THE KERNELS AND CONTINUITY IDEALS OF HOMOMORPHISMS FROM $\mathcal{C}_0(\Omega)$

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Abstract

We give a description of the continuity ideals and the kernels of homomorphisms from the algebras of continuous functions on locally compact spaces into Banach algebras. We also construct families of prime ideals satisfying a certain intriguing property in the algebras of continuous functions.

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1. Introduction

Let $\theta \colon A \to B$ be a homomorphism from a commutative Banach algebra A into a Banach algebra B. The *continuity ideal* of θ is defined to be the ideal

$$\mathcal{I}(\theta) = \{a \in A \mid \text{the map } b \mapsto \theta(ab), \ A \to B, \ \text{is continuous}\};$$

this ideal contains every ideal I in A on which θ is continuous. If $A = \mathcal{C}_0(\Omega)$ for some locally compact space Ω, then θ is continuous on $\mathcal{I}(\theta)$.

The aim of this paper is to characterize the ideals that are the kernels or the continuity ideals of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras. This is, in some sense, a final piece of the picture of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras.

The study of homomorphisms from $\mathcal{C}_0(\Omega)$ started with the theorem of Kaplansky [13] that every algebra norm on $\mathcal{C}_0(\Omega)$ majorizes the uniform norm. This essentially provides a description of all the continuous homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras.

Then Bade and Curtis [1] gave a detailed structural decomposition of discontinuous homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras. The following statement of their

theorem also includes some improvements from [6, 17] (see also [3, 18], noting that other improvements of this theorem can be found in [12]); see Section 2 for notation.

THEOREM 1.1. *Suppose that Ω is a locally compact space, and that θ is a discontinuous homomorphism from $C_0(\Omega)$ into a Banach algebra B. Set $B_0 = \overline{\theta(C_0(\Omega))}$.*

(i)   *The continuity ideal $\mathcal{I}(\theta)$ is the largest ideal of $C_0(\Omega)$ on which θ is continuous.*

(ii)   *There exists a nonempty finite subset $\{p_1, \ldots, p_n\}$ of $\Omega^\flat$ such that*

$$\bigcap_{i=1}^n J_{p_i} \subseteq \mathcal{I}(\theta) \subseteq \bigcap_{i=1}^n M_{p_i}.$$

(iii) *There exists a continuous homomorphism $\mu: C_0(\Omega) \to B_0$ such that*

$$B_0 = \mu(C_0(\Omega)) \oplus \operatorname{rad} B_0, \quad \mu\left(\bigcap_{i=1}^n M_{p_i}\right) \cdot \operatorname{rad} B_0 = \{0\},$$

and $\mu = \theta$ on a dense subalgebra of $C_0(\Omega)$ containing $\mathcal{I}(\theta)$.

(iv)   *Set $\nu = \theta - \mu$. Then ν maps into $\operatorname{rad} B_0$, and the restriction of ν to $\bigcap_{i=1}^n M_{p_i}$ is a homomorphism ν' onto a dense subalgebra of $\operatorname{rad} B_0$.*

(v)   *There exist linear maps $\nu_1, \ldots, \nu_n: C_0(\Omega) \to \operatorname{rad} B_0$ such that:*

  (a) $\nu = \nu_1 + \cdots + \nu_n$;

  (b) $\nu_i' = \nu_i|_{M_{p_i}}$ *is a nonzero homomorphism, whenever $1 \le i \le n$;*

  (c) $\nu_i(C_0(\Omega)) \cdot \nu_j(C_0(\Omega)) = \{0\}$ *whenever $1 \le i \ne j \le n$.*

(vi)   *The ideals $\ker \theta$ and $\mathcal{I}(\theta)$ are always intersections of prime ideals; we have*

$$\ker \theta = \mathcal{I}(\theta) \cap \ker \mu \quad \text{and} \quad \mathcal{I}(\theta) = \ker \nu' = \bigcap_{i=1}^n \ker \nu_i'.$$

For brevity, we define a *radical homomorphism* to be a homomorphism into a radical Banach algebra. The above result points out the important roles of prime ideals and of radical homomorphisms as building blocks for general (discontinuous) homomorphisms from $C_0(\Omega)$.

In the 1970s, Dales [2] and Esterle [6–8] independently proved that, assuming the continuum hypothesis, every ideal I which is the intersection of a *finite* number of nonmodular prime ideals in $C_0(\Omega)$ and such that $|C_0(\Omega)/I| = \mathfrak{c}$ is the kernel of a radical homomorphism from $C_0(\Omega)$ (for more details see [3]).

In fact, for some spaces Ω, the kernels of radical homomorphisms from $C_0(\Omega)$ are always finite intersections of nonmodular prime ideals [6, 15]. However, in [15], we showed that for most metrizable, noncompact, locally compact spaces Ω, for example $\mathbb{R}$, there exist radical homomorphisms from $C_0(\Omega)$ whose kernels are not the intersection of any finite number of prime ideals.

In this paper, we show that the kernels of radical homomorphisms from $C_0(\Omega)$ are always intersections of (relatively) compact families of nonmodular prime

ideals (Theorem 6.7(i)). In fact, we prove a more general result for continuity ideals of homomorphisms from Banach algebras into commutative Banach algebras (Corollary 4.11).

For the converse direction, we shall prove that, assuming the continuum hypothesis, every ideal I that is the intersection of a relatively compact family $\mathfrak{P}$ of nonmodular prime ideals and is such that $|\mathcal{C}_0(\Omega)/I| = \mathfrak{c}$ and every chain in the closure of $\mathfrak{P}$ is countable (thus, in particular, when $\mathfrak{P}$ itself is countable), is the kernel of a radical homomorphism from $\mathcal{C}_0(\Omega)$. A similar result holds for continuity ideals of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras: see Section 6. We conjecture that the countability condition in these converses is redundant (see the last paragraphs of Section 7).

REMARK 1.2. It was proved by Solovay and Woodin that the existence of discontinuous homomorphisms from $\mathcal{C}_0(\Omega)$ is not a theorem of Zermelo–Fraenkel set theory with the axiom of choice (see [4] for more details). All results that require the continuum hypothesis will be marked with (CH).

In the final section, we shall construct nontrivial examples of (relatively) compact families of prime ideals in $\mathcal{C}_0(\Omega)$ for metrizable locally compact spaces Ω with infinite limit level (Theorem 7.5). This, in particular, shows the complexity of the prime ideal structure of $\mathcal{C}_0(\Omega)$ even for countable compact subspaces Ω of $\mathbb{R}$.

2. Preliminary definitions and notation

Let A be a commutative algebra. The (*conditional*) *unitization* $A^{\#}$ of A is defined as the algebra A itself if A is unital, and as A with identity adjoined otherwise. The identity of $A^{\#}$ is denoted by $\mathbf{e}_A$.

A prime ideal or semiprime ideal in A must be a proper ideal. However, we consider A itself as *the intersection of the empty collection of prime ideals*.

Let I be an ideal in A. Define the *prime radical* $\sqrt{I}$ of I to be the intersection of all the prime ideals in A containing I, so that

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

For each element $a \in A^{\#}$, define the *quotient of I by a* to be the ideal

$$I : a = \{b \in A \mid ab \in I\}.$$

Clearly $I \subseteq I : a$.

