

TOTAL CHROMATIC NUMBER OF GRAPHS OF HIGH DEGREE

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Abstract

Using a new proof technique of the first author (by adding a new vertex to a graph and creating a total colouring of the old graph from an edge colouring of the new graph), we prove that the TCC (Total Colouring Conjecture) is true for any graph G of order n having maximum degree at least $n - 4$. These results together with some earlier results of M. Rosenfeld and N. Vijayaditya (who proved that the TCC is true for graphs having maximum degree at most 3), and A. V. Kostochka (who proved that the TCC is true for graphs having maximum degree 4) confirm that the TCC is true for graphs whose maximum degree is either very small or very big.

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1. Introduction

Throughout this paper, all graphs are finite, simple and undirected. Let G be a graph. We denote its vertex set, edge set, chromatic index and the maximum degree of its vertices by $V(G)$, $E(G)$, $\chi_1(G)$ and $\Delta(G)$ respectively. If $x \in V(G)$, we denote by $N(x)$ the neighbourhood of x and $d_G(x)$ (or simply $d(x)$) the degree of x . If $F \subseteq E(G)$, then $G - F$ is the graph obtained from G by deleting F from G . If $S \subseteq V(G)$, then $G[S]$ and $G - S$ denote the subgraphs of G induced by S and $V(G) \setminus S$ respectively. The null graph of order m is denoted by 0_m . Other terms and notation not defined in this paper can be found in [11].

A *total colouring* π of a graph G is a mapping $\pi: V(G) \cup E(G) \rightarrow \{1, 2, \dots\}$ such that

- (i) no two adjacent vertices or edges have the same image, and
- (ii) the image of each vertex x is distinct from the images of its incident edges.

The *total chromatic number* $\chi_2(G)$ of a graph G is the smallest integer k such that G has a total colouring π having image set $\{1, 2, \dots, k\}$.

From the definition of total chromatic number, it is clear that $\chi_2(G) \geq \Delta(G) + 1$. Behzad [1] and Vizing [8, 10] made the following conjecture.

TOTAL COLOURING CONJECTURE. *For any graph G , $\chi_2(G) \leq \Delta(G) + 2$.*

This conjecture was proved for complete graphs by Behzad, Chartrand and Cooper [3]; for graphs G having $\Delta(G) \leq 3$ by Rosenfeld [6] and Vijayaditya [7]; for graphs G having $\Delta(G) = 4$ by Kostochka [5]; for complete 3-partite graphs, complete balanced r -partite graphs by Rosenfeld [6]; and for complete r -partite graphs by Yap [12]. A survey on total colourings of graphs is given in a recent paper by Behzad [2]. The main result of this paper is stated in the abstract above.

We shall apply the following theorems.

THEOREM 1.1 (Rosenfeld [6], Vijayaditya [7], Kostochka [5]). *For any graph G having $\Delta(G) \leq 4$, $\chi_2(G) \leq \Delta(G) + 2$.*

(An alternate, slightly simpler proof of Theorem 1.2 for $\Delta(G) = 3$ can be found in [12].)

THEOREM 1.2 (Vizing [9]). *For any graph G having at most two vertices of maximum degree, $\chi_1(G) = \Delta(G)$.*

(This theorem follows from some results of Vizing: see [11, Theorem 3.3 and Corollary 3.6].)

THEOREM 1.3 (Chetwynd and Hilton [4]). *Let G be a connected graph of order n with three vertices of maximum degree. Then $\chi_1(G) = \Delta(G) + 1$ if and only if G has three vertices of degree $n - 1$ and the remaining vertices have degree $n - 2$ (this implies that n is odd).*

(A proof of Theorem 1.3 can also be found in [11, page 53].)

2. Proof of main results

We shall also apply the following lemmas. The first lemma requires no proof.

LEMMA 2.1. *For any subgraph H of a graph G , $\chi_2(H) \leq \chi_2(G)$.*

LEMMA 2.2. *Let G be a graph of order n and let $\Delta(G) = \Delta$. Suppose there exists $S \subseteq V(G)$ such that $G[S] = 0_r$, where $r = n - \Delta - 1$. If $G - S$ contains a matching M such that the graph G^* obtained by adding a new vertex $c^* \notin V(G)$ to $G - M$ and adding an edge joining c^* to each vertex in $G - S$ has chromatic index $\Delta + 1$, then $\chi_2(G) \leq \Delta + 2$.*

PROOF. We first note that $\Delta(G^*) = \Delta + 1$. Let π be a proper edge-colouring of G^* using the colours $1, 2, \dots, \Delta + 1$. Then we can turn π into a total colouring φ of G using the colours $1, 2, \dots, \Delta + 1, \Delta + 2$ as follows:

$$\begin{aligned} \varphi(v) &= \pi(c^*v) \text{ for any } v \in V(G - S); \\ \varphi(v) &= \Delta + 2 \text{ for any } v \in S; \\ \varphi(e) &= \pi(e) \text{ for any } e \in E(G - M); \text{ and} \\ \varphi(e) &= \Delta + 2 \text{ for any } e \in M. \end{aligned}$$

We now prove our main results.

