} = (x+y)/2$ within the K function.

We can now evaluate

$$\begin{aligned}\nabla^{\mu}\partial_{\mu}G_1(x,x') &= g^{\mu\nu}(x)\,\partial_{\mu}\partial_{\nu}G_1(x,y) - g^{\mu\nu}(x)\,\Gamma^{\lambda}_{\mu\nu}(x)\,\partial_{\lambda}G_1(x,y)\\ &= \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}G_1(x,y) + \frac{\Delta^{1/2}_{VM}(x,y)}{\sqrt{-g(\bar{z})}}\\ &\quad\times\left[\frac{1}{3}R^{\mu\ \nu}_{\lambda\ \rho}x^{\lambda}x^{\rho}\partial_{\mu}\partial_{\nu}K - \frac{2}{3}R^{\lambda}_{\ \sigma}x^{\sigma}\partial_{\lambda}K\right]\end{aligned} \tag{11.156}$$

Observe that

$$\partial_{\nu}G_1(x,y) = \frac{\Delta^{1/2}_{\mathrm{VM}}(x,y)}{\sqrt{-g(\bar{z})}}\left[\partial_{\nu}K + \left(\frac{1}{2}\partial_{\nu}\ln\left[\Delta_{\mathrm{VM}}(x,y)\right] - \frac{1}{2}\partial_{\nu}\ln\left[-g(\bar{z})\right]\right)K\right] \tag{11.157}$$

$$\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}G_1(x,y) = \eta^{\mu\nu}\frac{\Delta^{1/2}_{\mathrm{VM}}(x,y)}{\sqrt{-g(\bar{z})}}\mathcal{J}_{\mu\nu} \tag{11.158}$$

$$\begin{aligned}\mathcal{J}_{\mu\nu} &= \partial_{\mu}\partial_{\nu}K + \left(\partial_{\nu}\ln\left[\Delta_{\mathrm{VM}}(x,y)\right] - \partial_{\nu}\ln\left[-g(\bar{z})\right]\right)\partial_{\mu}K\\ &\quad + \frac{1}{2}\left(\partial_{\mu}\partial_{\nu}\ln\left[\Delta_{\mathrm{VM}}(x,y)\right] - \partial_{\mu}\partial_{\nu}\ln\left[-g(\bar{z})\right]\right)K\end{aligned} \tag{11.159}$$

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Now that all derivatives have been made explicit, there is no loss of generality if we specialize to the case $x = -y = u/2$. We have

$$\partial_\nu \ln\left[\Delta_{\rm VM}\left(\frac{u}{2},\frac{-u}{2}\right)\right] = \frac{1}{3}R_{\nu\sigma}u^\sigma \tag{11.160}$$

$$\partial_\nu \ln\left[-g\left(\bar{z}\right)\right] = 0 \tag{11.161}$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu \ln\left[\Delta_{\rm VM}\left(x,y\right)\right] = \frac{1}{3}R \tag{11.162a}$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu \ln\left[-g\left(\bar{z}\right)\right] = \frac{-1}{6}R \tag{11.162b}$$

$$\begin{aligned}\nabla^\mu\partial_\mu G_1\left(\frac{u}{2},\frac{-u}{2}\right) &= \frac{\Delta_{\rm VM}^{1/2}\left(\frac{u}{2},-\frac{u}{2}\right)}{\sqrt{-g\left(P\right)}}\left\{\eta^{\mu\nu}\partial_\mu\partial_\nu K + \frac{1}{4}RK + \frac{1}{12}R^{\mu\ \nu}_{\lambda\ \rho}u^\lambda u^\rho\partial_\mu\partial_\nu K\right\}\\ &= \frac{\Delta_{\rm VM}^{1/2}\left(\frac{u}{2},-\frac{u}{2}\right)}{\sqrt{-g\left(P\right)}}\int\frac{d^4p}{\left(2\pi\right)^4}\,e^{ipu}\,\left\{-p^2 + ip^\mu\frac{\partial}{\partial X^\mu}\right.\\ &\quad + \frac{\eta^{\mu\nu}}{4}\frac{\partial^2}{\partial X^\mu\partial X^\nu}\frac{1}{6}R + \frac{1}{12}R^{\mu\ \nu}_{\lambda\ \rho}p_\mu p_\nu\frac{\partial}{\partial p_\lambda}\frac{\partial}{\partial p_\rho}\\ &\quad \left.-\frac{1}{6}R^{\ \nu}_{\lambda}p_\nu\frac{\partial}{\partial p_\lambda}\right\}F\left(X,p\right)_{X=0}\end{aligned} \tag{11.163}$$