For the definition of *universal* algebras, see [3, Definition 5.7.8]. For example, the integral domain $L^1(\mathbb{R}^+, \omega)$ is universal for each radical weight ω bounded near the origin [3, Theorem 5.7.25]. Indeed, the class of universal, commutative, radical Banach algebras has been characterized in [10] (see also [3, Theorem 5.7.28]).

For a discussion of the theory of the algebras of continuous functions, see any of [3, 5, 11]. Here we just give some facts that are needed in our discussion.

Let Ω be a locally compact space; the convention is that locally compact spaces and compact spaces are Hausdorff and nonempty. The *one-point compactification* of Ω is denoted by $\Omega^\flat$. Denote by $\mathcal{C}_0(\Omega)$ the *algebra of (complex-valued) continuous functions that vanish at infinity* on Ω. Denote by $\mathcal{C}_c(\Omega)$ the subalgebra of $\mathcal{C}_0(\Omega)$ consisting of *functions with compact support*. For all $p \in \Omega$, define

$$J_p = \{f \in \mathcal{C}_0(\Omega) \mid f \text{ is zero on a neighbourhood of } p\},$$
$$M_p = \{f \in \mathcal{C}_0(\Omega) \mid f(p) = 0\}.$$

When p is the point (at infinity) adjoined to Ω to obtain $\Omega^\flat$, we also set

$$J_p = \mathcal{C}_c(\Omega) \quad \text{and} \quad M_p = \mathcal{C}_0(\Omega).$$

For each prime ideal P in $\mathcal{C}_0(\Omega)$, there exists a unique point $p \in \Omega^\flat$ such that $J_p \subseteq P \subseteq M_p$, and we say that P *is supported at the point* p. It can be seen that P is modular if and only if its support point belongs to Ω.

We shall frequently use the important fact that, for each prime ideal P in $\mathcal{C}_0(\Omega)$, the set of prime ideals containing P is a chain with respect to the inclusion relation.

For each function f continuous on Ω, the *zero set* of f is denoted by $\mathbf{Z}(f)$. The set of zero sets of continuous functions on Ω is denoted by $\mathbf{Z}[\Omega]$.

A *z-filter* $\mathcal{F}$ on Ω is a nonempty proper subset of $\mathbf{Z}[\Omega]$ such that:

(i) $Z_1 \cap Z_2$ belongs to $\mathcal{F}$ whenever both Z_1 and Z_2 belong to $\mathcal{F}$;
(ii) if $Z_1 \in \mathcal{F}$, $Z_2 \in \mathbf{Z}[\Omega]$ and $Z_1 \subseteq Z_2$, then Z_2 also belongs to $\mathcal{F}$.

Each *z-filter* $\mathcal{F}$ corresponds to an ideal

$$\{f \in \mathcal{C}(\Omega) \mid \mathbf{Z}(f) \in \mathcal{F}\},$$

denoted by $\mathbf{Z}^{-1}[\mathcal{F}]$.

Let κ be an infinite cardinal and let $\mathcal{U}$ be a free ultrafilter on κ. Define $M_\mathcal{U}$ to be the maximal ideal of $\mathbb{C}^\kappa$ consisting of all elements $f \in \mathbb{C}^\kappa$ such that $\{\sigma \in \kappa \mid f(\sigma) = 0\} \in \mathcal{U}$. The field $\mathbb{C}^\kappa / M_\mathcal{U}$ is called an *ultrapower*, and is denoted by $\mathbb{C}^\kappa / \mathcal{U}$. An element $f + M_\mathcal{U} \in \mathbb{C}^\kappa / \mathcal{U}$ is an *infinitesimal* if

$$\{\sigma \in \kappa \mid |f(\sigma)| < \varepsilon\} \in \mathcal{U} \quad \forall \varepsilon > 0.$$

The subalgebra of infinitesimals of $\mathbb{C}^\kappa / \mathcal{U}$ is denoted by $(\mathbb{C}^\kappa / \mathcal{U})^\circ$.

For a well-ordered set Λ, denote by $o(\Lambda)$ the ordinal order isomorphic to Λ.

3. Relatively compact families of prime ideals

In this section, let A be a commutative algebra.

DEFINITION 3.1 [15, Definition 3.1]. An indexed family $(P_i)_{i \in S}$ of prime ideals in A is *pseudofinite* if $a \in P_i$ for all but finitely many $i \in S$ whenever $a \in \bigcup_{i \in S} P_i$.

For a pseudofinite family $(P_i : i \in S)$ of prime ideals where S is infinite, it is obvious that $\bigcup_{i \in T} P_i = \bigcup_{i \in S} P_i$ is either a prime ideal in A or the whole of A for each infinite subset T of S.

DEFINITION 3.2. A family $\mathfrak{C}$ of prime ideals in A is *relatively compact* if every sequence of prime ideals in $\mathfrak{C}$ contains a pseudofinite subsequence. The family $\mathfrak{C}$ is *compact* if it is relatively compact and contains the union of all of its pseudofinite sequences.

Obviously, the union of finitely many pseudofinite families is relatively compact. In the rest of this section, we shall indeed relate our notion of compactness to the usual meaning of this terminology.

Denote by Π the set of prime ideals in A. For all $a_1, a_2, \ldots, a_m$, and b in A, define

$$\mathcal{U}^b_{a_1,\ldots,a_m} = \{P \in \Pi \mid \text{all } a_i \in P \text{ and } b \notin P\}.$$

Then the collection of all such sets $\mathcal{U}^b_{a_1,\ldots,a_m}$ is a base for a topology τ. Indeed, by the primeness, we have

$$\mathcal{U}^{bd}_{a_1,\ldots,a_m,c_1,\ldots,c_n} = \mathcal{U}^b_{a_1,\ldots,a_m} \cap \mathcal{U}^d_{c_1,\ldots,c_n}.$$

It is also easy to see that τ is Hausdorff. We claim that $\mathcal{U}^u_0$ is τ-compact for all $u \in A$. Indeed, we see that $\{\mathcal{U}^u_a, \mathcal{U}^a_0 \mid a \in A\}$ is a subbasis for the relative τ-topology on $\mathcal{U}^u_0$, so by Alexander's lemma, we need only show that each cover of $\mathcal{U}^u_0$ by sets in this subbasis has a finite subcover. Let E, F be subsets of A such that

$$\mathcal{U}^u_0 = \bigcup_{a \in E} \mathcal{U}^u_a \cup \bigcup_{b \in F} \mathcal{U}^b_0.$$

Set $S = \{u^m a_1 \cdots a_n \mid m, n \in \mathbb{N}, \ a_1, \ldots, a_n \in E\}$, and let I be the ideal generated by F. Assume towards a contradiction that $S \cap I = \emptyset$. Then since S is closed under multiplication, there exists a prime ideal P such that $P \supseteq I$ and $P \cap S = \emptyset$; this implies that $P \in \mathcal{U}^u_0$ but $P \notin \bigcup_{a \in E} \mathcal{U}^u_a \cup \bigcup_{b \in F} \mathcal{U}^b_0$, a contradiction. Thus $S \cap I \neq \emptyset$, so there exist $k, m, n \in \mathbb{N}, a_1, \ldots, a_m \in E, b_1, \ldots, b_n \in F$, and $c_1, \ldots, c_n \in A$ such that

$$u^k a_1 \cdots a_m = b_1 c_1 + \cdots + b_n c_n.$$

We can then deduce that

$$\mathcal{U}^u_0 = \bigcup_{i=1}^m \mathcal{U}^u_{a_i} \cup \bigcup_{j=1}^n \mathcal{U}^{b_j}_0.$$

Thus τ is locally compact.