THEOREM 2.3. *For any graph G of order n having $\Delta(G) = n - 3$, $\chi_2(G) \leq n - 1$.*

PROOF. By Lemma 2.1, we can assume that G is maximal, that is, for any two nonadjacent vertices x and y of G , either $d(x) = n - 3$ or $d(y) = n - 3$.

Suppose x and y are two nonadjacent vertices of G , and $d(x) = n - 3$. Let $H = G - \{x, y\}$, $V_1 = \{z \in V(G) \mid d(z) = n - 3\}$, and M be a matching in H such that $|V(M) \cap V_1|$ is maximum among all matchings in H .

We first prove that V_1 contains at most one M -unsaturated vertex different from x and y . Suppose otherwise. Let u and v be two M -unsaturated vertices in V_1 . Clearly $uv \notin E(G)$. By Theorem 1.1, we can assume that $\Delta(G) \geq 5$. Thus there exist at least three vertices a_1, a_2, a_3 in H such that $a_i u \in E(H)$ for $i = 1, 2, 3$. Clearly such a_i is M -saturated. Thus there exist distinct vertices b_1, b_2, b_3 in H such that $a_1 b_1, a_2 b_2, a_3 b_3 \in M$. (Note that $b_i \neq v$ for any $i = 1, 2, 3$ but b_i can be a a_j for some $j \neq i$.) If $b_i v \in E(H)$ for

some $i = 1, 2, 3$, then $M' = (M \setminus \{a_i b_i\}) \cup \{a_i u, b_i v\}$ is a matching in H such that $|V(M') \cap V_1| > |V(M) \cap V_1|$, a contradiction to our assumption. (We shall call such an argument the “ENLARGE-MATCHING ARGUMENT.”) Hence $b_i v \notin E(H)$ for all $i = 1, 2, 3$. This implies that $d(v) \leq n - 4$ (in fact, since u is also not adjacent to v in G , $d(v) \leq n - 5$), which is false. Hence V_1 contains at most one M -unsaturated vertex different from x and y .

Finally, let G^* be the graph obtained by adding a new vertex $c^* \notin V(G)$ to $G - M$, and adding an edge joining c^* to each vertex in $G - \{x, y\}$. Then $\Delta(G^*) = n - 2$ and G^* has at most two vertices of (maximum) degree $n - 2$, namely, c^* and z , where z is an M -unsaturated vertex in V_1 (if V_1 contains such an M -unsaturated vertex). Hence, by Theorem 1.2, $\chi_1(G^*) = n - 2$. Theorem 2.3 now follows from Lemma 2.2.

THEOREM 2.4. *For any graph G of order n having $\Delta(G) = n - 4$, $\chi_2(G) \leq n - 2$.*

PROOF. By Lemma 2.1, we can assume that G is maximal, that is, for any two nonadjacent vertices x and y of G , either $d(x) = n - 4$ or $d(y) = n - 4$. By Theorem 1.1, we can assume that $\Delta(G) \geq 5$, that is, $n \geq 9$.

Let $V_1 = \{v \in V(G) \mid d(v) = n - 4\}$. We first settle the case that G contains three vertices x, y and z such that $G[\{x, y, z\}] = 0_3$.

Let M be a matching in $H = G - \{x, y, z\}$ such that $|V(M) \cap V_1|$ is maximum among all matchings in H . We first prove that V_1 contains at most two M -unsaturated vertices different from x, y and z . Suppose otherwise. Let u, v and w be three M -unsaturated vertices in $V_1 - \{x, y, z\}$. Clearly $G[\{u, v, w\}] = 0_3$. Since $\Delta(G) \geq 5$, there exist at least two vertices a_1 and a_2 in H such that $a_1 u, a_2 u \in E(H)$. It is clear that a_1 and a_2 are M -saturated. Let $b_1, b_2 \in V(H)$ be such that $a_1 b_1, a_2 b_2 \in M$. (Note that $b_1, b_2 \neq v, w$, but b_i can be a_j for some $j \neq i$.) Using the “Enlarge-Matching Argument”, we can show that $b_1 v, b_2 v \notin E(H)$. Hence $b_1, b_2, u, w \notin N(v)$. This implies that $d(v) \leq n - 5$ which is false. Hence $V_1 - \{x, y, z\}$ contains at most two M -unsaturated vertices, u and v say.