Therefore the mass-shell constraint and the transport equation, evaluated at the origin of a RNC system, read

$$\begin{aligned}&\left[-p^2 - m^2 - \left(\xi - \frac{1}{6}\right)R + \frac{1}{12}R^{\mu\ \nu}_{\lambda\ \rho}p_\mu p_\nu\frac{\partial}{\partial p_\lambda}\frac{\partial}{\partial p_\rho}\right.\\ &\quad\left. - \frac{1}{6}R^{\ \nu}_{\lambda}p_\nu\frac{\partial}{\partial p_\lambda} + \frac{\eta^{\mu\nu}}{4}\frac{\partial^2}{\partial X^\mu\partial X^\nu}\right]F = 0\end{aligned} \tag{11.164}$$

$$p^\mu\frac{\partial}{\partial X^\mu}F = 0 \tag{11.165}$$

To obtain the corresponding expressions in an arbitrary coordinate system, we must replace the ordinary derivatives by the covariant derivatives

$$\nabla_\mu F = \left[\frac{\partial}{\partial X^\mu} + \Gamma^\lambda_{\mu\rho}\,p_\lambda\frac{\partial}{\partial p_\rho}\right]F \tag{11.166}$$

$$\nabla_\nu\nabla_\mu F = \partial_\nu\nabla_\mu F - \Gamma^\lambda_{\nu\mu}\nabla_\lambda F + \Gamma^\lambda_{\nu\rho}\,p_\lambda\frac{\partial}{\partial p_\rho}\nabla_\mu F \tag{11.167}$$

At the origin of the RNC

$$\nabla_\nu\nabla_\mu F\left(X,p\right)_{X=0} = \partial_\nu\partial_\mu F + \frac{1}{3}\left[R^\lambda_{\mu\nu\rho} + R^\lambda_{\rho\nu\mu}\right]p_\lambda\frac{\partial}{\partial p_\rho} \tag{11.168}$$

so the covariant mass-shell constraint is

$$\begin{aligned}&\left[-p^2 - m^2 - \left(\xi - \frac{1}{6}\right)R + \frac{1}{12}\left[R^{\mu\ \nu}_{\lambda\ \rho}p_\mu p_\nu\frac{\partial}{\partial p_\lambda}\frac{\partial}{\partial p_\rho} - R^{\ \nu}_{\lambda}p_\nu\frac{\partial}{\partial p_\lambda}\right]\right.\\ &\quad\left. + \frac{g^{\mu\nu}}{4}\nabla_\nu\nabla_\mu\right]F = 0\end{aligned} \tag{11.169}$$

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and the covariant transport equation is

$$p^{\mu}\frac{\partial F}{\partial X^{\mu}}+p^{\mu}\Gamma^{\lambda}_{\mu\rho}\,p_{\lambda}\frac{\partial F}{\partial p_{\rho}}=0 \tag{11.170}$$

If we think of this as a classical Liouville equation, it describes particles moving along the geodesics of the background spacetime. The geodesics are parameterized by $s=\tau/m$, where m is the mass of the particles and τ is their proper time, and the 4-velocity is $u^{\mu}=p^{\mu}/m$.

Higher spin fields

We now discuss the generalization of the quantum kinetic theory for scalar fields in curved spacetimes to fields of higher spin. For concreteness, we shall discuss the case of a Dirac spinor, but the central ideas apply to fields of any spin.

To begin, the notion of a local Lorentz transformation is introduced to define spinor fields in curved spacetimes. To do this, we need a moving frame, or, in four-dimensional spacetimes, a vierbein. A vierbein is a set of four vector fields e^{μ}_{a} such that at every point

$$g_{\mu\nu}e^{\mu}_{a}e^{\nu}_{b}=\eta_{\mu\nu}\quad\text{and}\quad\eta^{ab}e^{\mu}_{a}e^{\nu}_{b}=g^{\mu\nu} \tag{11.171}$$