The one-point compactification of (Π, τ) can be considered as the set $\Pi \cup \{A\}$; a basis of neighbourhood for A is given by the collection of all

$$\mathcal{U}_{a_1,\ldots,a_m} = \{A\} \cup \{P \in \Pi \mid \text{all } a_i \in P\} \quad \text{where } a_1, a_2, \ldots, a_m \in A.$$

PROPOSITION 3.3. *Let A be a commutative algebra. Denote by Π the set of prime ideals in A. Define a topology τ as above. Then (Π, τ) is a totally disconnected locally compact space, and every [relatively] compact family of prime ideals in A is a [relatively] sequentially τ-compact subset of $\Pi \cup \{A\}$.*

PROOF. It remains to prove the last assertion. We claim that a pseudofinite sequence (P_n) of prime ideals in A is τ-convergent in $\Pi \cup \{A\}$. In fact, set $P = \bigcup_{n=1}^{\infty} P_n$. Then either $P \in \Pi$ or $P = A$, and in both cases (P_n) τ-converges to P. $\qquad\square$

REMARK 3.4. If A is either unital or $\mathcal{C}_0(\Omega)$ for some locally compact space Ω, then the union $\bigcup_{n=1}^{\infty} P_n$ is in fact a prime ideal in A, for each pseudofinite sequence (P_n) of prime ideals in A. Thus in the above proposition, we can replace $\Pi \cup \{A\}$ by Π.

REMARK 3.5. Let us instead consider a topology σ on $\Pi \cup \{A\}$ generated by

$$\mathcal{U}_{a_1, \ldots, a_m}^{Q} = \{P \in \Pi \cup \{A\} \mid P \subseteq Q \text{ and all } a_i \in P\},$$

where Q is either a semiprime ideal in A or A itself, and $a_1, \ldots, a_m \in Q$. Then a sequence of prime ideals in A is pseudofinite if and only if it is convergent in $(\Pi \cup \{A\}, \sigma)$, and so a family of prime ideals in A is [relatively] compact if and only if it is [relatively] sequentially compact in $(\Pi \cup \{A\}, \sigma)$. However, in general, σ is neither Hausdorff nor locally compact. In the case where $A = \mathcal{C}_0(\Omega)$, then $(\Pi \cup \{A\}, \sigma)$ is Hausdorff (but not locally compact).

4. Homomorphisms from general commutative Banach algebras

Let $\theta : A \to B$ be a homomorphism from a commutative Banach algebra A into a Banach algebra B. Let $(a_n : n \in \mathbb{N})$ be a sequence in A. Then

$$\mathcal{I}(\theta) : a_1 a_2 \cdots a_n \subseteq \mathcal{I}(\theta) : a_1 a_2 \cdots a_{n+1} \quad \forall n \in \mathbb{N}.$$

It follows easily from the stability lemma (see [3, Corollary 5.2.7] or [18, Lemma 1.6] for the statement and proof) that there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{I}(\theta) : a_1 a_2 \cdots a_n = \mathcal{I}(\theta) : a_1 a_2 \cdots a_{n+1} \quad \forall n \geq n_0.$$

Thus $\mathcal{I}(\theta)$ is an *abstract continuity ideal* in the following sense.

DEFINITION 4.1. Let A be a commutative algebra. An ideal I is an *abstract continuity ideal* if, for each sequence (a_n) in A, there exists $n_0 \in \mathbb{N}$ such that

$$I : a_1 a_2 \cdots a_n = I : a_1 a_2 \cdots a_{n+1} \quad \forall n \geq n_0.$$

PROPOSITION 4.2. *Let $\mathfrak{P}$ be a relatively compact family of prime ideals in a commutative algebra A. Then $\bigcap_{P \in \mathfrak{P}} P$ is an abstract continuity ideal in A.*

PROOF. Set $I = \bigcap_{P \in \mathfrak{P}} P$. Assume towards a contradiction that I is not an abstract continuity ideal. Then there exists a sequence (a_n) in A such that

$$I : a_1 a_2 \cdots a_n \subset I : a_1 a_2 \cdots a_{n+1} \quad \forall n \in \mathbb{N}.$$

For all n, we see that

$$I : a_1 \cdots a_n = \bigcap_{\substack{P \in \mathfrak{P} \\ a_1 \cdots a_n \notin P}} P.$$

Thus it follows that there exists $P_n \in \mathfrak{P}$ such that $a_1 \cdots a_n \notin P_n$ but $a_1 \cdots a_{n+1} \in P_n$. The relative compactness implies that there exist $n_1, n_2, \ldots,$ such that $n_1 < n_2 < \cdots$ and (P_{n_i}) is pseudofinite. However, we see that $a_1 a_2 \cdots a_{n_2} \in P_{n_1}$, but $a_1 a_2 \cdots a_{n_2} \notin P_{n_i}$ for all $i \geq 2$; this contradicts the pseudofiniteness. □

The remainder of this section is devoted to a converse of the above proposition. Let I be an abstract continuity ideal of a commutative algebra A. Denote by $\mathfrak{P}$ the set of prime ideals of the form $I : a$ for some $a \in A$. The following is a generalization of [15, Lemma 4.3].

LEMMA 4.3. *For each cardinal $\kappa \leq |\mathfrak{P}|$, there exists a subfamily $\mathcal{G} \subseteq \mathfrak{P}$ with the properties that $|\mathcal{G}| \geq \kappa$ and that $|\{P \in \mathcal{G} \mid a \notin P\}| < \kappa$ for all $a \in \bigcup_{P \in \mathcal{G}} P$.*

PROOF. For each $a \in A \cup \{\mathbf{e}_A\}$, let $\mathfrak{P}_a$ be the set of prime ideals of the form $I : ab$ for some $b \in A$. We claim that there exists $a_0 \in A \cup \{\mathbf{e}_A\}$ such that $|\mathfrak{P}_{a_0}| \geq \kappa$ and such that either $|\mathfrak{P}_{a_0 a}| < \kappa$ or $I : a_0 a = I : a_0$, for each $a \in A$. Indeed, assume that the contrary holds. Then, since $|\mathfrak{P}_{\mathbf{e}_A}| \geq \kappa$, by induction, there exists a sequence $(a_n) \subseteq A$ such that $|\mathfrak{P}_{a_1 \cdots a_n}| \geq \kappa$ and such that

$$I : a_1 \cdots a_n \subset I : a_1 \cdots a_{n+1} \quad \forall n \in \mathbb{N}.$$

This contradicts the definition of an abstract continuity ideal. Hence the claim holds.