Let G^* be the graph obtained by adding a new vertex $c^* \notin V(G)$ to $G - M$, and adding an edge joining c^* to each vertex in $G - \{x, y, z\}$. Then G^* contains at most three vertices of (maximum) degree $n - 3$, namely, c^*, u and v . Hence by Theorems 1.2 and 1.3, $\chi_1(G^*) = n - 3$. Thus, by Lemma 2.2, $\chi_2(G) \leq n - 2$.

From now on we assume that G does not induce 0_3 .

Let x_1, y_1 and x_2, y_2 be two pairs of nonadjacent vertices in G such that $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$, $d(x_1) = n - 4$ and $d(x_2) = n - 4$. Let M_1 be a matching

in $H_1 = G - \{x_1, y_1\}$ such that

- (1) $|V(M_1) \cap V_1|$ is maximum among all matchings in H_1 .

Using an argument similar to the proof of Theorem 2.3 we can prove that V_1 contains at most one M_1 -unsaturated vertex, say u , different from x_1 and y_1 .

Next, let $G_1 = G - M_1$ and let

$$V_2 = \begin{cases} \{v \in V(G_1) \mid d_{G_1}(v) = n - 5\} \cup \{x_1\} & \text{if } d(y_1) \leq n - 6, \\ \{v \in V(G_1) \mid d_{G_1}(v) = n - 5\} \cup \{x_1, y_1\} & \text{if } d(y_1) \geq n - 5. \end{cases}$$

Let M_2 be a matching in $H_2 = G_1 - \{x_2, y_2\}$ such that

- (2) $u \in V(M_2)$ (if H_1 contains an M_1 -unsaturated vertex u) and $|V(M_2) \cap V_2|$ is maximum among all matchings in H_2 .

(Note that here we assume that $u \neq x_2$ or y_2 . However, if $u = x_2$ say, then since G does not induce O_3 , G contains a vertex $x'_2 \neq u, x_1, y_1$, such that x'_2 and y_2 are not adjacent. Hence we can replace x_2 by x'_2 .)

We now show that V_2 contains at most one M_2 -unsaturated vertex different from x_1 and y_1 . Suppose otherwise. Let v and w be two M_2 -unsaturated vertices in $V_2 - \{x_1, y_1\}$. Clearly $vw \notin E(H_2)$. We can in fact assume that $vw \notin E(G)$, that is, $vw \notin M_1$. (Suppose $vw \in M_1$. Since $d(v) \geq 5$, there exist at least two vertices a_1 and a_2 in H_2 such that $a_1v, a_2v \in E(H_2)$. Clearly a_1 and a_2 are M_2 -saturated. Let $b_i \in V(H_2)$ be such that $a_ib_i \in M_2, i = 1, 2$. We note that $b_i \neq w$ but b_i can be a_j for $j \neq i$. By the ‘‘Enlarge-Matching Argument’’, $b_iw \notin E(H_2)$, in fact, since $vw \in M_1$, we have $b_iw \notin E(G), i = 1, 2$. It is clear that $M'_2 = (M_2 \setminus \{a_ib_i\}) \cup \{a_1v\}$ is a matching in H_2 such that $|V(M'_2) \cap V_2| \geq |V(M_2) \cap V_2|$. By (2), $|V(M'_2) \cap V_2| = |V(M_2) \cap V_2|$ and $d_{G_1}(b_1) \geq n - 5$, from which it follows that the two M'_2 -unsaturated vertices b_1, w in V_2 are not adjacent in G . We note that if $u = b_1$, then the condition (2) that $u \in V(M'_2)$ is violated. However, if this happens, we can replace a_1 and b_1 by a_2 and b_2 .)

We continue our proof by considering three cases separately.

CASE 1: $n \geq 11$. In this case there exist at least four vertices a_1, a_2, a_3, a_4 in H_2 such that $a_iv \in E(H_2), i = 1, 2, 3, 4$. Clearly each a_i is M_2 -saturated. Let $b_i \in V(H_2)$ be such that $a_ib_i \in M_2, i = 1, 2, 3, 4$. By the ‘‘Enlarge-Matching Argument’’, $b_iw \notin E(H_2)$. Hence $v, b_1, b_2, b_3, b_4 \notin N_{H_2}(w)$. This implies that $d(w) \leq n - 5$ which is false.