The components of the vierbein transform as contravariant vectors under general coordinate transformations. The vierbein changes under a local Lorentz transformation as

$$e^{\mu}_{a}\rightarrow\xi^{\mu}_{a}=\Lambda^{b}_{a}e^{\mu}_{b} \tag{11.172}$$

A Dirac spinor ψ is a set of four (world) scalar fields which transform as a spinor under the local Lorentz transformation Λ^{b}_{a}. In general the quantity obtained by taking the ordinary derivatives of a spinor field is not a spinor. We define instead the covariant derivative

$$\nabla_{\mu}\psi=\partial_{\mu}\psi-\Gamma_{\mu}\psi \tag{11.173}$$

where

$$\Gamma_{\mu}=\frac{1}{2}\Sigma^{ab}e_{a\nu}e^{\nu}_{b;\mu} \tag{11.174}$$

Σ^{ab} is the Lorentz generator appropriate to the representation to which ψ belongs. $\nabla_{\mu}\psi$ is a spinor of the same order as ψ.

The propagator $S(x,x')$ transforms as the product $\psi(x)\,\bar{\psi}(x')$. We want to express it in terms of a Wigner function F defined at the geodesic midpoint $\bar{z}$ between x and x', which transforms as $\psi(\bar{z})\,\bar{\psi}(\bar{z})$. To do this, we introduce the so-called parallel transport matrices $A(x,\bar{z})$, which transform as $\psi(x)\,\bar{\psi}(\bar{z})$, and write

$$S(x,x')=A(x,\bar{z})\,S^{(\bar{z})}(x,x')\,\overline{A(x',\bar{z})} \tag{11.175}$$

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The matrices $A(x,\bar{z})$ are parallel transported along the geodesic from $\bar{z}$ to x. In RNC around $\bar{z}$ this means

$$x^\mu A(x,0)_{;\mu} = 0 \tag{11.176}$$

with the boundary condition $A(0,0) = 1$. This equation allows us to write the parallel transport matrices in terms of the spin connection

$$A(x,0) = 1 + x^\mu \Gamma_\mu(\bar{z}) + \ldots \tag{11.177}$$

The object $S^{(\bar{z})}(x,x')$ may be treated with the methods we have used for scalar fields; indeed, it is a world biscalar, though a local bispinor at $\bar{z}$. We refer the reader to [CaHaHu88] for further details.

Higher spin fields in non-abelian theories require the combination of methods presented in all sections of this chapter. The gauge fields themselves pose a particular problem, since their transformation law is not homogeneous. In this case the simplest strategy is the so-called background field method [DeW81, Abb81, Hart93, Alx99, PesSch95]. The gauge field A^μ is split into a c-number background V^μ and a quantum fluctuation W^μ. Under a gauge transformation, V^μ transforms as a gauge field, and W^μ as a field on the adjoint representation. A gauge-fixing term

$$\frac{1}{2\alpha}\left(D_V^\mu W_\mu\right)^2 \tag{11.178}$$

is added to the action, where α is the gauge-fixing parameter and $D_V^\mu W_\nu$ is the gauge covariant derivative with connection V^μ, namely

$$D_{V\mu} W^\nu = \partial_\mu W^\nu - i\left[V_\mu, W^\nu\right] \tag{11.179}$$

The action (where we must also add the corresponding ghost terms) is invariant under joint gauge transformations of V^μ and W^μ, but non-invariant under gauge transformations of W^μ alone. This is enough to make the W propagator well defined.

The quantum field W has a homogeneous transformation law, and may be handled as any other higher spin field. In curved spacetime, of course, we would not be concerned with the world-vector W^μ but with the four world-scalars $W^a = e^a_\mu W^\mu$, which transform as a vector under local Lorentz transformations.

11.2.4 A note on the literature

For original literature on Wigner functions in gauge backgrounds we recommend Heinz [Hei83] and Winter [Win84], and for Wigner functions in curved spacetimes, Winter [Win85] and Calzetta, Habib and Hu [CaHaHu88]. These methods were elaborated by many authors; other relevant references are [ElGyVa86, Mro89, Fon94, Gei96, Gei97, Son97, BlaIan99, BlaIan02, LitMan02]. Our

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exposition on gauge backgrounds is greatly influenced by [WRSG02, WRSG03], and on curved spacetimes by [CaHaHu88].

We recommend [BirDav82] as an entry point to the literature on higher spin fields in curved spacetimes, and [ChWiDi77] for further information on geometry and analysis on group manifolds.