Put $\mathcal{G} = \mathfrak{P}_{a_0}$; this obviously satisfies $|\mathcal{G}| \geq \kappa$. Suppose that $a \in A$ and that $\mathcal{G}' = \{P \in \mathcal{G} \mid a \notin P\}$ has cardinality at least κ. Then, for each $P \in \mathcal{G}'$, we have $P : a = P$ because $a \notin P$. Thus $\mathcal{G}' \subseteq \mathfrak{P}_{a_0 a}$, and hence $|\mathfrak{P}_{a_0 a}| \geq \kappa$. Therefore, by the claim, we must have $I : a_0 a = I : a_0$. We now show that $\mathcal{G}' = \mathcal{G}$. Assume towards a contradiction that $\mathcal{G}' \neq \mathcal{G}$, and take $P \in \mathcal{G} \setminus \mathcal{G}'$, say $P = I : a_0 a_1$ for some $a_1 \in A$. Then $a_1 \in I : a_0 a = I : a_0$ since $a \in P$, so that $a_0 a_1 \in I$. This implies that $P = A$, a contradiction. This proves that $\mathcal{G}$ has the desired property. □

LEMMA 4.4. *The prime radical $\sqrt{I}$ of the abstract continuity ideal I is the intersection of the prime ideals in $\mathfrak{P}$.*

PROOF. This is based on the commutative prime kernel theorem due to Sinclair (see [3, Theorem 5.3.15] or [18, Theorem 11.4]; the proof in [15, Lemma 4.1] works almost verbatim). □

LEMMA 4.5. *Every element in $\mathfrak{P}$ contains a minimal element.*

PROOF. Assume towards a contradiction that there is a sequence $(P_n = I : f_n) \subseteq \mathfrak{P}$ such that

$$P_1 \supset P_2 \supset \cdots \supset P_n \supset \cdots.$$

For all n, choose $a_n \in A$ such that $a_n \in P_n \setminus P_{n+1}$. Then we see that $a_1 \cdots a_n f_n \in I$ but $a_1 \cdots a_n f_{n+1} \notin I$. Thus

$$I : a_1 a_2 \cdots a_n \subset I : a_1 a_2 \cdots a_{n+1} \quad \forall n \in \mathbb{N};$$

this contradicts I being an abstract continuity ideal. □

LEMMA 4.6. *Let P be in $\mathfrak{P}$. Then there exists $a \in A \setminus P$ such that $a \in Q$ for all $Q \in \mathfrak{P}$ with $Q \nsubseteq P$.*

PROOF. Assume that the contrary holds. Pick $a_1 \notin P$. Suppose that we have already picked $a_1, \ldots, a_n \notin P$. By the assumption, we can find $Q_n \in \mathfrak{P}$ such that $a_1 \ldots a_n \notin Q_n$ and $Q_n \nsubseteq P$. We can then choose $a_{n+1} \in Q_n \setminus P$. The induction can be continued. We see that $(a_n), (Q_n)$ as constructed satisfy $a_1 \cdots a_n \notin Q_n$ but $a_1 \cdots a_{n+1} \in Q_n$ for all $n \in \mathbb{N}$. Let $Q_n = I : f_n$. Then we see that $f_n \in I : a_1 a_2 \cdots a_{n+1} \setminus I : a_1 a_2 \cdots a_n$; this contradicts I being an abstract continuity ideal. □

LEMMA 4.7. *Let $\{P_\alpha \mid \alpha \in S\}$ be a subfamily of $\mathfrak{P}$. Then $\bigcap_{\alpha \in S} P_\alpha$ is also an abstract continuity ideal.*

PROOF. Assume that the contrary holds. Then there exists (a_n) such that

$$\left(\bigcap_{\alpha \in S} P_\alpha \right) : a_1 a_2 \cdots a_n \subset \left(\bigcap_{\alpha \in S} P_\alpha \right) : a_1 a_2 \cdots a_{n+1} \quad \forall n \in \mathbb{N}.$$

Choose $b_n \in A$ such that $a_1 \cdots a_n b_n \notin \bigcap_{\alpha \in S} P_\alpha$ but $a_1 \cdots a_{n+1} b_n \in \bigcap_{\alpha \in S} P_\alpha$ for all n. Then choose $\alpha_n \in S$ such that $a_1 \cdots a_n b_n \notin P_{\alpha_n}$. We have $P_{\alpha_n} = I : f_{\alpha_n}$ for some $f_{\alpha_n} \in A$. We see that $a_1 \cdots a_n b_n f_{\alpha_n} \notin I$ but $a_1 \cdots a_{n+1} b_n f_{\alpha_n} \in I$. Thus

$$I : a_1 a_2 \cdots a_n \subset I : a_1 a_2 \cdots a_{n+1} \quad \forall n \in \mathbb{N};$$

this contradicts I being an abstract continuity ideal. □

LEMMA 4.8. *Let J be a semiprime ideal in A. Let $a, b \in A$ be such that $J : a$ and $J : b$ are prime ideals. Then the following are equivalent.*

(a) $J : a \subseteq J : b$.
(b) $ab \notin J$.
(c) $J : a = J : b$.

PROOF. To show that (a) implies (b), we note that since J is semiprime and $J : b$ is a proper ideal in A, we must have $b \notin J : b$. So $b \notin J : a$, and therefore $ab \notin J$.

To show that (b) implies (c), we observe that condition (b) implies that $a \notin J : b$. Let $f \in J : a$. Then $fa \in J \subseteq J : b$, and so $f \in J : b$ by the primeness of $J : b$. Thus $J : a \subseteq J : b$. Similarly, $J : b \subseteq J : a$. □

REMARK 4.9. In the case where I is semiprime, since $\mathfrak{P}$ is the set of prime ideals of the form $I : a$, the above lemma shows that, if $P = I : a \in \mathfrak{P}$, then P is minimal in $\mathfrak{P}$, $a \in A \setminus P$, and $a \in Q$ whenever $Q \in \mathfrak{P} \setminus \{P\}$.

We can now state the main result of this section.

THEOREM 4.10. *Suppose that I is an abstract continuity ideal of a commutative algebra A. Denote by $\mathfrak{P}_0$ the set of minimal ideals among the prime ideals of the form $I : a$ for some $a \in A$. Then:*

(i) $\sqrt{I} = \bigcap_{P \in \mathfrak{P}_0} P$;

(ii) $\mathfrak{P}_0$ *is a relatively compact family of prime ideals.*

PROOF. The first assertion follows from Lemmas 4.4 and 4.5. For the second, let $(P_n) \subseteq \mathfrak{P}_0$. We can assume that P_n where $n \in \mathbb{N}$ are distinct. Set $J = \bigcap_{n=1}^{\infty} P_n$. By Lemma 4.6, there exists $a_n \in \bigcap_{i \neq n} P_i \setminus P_n$, and so $P_n = J : a_n$. Let $a \in A$ be such that $J : a$ is a prime ideal. We claim that $J : a \in \{P_n\}$. Indeed, we see that $a \notin J$, and thus $a \notin P_{n_0}$ for some n_0. So $aa_{n_0} \notin J$. By Lemma 4.8, we deduce that $J : a = P_{n_0}$. It then follows from Lemmas 4.7 and 4.3 (applied to J, with κ being the first infinite cardinal $\aleph_0$) that (P_n) must have a pseudofinite subsequence. □

COROLLARY 4.11. *Let $\theta : A \to B$ be a homomorphism from a Banach algebra A into a commutative Banach algebra B. Then $\sqrt{\mathcal{I}(\theta)}$ is the intersection of a relatively compact family of prime ideals of the form $\mathcal{I}(\theta) : a$ for some $a \in A$.*

REMARK 4.12. Since $\ker \theta \subseteq \mathcal{I}(\theta)$ and $A/\ker \theta$ is commutative, all the previous definitions and results still make sense in this case.