CASE 2: $n = 10$. In this case there exist at least three vertices a_1, a_2, a_3 in H_2 such that $a_iv \in E(H_2)$. Clearly each a_i is M_2 -saturated. Let $b_i \in V(H_2)$ be such that $a_ib_i \in M_2, i = 1, 2, 3$. By the ‘‘Enlarge-Matching Argument,’’ $b_iw \notin E(H_2), i = 1, 2, 3$. Suppose $\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\} = \emptyset$. Since $d_{H_2}(w) \geq 3$ and $v, b_1, b_2, b_3 \notin N_{H_2}(w)$, we have $a_iw \in E(H_2)$ and thus

$b_i v \notin E(H_2)$, $i = 1, 2, 3$. (Otherwise $M'_2 = (M_2 \setminus \{a_i b_i\}) \cup \{b_i v, a_i w\}$ is a matching in H_2 such that $|V(M'_2) \cap V_2| > |V(M_2) \cap V_2|$, which contradicts (2).) Hence $G[\{v, w, b_i\}] = 0_3$ for at least one $i \in \{1, 2, 3\}$ which is false. Hence, without loss of generality, we can assume that $b_1 = a_2$ and $b_2 = a_1$. Now let $\{c_1, c_2\} = V(G) \setminus \{x_2, y_2, v, w, a_1, a_2, a_3, b_3\}$. Since $d_{H_2}(w) \geq 3$ and $v, a_1, a_2, b_3 \notin N_{H_2}(w)$. we have $a_3 w, c_1 w, c_2 w \in E(H)$. Hence $c_1 c_2 \in M_2$ which in turn implies that $c_1 v, c_2 v \notin E(H_2)$. Also it is clear that $b_3 v \notin E(H_2)$. The partial subgraph of H_2 is depicted in Figure 1(a) in which dotted lines indicate nonadjacency in H_2 . Now if $b_3 a_1 \in E(H_2)$, then $\{b_3 a_1, a_2 v, a_3 w, c_1 c_2\}$ is a perfect matching in H_2 , contradicting our assumption. Hence $b_3 a_1 \notin E(H_2)$. Similarly, $b_3 a_2, b_3 c_1, b_3 c_2 \notin E(H_2)$. Next, since $H_2[\{v, w, b_3\}] = 0_3$, and G does not induce 0_3 , we have either $b_3 w \in M_1$ or $b_3 v \in M_1$. Suppose $b_3 w \in M_1$. Since $H_2[\{v, b_3, c_1\}] = 0_3$ and G does not induce 0_3 , we have $c_1 v \in M_1$. Consequently $G[\{v, b_3, c_2\}] = 0_3$ (see Figure 1(b); the wavy lines denote edges in M_1), contradicting our assumption. On the other hand if $b_3 v \in M_1$, then by a similar argument, we have either $G[\{w, b_3, a_1\}] = 0_3$ or $G[\{w, b_3, a_2\}] = 0_3$, again contradicting our assumption.