PROOF. It can be seen that, for all $a \in A$,

$$(\mathcal{I}(\theta)/\ker \theta) : (a + \ker \theta) = (\mathcal{I}(\theta) : a)/\ker \theta.$$

So by the stability lemma, we see that $\mathcal{I}(\theta)/\ker \theta$ is an abstract continuity ideal in $A/\ker \theta$. The rest follows from the theorem. □

COROLLARY 4.13. *Let $\theta : A \to B$ be an epimorphism from a Banach algebra A onto a commutative Banach algebra B. Then $\sqrt{\mathcal{I}(\theta)}$ is the intersection of a finite number of prime ideals of the form $\mathcal{I}(\theta) : a$ for some $a \in A$ and there exists $k \in \mathbb{N}$ such that $\sqrt{\mathcal{I}(\theta)} = \{a \in A \mid a^k \in \mathcal{I}(\theta)\}$.*

PROOF. This is proved in the same way as [15, Corollary 4.7]. □

LEMMA 4.14. *Let I be an abstract continuity ideal of $C_0(\Omega)$ for a locally compact space Ω. Then I is either a semiprime ideal or the whole of $C_0(\Omega)$.*

PROOF. The proof is the same as the proof that the continuity ideal of a discontinuous homomorphism from $C_0(\Omega)$ into a Banach algebra is semiprime ([6, 17], see also [3, Theorem 5.4.31]). □

COROLLARY 4.15. *Let Ω be a locally compact space.*

(i)   *Let I be an abstract continuity ideal in $C_0(\Omega)$. Denote by $\mathfrak{P}$ the set of prime ideals of the form $I : f$ for some $f \in C_0(\Omega)$. Then:*

   (a) $I = \bigcap_{P \in \mathfrak{P}} P$;

   (b) $\mathfrak{P}$ *is a relatively compact family of prime ideals.*

(ii)   *Conversely, let $\mathfrak{P}$ be a relatively compact family of prime ideals in $C_0(\Omega)$. Then $\bigcap_{P \in \mathfrak{P}} P$ is an abstract continuity ideal in $C_0(\Omega)$.*

COROLLARY 4.16. *Let Ω be a locally compact space. Then each homomorphism from $C_0(\Omega)$ into a Banach algebra is continuous on the intersection of a relatively compact family of prime ideals of the form $\mathcal{I}(\theta) : f$ for some $f \in C_0(\Omega)$.*

5. (Relatively) compact families of prime ideals in $C_0(\Omega)$

In this section, let Ω be a locally compact space, and let $\mathfrak{P}$ be a nonempty relatively compact family of prime ideals in $C_0(\Omega)$. Denote by $\mathfrak{Q}$ the collection of all the ideals that are unions of *countably* many ideals in $\mathfrak{P}$. We call $\mathfrak{Q}$ the *closure* of $\mathfrak{P}$; we shall show that it is indeed the smallest compact family of prime ideals containing $\mathfrak{P}$.

Note that an ideal in $\mathfrak{Q}$ is automatically prime in $C_0(\Omega)$, and that the union of each pseudofinite sequence of prime ideals in $C_0(\Omega)$ is again a prime ideal in $C_0(\Omega)$; this follows from the next lemma.

LEMMA 5.1. *The union of finitely many prime ideals in $C_0(\Omega)$ either is one of the given prime ideals or is not even a linear space. The union of countably many prime ideals in $C_0(\Omega)$ is not equal to $C_0(\Omega)$.*

PROOF. We prove the second clause only; the proof of the first is similar. Let $(P_n : n \in \mathbb{N})$ be a sequence of prime ideals in $C_0(\Omega)$. For all $n \in \mathbb{N}$, choose $f_n \in C_0(\Omega) \setminus P_n$. We can assume that $0 \le f_n \le 2^{-n}$. Set $f = \sum_{n=1}^{\infty} f_n$. Then $f \in C_0(\Omega)$ but $f \notin P_n$ since $f \ge f_n$ for all $n \in \mathbb{N}$. □

LEMMA 5.2. *Each chain in $\mathfrak{Q}$ is well ordered with respect to inclusion; that is, each nonempty chain in $\mathfrak{Q}$ has a smallest element.*

PROOF. Assume that the contrary holds. Then we can find an infinite chain in $\mathfrak{Q}$: $\cdots \subset Q_n \subset \cdots \subset Q_1$. For all n, choose $P_n \in \mathfrak{P}$ such that $P_n \subseteq Q_n$ but $P_n \nsubseteq Q_{n+1}$. By the relative compactness of $\mathfrak{P}$, we can assume without loss of generality that $(P_n : n \in \mathbb{N})$ is a pseudofinite sequence. Set $Q = \bigcup_{n=1}^{\infty} P_n$. Then $Q \in \mathfrak{Q}$, and for each $n \in \mathbb{N}$, either $Q_n \subseteq Q$ or $Q \subseteq Q_n$ (since both contain P_n). Since $P_{n-1} \nsubseteq Q_n$, we

have $Q_n \subseteq Q$ for all $n \geq 2$. Choose $a \in Q_2 \setminus Q_3$. Then $a \notin Q_n$, and so $a \notin P_n$ for all $n \geq 3$. However, $a \in Q = \bigcup_{n=1}^{\infty} P_n$. This contradicts the pseudofiniteness of (P_n). $\square$

LEMMA 5.3. $\mathfrak{Q}$ *is compact.*

PROOF. Let (Q_n) be a sequence in $\mathfrak{Q}$. Let $P_n \in \mathfrak{P}$ such that $P_n \subseteq Q_n$. Since $\mathfrak{P}$ is relatively compact, we can assume without loss of generality that (P_n) is pseudofinite; its union is denoted by Q. We see that either $Q_n \subseteq Q$ or $Q \subseteq Q_n$. If there are infinitely many Q_n contained in Q, then those Q_n form a pseudofinite sequence, following from the pseudofiniteness of (P_n). On the other hand, if there are infinitely many Q_n containing Q, then those Q_n form a chain, and the previous lemma enables us to find a nondecreasing sequence of ideals. Thus $\mathfrak{Q}$ is relatively compact. The result then follows from the definition of $\mathfrak{Q}$. $\square$

LEMMA 5.4. $\mathfrak{Q}$ *is the set of unions of pseudofinite sequences of ideals in* $\mathfrak{P}$.

PROOF. We only need to prove that each ideal $Q \in \mathfrak{Q}$ is the union of a pseudofinite sequence in $\mathfrak{P}$. For this purpose, we can suppose without loss of generality that $\mathfrak{P}$ is countable and that Q is the union of $\mathfrak{P}$. It is obvious that, in this case, any chain in $\mathfrak{Q}$ is countable.

Case 1: Q is the union of a chain of ideals in $\mathfrak{Q} \setminus \{Q\}$. By the countability and well-ordering of the chain, there exists a chain $Q_1 \subset Q_2 \subset \cdots \subset Q$ such that $Q = \bigcup_{n=1}^{\infty} Q_n$. For all n, choose $P_n \in \mathfrak{P}$ such that $P_n \subseteq Q_{n+1}$ but $P_n \nsubseteq Q_n$. Since $\mathfrak{P}$ is relatively compact, without loss of generality, we can assume that (P_n) is pseudofinite; its union is denoted by Q'. Then $Q' \subseteq Q$, and for each $n \geq 2$, either $Q_n \subseteq Q'$ or $Q' \subseteq Q_n$ (since both contain P_{n-1}). Since $P_n \nsubseteq Q_n$, we must have $Q_n \subseteq Q'$ for all $n \geq 2$. So $Q = Q'$.

Case 2: Q is not the union of any chain of ideals in $\mathfrak{Q} \setminus \{Q\}$. Then any $P \in \mathfrak{Q} \setminus \{Q\}$ is contained in a maximal element of $\mathfrak{Q} \setminus \{Q\}$. Since Q cannot be the union of any finite number of prime ideals properly contained in Q, either $\mathfrak{Q} = \{Q\}$, which implies that $Q \in \mathfrak{P}$, or there exist infinitely many maximal elements of $\mathfrak{Q} \setminus \{Q\}$. In the latter case, let Q_n, where $n \in \mathbb{N}$, be distinct maximal elements of $\mathfrak{Q} \setminus \{Q\}$. Choose $P_n \in \mathfrak{P}$ such that $P_n \subseteq Q_n$. By the relative compactness of $\mathfrak{P}$ and without loss of generality, we can assume that $(P_n : n \in \mathbb{N})$ is a pseudofinite sequence. Set $Q' = \bigcup_{n=1}^{\infty} P_n$. Then $Q' \in \mathfrak{Q}$, and for each $n \in \mathbb{N}$, either $Q_n \subseteq Q'$ or $Q' \subseteq Q_n$. Assume towards a contradiction that $Q' \neq Q$. The maximality of the ideal Q_n in $\mathfrak{Q} \setminus \{Q\}$ implies that $Q' \subseteq Q_n$ for all $n \in \mathbb{N}$. This implies that $\{Q_n \mid n \in \mathbb{N}\}$ form a chain, contradicting the maximality and distinction of the ideals Q_n. Thus $Q' = Q$.

In both cases, Q is the union of a pseudofinite sequence in $\mathfrak{P}$. $\square$

In summary, we have the following proposition.

PROPOSITION 5.5. *The closure $\mathfrak{Q}$ of $\mathfrak{P}$ satisfies the following conditions.*

(i) $\mathfrak{Q}$ *is the set of unions of pseudofinite sequences of ideals in* $\mathfrak{P}$.
(ii) $\mathfrak{Q}$ *is compact.*

(iii) *Every chain in $\mathfrak{Q}$ is well ordered with respect to the inclusion.*

(iv) $\bigcap_{P \in \mathfrak{P}} P = \bigcap_{Q \in \mathfrak{Q}} Q.$

From (i), we see that $\mathfrak{Q}$ is the smallest compact family of prime ideals containing $\mathfrak{P}$. Property (iii) also shows that the intersection of $\mathfrak{P}$ is equal to the intersection of its minimal elements.

In the remainder of the section, we consider $\mathfrak{Q}$ to be any compact family of prime ideals in $\mathcal{C}_0(\Omega)$.

LEMMA 5.6. *Let P be in $\mathfrak{Q}$. Then there exists $a \in A \setminus P$ such that $a \in Q$ for all $Q \in \mathfrak{Q}$ with $Q \nsubseteq P$.*

PROOF. Assume that the contrary holds. As in Lemma 4.6, we can construct $(a_n) \subseteq A$ and $(Q_n) \subseteq \mathfrak{Q}$ such that $a_1 \ldots a_n \notin Q_n$ but $a_1 \ldots a_{n+1} \in Q_n$ for all $n \in \mathbb{N}$. By compactness, (Q_n) has a pseudofinite subsequence (Q_{n_i}). However, $a_1 \ldots a_{n_2} \in Q_{n_1}$ but $a_1 \ldots a_{n_2} \notin Q_{n_i}$ for all $i \geq 2$; this contradicts the pseudofiniteness. $\square$

We say that an ideal Q is a *roof* of $\mathfrak{Q}$ if it is the union of the ideals in a maximal chain in $\mathfrak{Q}$. A roof must be either a prime ideal in $\mathcal{C}_0(\Omega)$ or $\mathcal{C}_0(\Omega)$ itself.

LEMMA 5.7. *$\mathfrak{Q}$ has only finitely many roofs. Also, there are only finitely many maximal modular ideals in $\mathcal{C}_0(\Omega)$ such that each of them contains an ideal in $\mathfrak{Q}$.*

PROOF. We shall prove that there are only finitely many *disjoint* maximal chains in $\mathfrak{Q}$; the lemma then follows. Assume to the contrary that $\mathfrak{C}_n$, where $n \in \mathbb{N}$, are disjoint maximal chains in $\mathfrak{Q}$. Pick $Q_n \in \mathfrak{C}_n$. Without loss of generality, we can suppose that (Q_n) is pseudofinite; its union is denoted by Q. We see that $Q \in \mathfrak{Q}$, and since $Q \supseteq Q_n$ and $\mathfrak{C}_n$ is maximal, we have $Q \in \mathfrak{C}_n$ for all $n \in \mathbb{N}$, contradicting the disjointness of the chains $\mathfrak{C}_n$. $\square$

For a function $f \in \mathcal{C}_0(\Omega)$, we can define a new function $f/\sqrt{|f|}$, also in $\mathcal{C}_0(\Omega)$, as follows:

$$\left(\frac{f}{\sqrt{|f|}} \right)(x) = \begin{cases} \dfrac{f(x)}{\sqrt{|f(x)|}} & \text{when } x \in \Omega \text{ and } f(x) \neq 0 \\ 0 & \text{when } x \in \Omega \text{ and } f(x) = 0. \end{cases}$$

REMARK 5.8. If P is a prime ideal in $\mathcal{C}_0(\Omega)$ and $f \in P$, then the function $f/\sqrt{|f|}$ defined above is also in P.

The following lemma and proposition are inspired by a suggestion of an anonymous referee of an initial version of [15].

LEMMA 5.9. *Suppose that $\mathfrak{Q}$ is a compact family of prime ideals in $\mathcal{C}_0(\Omega)$ with a maximum element Q. Set $I = \bigcap_{P \in \mathfrak{Q}} P$. Let $a \in \mathcal{C}_0(\Omega) \setminus Q$ and let $b \in Q$. Then there exists $s \in Q$ such that $as - b \in I$.*

PROOF. It is standard that for each prime ideal $P \subseteq Q$, there exists $s \in Q$ such that $as - b \in P$.