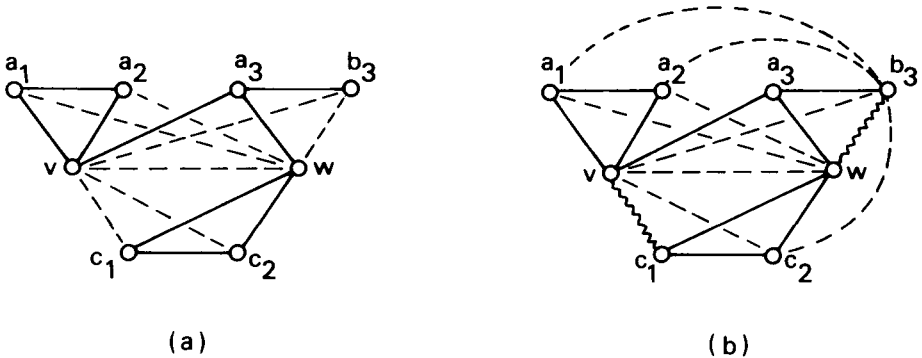


FIGURE 1

CASE 3: $n = 9$. In this case there exist at least two vertices a_1, a_2 in H_2 such that $a_1 v, a_2 v \in E(H_2)$. Clearly a_1 and a_2 are M_2 -saturated. Let $b_1, b_2 \in V(H_2)$ be such that $a_1 b_1, a_2 b_2 \in M_2$. By the ‘Enlarge-Matching Argument’, $b_1 w, b_2 w \notin E(H_2)$. Suppose $\{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset$. Since $d_{H_2}(w) \geq 2$ and $v, b_1, b_2 \notin N_{H_2}(w)$, we have either $a_1 w \in E(H_2)$ or $a_2 w \in E(H_2)$. Without loss of generality, we assume that $a_1 w \in E(H_2)$. Then by the ‘Enlarge-Matching Argument’, $b_1 v \notin E(H_2)$. Hence $H_2[\{v, w, b_1\}] = 0_3$. Since G does not induce 0_3 , we have either $b_1 w \in M_1$ or $b_1 v \in M_1$. If $b_1 w \in M_1$, then

$b_2v \in E(H_2)$ because $G[\{v, w, b_2\}] \neq O_3$. Hence by the “Enlarge-Matching Argument”, $a_2w \notin E(H_2)$. Let c be the remaining vertex in H_2 . Since $d_{H_2}(w) \geq 2$, $wc \in E(H_2)$. Hence $M'_2 = \{wc, a_1b_1, a_2b_2\}$ forms a matching in H_2 , and v is the only M'_2 -unsaturated vertex in H_2 , contradicting our assumption. On the other hand if $\{a_1, a_2\} \cap \{b_1, b_2\} \neq \emptyset$, then $b_1 = a_2$ and $b_2 = a_1$. By symmetry, H_2 has two vertices c_1, c_2 such that $c_1c_2 \in M_2$ and $wc_1, wc_2 \in E(H_2)$. By the “Enlarge-Matching Argument,” $a_1, a_2 \notin N_{H_2}(w)$, $c_1, c_2 \notin N_{H_2}(v)$. Let c be the remaining vertex in H_2 . Then $vc, wc \notin E(H_2)$ and since G does not induce O_3 , either $vc \in M_1$ or $wc \in M_1$. Without loss of generality, assume that $vc \in M_1$. Since $a_1, a_2 \notin N_{H_2}(w)$, at least one of a_1 and a_2 is not adjacent to w in G , say $wa_1 \notin E(G)$; then $ca_1 \in E(G)$, otherwise $G[\{a_1, c, w\}] = O_3$. Hence $M'_2 = \{ca_1, va_2, c_1c_2\}$ forms a matching in H_2 , and w is the only M'_2 -unsaturated vertex in H_2 , again contradicting our assumption.

Finally, let G^* be the graph obtained by adding a new vertex $c^* \notin V(G)$ to $G - M_1 - M_2$ and adding an edge joining c^* to each vertex in $G - \{x_1, y_1, x_2, y_2\}$. Then $\Delta(G^*) = n - 4$ and G^* contains at most three vertices of (maximum) degree $n - 4$, namely, c^* , u (an M_1 -unsaturated vertex) and v (an M_2 -unsaturated vertex). Consequently, by Theorems 1.2 and 1.3, $\chi_1(G^*) = n - 4$. Let π be a proper edge-colouring of G^* using the colours $1, 2, \dots, n - 4$. Then π can be turned into a total colouring φ of G using the colours $1, 2, \dots, n - 4, n - 3, n - 2$ as follows:

$$\begin{aligned} \varphi(v) &= \pi(c^*v) \text{ for any } v \in V(G - \{x_1, y_1, x_2, y_2\}); \\ \varphi(x_1) &= n - 3 = \varphi(y_1), \varphi(x_2) = n - 2 = \varphi(y_2); \\ \varphi(e) &= \pi(e) \text{ for any } e \in E(G - M_1 - M_2); \\ \varphi(e) &= n - 3 \text{ for any } e \in M_1; \text{ and} \\ \varphi(e) &= n - 2 \text{ for any } e \in M_2. \end{aligned}$$

REMARKS. Since the TCC is true for complete graphs (Behzad, Chartrand and Cooper [3]), by Lemma 2.1 it is also true for graphs G of order n having $\Delta(G) = n - 1$. Furthermore, suppose G is a graph of order n having $\Delta(G) = n - 2$. Then applying the proof technique of Theorem 2.3, we can show that $\chi_2(G) \leq n$. Combining these facts and Theorems 2.3, 2.4, we have known that the TCC is true for graphs G of order n having $\Delta(G) \geq n - 4$.

Acknowledgement

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Note Added in Proof

A. J. W. Hilton informed the first author that he and A. G. Chetwynd had also used (independently) the same new proof technique in their study of total colourings of regular graphs of high degree.

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