Assume towards a contradiction that for all $s \in Q$, $as - b \notin I$. Initially, set $s_1 = 0$, $b_1 = b - as_1 = b$, and $\mathfrak{Q}_1 = \{P \in \mathfrak{Q} \mid b_1 \notin P\}$. Suppose that we have already constructed $s_n \in Q$, $b_n = b - as_n$, and $\mathfrak{Q}_n = \{P \in \mathfrak{Q} \mid b_n \notin P\}$ such that

$$\mathfrak{Q}_n \subset \cdots \subset \mathfrak{Q}_1.$$

Since $b_n \notin I$ by the assumption, $\mathfrak{Q}_n \neq \emptyset$. Choose $P_n \in \mathfrak{Q}_n$. We see that $b_n \in Q$, and so $b_n/\sqrt{|b_n|} \in Q$. Thus there exists $s' \in Q$ such that $as' - (b_n/\sqrt{|b_n|}) \in P_n$. Now we set $s_{n+1} = s_n + s'\sqrt{|b_n|}$,

$$b_{n+1} = b - as_{n+1} = \left(\frac{b_n}{\sqrt{|b_n|}} - as' \right) \sqrt{|b_n|},$$

and $\mathfrak{Q}_{n+1} = \{P \in \mathfrak{Q} \mid b_{n+1} \notin P\}$. We see that $\mathfrak{Q}_{n+1} \subset \mathfrak{Q}_n$; $P_n \in \mathfrak{Q}_n \setminus \mathfrak{Q}_{n+1}$. Thus the construction can be continued inductively.

In particular, when $m, n \in \mathbb{N}$, we have $b_m \in P_n$ if $m > n$, but $b_m \notin P_n$ if $m \leq n$. The compactness implies that there exists a pseudofinite subsequence (P_{n_i}). However, this contradicts the fact that $b_{n_2} \in P_{n_1}$, but $b_{n_2} \notin P_{n_i}$ for all $i \geq 2$. $\qquad \square$

PROPOSITION 5.10. *Suppose that Ω is noncompact and $\mathfrak{Q}$ is a compact family of nonmodular prime ideals in $\mathcal{C}_0(\Omega)$ with a maximum element Q. Set $I = \bigcap_{P \in \mathfrak{Q}} P$. Let $P \in \mathfrak{Q}$, and let A be a subalgebra of $\mathcal{C}_0(\Omega)$. Suppose that $\mathcal{C}_0(\Omega) = A + P$ and $A \cap P$ is the intersection of a subfamily of $\mathfrak{Q}$. Let B be a subalgebra of A such that B is maximal with respect to the property that $B \cap Q \subseteq I$. Then $\mathcal{C}_0(\Omega) = B + Q$.*

PROOF. When $a - b \in I$ we also write $a = b \bmod I$.

Note that $I \subseteq A \cap P$. By maximality of B we see that $I \subseteq B$, so indeed $B \cap Q = I$. We can also check, using the maximality of B again, that $B \neq I$. Moreover, I is nonmodular in B; for otherwise, Q would be modular in $B + Q$ (because $B/I \cong (B + Q)/Q$), and from the primeness of Q we could deduce that Q is modular in $\mathcal{C}_0(\Omega)$, which would be a contradiction. Thus I is indeed a nonmodular prime ideal in B, and hence, a prime ideal in $B^\#$.

We claim that for each $a \in A \setminus Q$ and each $b \in A \cap Q$, there exists $s \in A \cap Q$ such that $as - b \in I$. Indeed, by the previous lemma, there exists $s \in Q$ such that $as - b \in I$. Write $s = c + p$ where $c \in A$ and $p \in P$. Then $ap + (ac - b) \in I \subseteq A$ implies that $ap \in A \cap P$. Since $a \notin Q$ and $A \cap P$ is the intersection of a family of prime ideals contained in Q, we must have $p \in A \cap P$. Thus $s = c + p \in A \cap Q$.

We shall prove that $A = B + (A \cap Q)$; the proposition then follows. Indeed, assume towards a contradiction that $B + (A \cap Q) \subset A$. Let $a \in A$ such that $a \notin B + Q$. By the maximality of B, there exists a polynomial $q(X)$ in $B^\#[X] \setminus I[X]$ such that $q(a) \in Q$. Let $q(X)$ be such a polynomial of smallest degree. By multiplying with some element in $B \setminus I$, we can further suppose that the coefficients of $q(X)$ are in B. Then we see that $q'(a) \in A \setminus Q$, where $q'(X)$ is the formal derivative. Let $s \in \mathcal{C}_0(\Omega)$. Then

$$q(a + q'(a)s) = q(a) + q'(a)^2 s + \cdots + q'(a)^n q^{(n)}(a) \frac{s^n}{n!}$$

where $q^{(k)}(X)$ is the formal kth derivative of $q(X)$. By the claim, there exists $d \in A \cap Q$ such that

$$q(a) = q'(a)^2 d \bmod I. \tag{5.1}$$

So for all $s \in \mathcal{C}_0(\Omega)$,

$$q(a + q'(a)s) = q'(a)^2 \Big(d + s + \cdots + q'(a)^{n-2} q^{(n)}(a) \frac{s^n}{n!} \Big) \bmod I.$$

Since $(A/(A \cap P)) \cong \mathcal{C}_0(\Omega)/P$ is radical and $(\mathcal{C}_0(\Omega)/P)^\#$ is Henselian, there exists $s \in A$ such that

$$d + s + \cdots + q'(a)^{n-2} q^{(n)}(a) \frac{s^n}{n!} \in A \cap P \subseteq Q$$

(by [3, Theorem 2.4.30 and Proposition 1.6.3]). It follows that $s \in Q$ (so $s \in A \cap Q$) and

$$q(a + q'(a)s) \in A \cap P.$$

If we set $b = a + q'(a)s$, then $b \in A \setminus (B + Q)$, $q(b) \in P$, and $q(X)$ is a polynomial of smallest degree among all the polynomials in $B^\#[X] \setminus I[X]$ such that $q(b) \in Q$.

Hence, without loss of generality, we can assume from the start that $q(a) \in P$. In this case, $d \in P$ by (5.1), so $d \in A \cap P$. Again, since $\mathcal{C}_0(\Omega)/I$ is radical and $\mathcal{C}_0(\Omega)^\#/I$ is Henselian, there exists $t \in \mathcal{C}_0(\Omega)$ such that

$$d + t + \cdots + q'(a)^{n-2} q^{(n)}(a) \frac{t^n}{n!} \in I \subseteq A \cap P.$$

Now $A \cap P$ is an ideal in $\mathcal{C}_0(\Omega)$, and so $t \in A \cap P$. Set $c = a + q'(a)t$. Then $c \in A \setminus (B + Q)$, $q(c) \in I$, and $q(X)$ is a polynomial of smallest degree among all the polynomials in $B^\#[X] \setminus I[X]$ such that $q(c) \in Q$. The maximality of B implies that there exists a polynomial $p(X)$ with coefficients in $B^\#$ such that $p(c) \in Q \setminus I$. We see that there exist $u \in B \setminus I$ and a polynomial $h(X)$ with coefficients in $B^\#$ such that $up(X) = q(X)h(X) \bmod I$; this is possible since I is prime in $B^\#$. Then $up(c) = q(c)h(c) = 0 \bmod I$. Since $u \notin Q$ (otherwise, $u \in B \cap Q = I$) and I is the intersection of some prime ideals contained in Q, we deduce that $p(c) \in I$; a contradiction. □

A special case of the previous proposition is when $A = \mathcal{C}_0(\Omega)$.

6. Homomorphisms from $\mathcal{C}_0(\Omega)$

In this section, we shall show the connection between continuity ideals as well as the kernels of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras and intersections of (relatively) compact families of prime ideals. One direction is an immediate consequence of the results in Section 4, so most of this section concerns the converse.

We shall need some basic complex algebraic-geometry results; we are indebted to [14] for these. For a set $S \subseteq \mathbb{C}[Z_1, Z_2, \ldots, Z_n]$, denote by $\mathcal{V}(S)$ the *variety* (that

is, common zero set) of S in $\mathbb{C}^n$. For each prime ideal Q in $\mathbb{C}[Z_1, \ldots, Z_n]$, the variety $\mathcal{V}(Q)$ is *irreducible*. The topology considered on complex spaces will be the Euclidean topology. We shall need the fact that, for each irreducible variety V and each variety W not containing V, the set $V \setminus W$ is dense and (relatively) open in V [14, Ch. IV, Theorem 2.11].

NOTATION. For clarity, we shall use X_i, Y_j for variables, x_i, y_j for complex numbers, and a_i, b_j for elements of an algebra. When there is no ambiguity, we shall use boldface characters to denote tuples of elements of the same type; for example, we set

$$\boldsymbol{X} = (X_1, X_2, \ldots, X_m) \quad \text{or} \quad \boldsymbol{y} = (y_1, \ldots, y_n).$$

In the case where $\boldsymbol{X} = (X_1, \ldots, X_m)$, we also denote by $\mathbb{C}_{\boldsymbol{X}}$ the corresponding space $\mathbb{C}^m$.

LEMMA 6.1. *Let* $m, n \in \mathbb{N}$, *and let* Q *be a prime ideal in* $\mathbb{C}[\boldsymbol{X}, \boldsymbol{Y}]$, *where* $\boldsymbol{X} = (X_1, \ldots, X_m)$ *and* $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$. *Consider* $Q_{\boldsymbol{X}} = Q \cap \mathbb{C}[\boldsymbol{X}]$ *as a prime ideal in* $\mathbb{C}[\boldsymbol{X}]$. *Let* V *be the variety of* Q, *and let* $V_{\boldsymbol{X}}$ *be the variety of* $Q_{\boldsymbol{X}}$. *Let* π *be the natural projection* $\mathbb{C}_{\boldsymbol{X}, \boldsymbol{Y}} \to \mathbb{C}_{\boldsymbol{X}}$. *Then* $\pi : V \to V_{\boldsymbol{X}}$ *and there exists a dense open subset* U *of* V *such that* $\pi : U \to V_{\boldsymbol{X}}$ *is an open map.*

PROOF. Obviously, π maps V to $V_{\boldsymbol{X}}$. Without loss of generality, take a transcendental basis $(X_1, \ldots, X_k)$ for $\mathbb{C}[\boldsymbol{X}]$ modulo $Q_{\boldsymbol{X}}$. We consider $\mathbb{C}^k = \mathbb{C}_{X_1, \ldots, X_k}$. Denote by π_1 the natural projection $\mathbb{C}_{\boldsymbol{X}, \boldsymbol{Y}} \to \mathbb{C}^k$, and by π_2 the natural projection $\mathbb{C}_{\boldsymbol{X}} \to \mathbb{C}^k$.

By [15, Lemma 6.3], there exist dense open subsets U of V and $U_{\boldsymbol{X}}$ of $V_{\boldsymbol{X}}$, such that $\pi_1 : U \to \mathbb{C}^k$ and $\pi_2 : U_{\boldsymbol{X}} \to \mathbb{C}^k$ are open maps. Inspecting the proof of [15, Lemma 6.3], we see that $U_{\boldsymbol{X}}$ can be chosen as $V_{\boldsymbol{X}} \setminus V_0$, where V_0 is a proper subvariety of $V_{\boldsymbol{X}}$, and that π_2 is a local homeomorphism from $U_{\boldsymbol{X}}$ onto an open subset of $\mathbb{C}^k$. Since V_0 has dimension at most $k - 1$ [14, Ch. IV], by shrinking $U_{\boldsymbol{X}}$ if necessary, we can further require that

$$\pi_2(U_{\boldsymbol{X}}) \cap \pi_2(V_{\boldsymbol{X}} \setminus U_{\boldsymbol{X}}) = \emptyset. \tag{6.1}$$

Let $W = \pi_1(U) \cap \pi_2(U_{\boldsymbol{X}})$. Then W is an open set in $\mathbb{C}^k$. It can be seen that W is dense in $\pi_1(U)$. Set

$$U' = U \cap \pi_1^{-1}(W).$$

Then U' is a dense open subset of U, by the openness of $\pi_1 : U \to \mathbb{C}^k$, and hence a dense open subset of V.

We claim that $\pi : U' \to V_{\boldsymbol{X}}$ is an open map. Indeed, let $(\boldsymbol{x}, \boldsymbol{y}) \in U'$ be arbitrary. Then $(x_1, \ldots, x_k) \in W$, and so $\boldsymbol{x} \in U_{\boldsymbol{X}}$ by (6.1). Choose $\Delta_{\boldsymbol{X}}$ to be a neighbourhood of $\boldsymbol{x}$ in $U_{\boldsymbol{X}}$ such that π_2 is a homeomorphism from $\Delta_{\boldsymbol{X}}$ onto an open subset of $\mathbb{C}^k$. Let Δ be any neighbourhood of $(\boldsymbol{x}, \boldsymbol{y})$ in U' contained in $\pi^{-1}(\Delta_{\boldsymbol{X}}) \cap U'$. Then we see that $\pi(\Delta) \subseteq \Delta_{\boldsymbol{X}}$ and $\pi_2(\pi(\Delta)) = \pi_1(\Delta)$, which is open in $\mathbb{C}^k$ by the openness of π_1 on U'. It follows that $\pi(\Delta)$ is open in $\Delta_{\boldsymbol{X}}$ and thus in $V_{\boldsymbol{X}}$. □

PROPOSITION 6.2. *Let $A = C_0(\Omega)$ for a locally compact space Ω, and let $\mathfrak{Q}$ be a nonempty compact family of nonmodular prime ideals in A. Suppose that every chain in $\mathfrak{Q}$ is countable. Then there exist a cardinal κ, a free ultrafilter $\mathcal{U}$ on κ, and a homomorphism $\theta_P \colon A \to (\mathbb{C}^\kappa / \mathcal{U})^\circ$ for each $P \in \mathfrak{Q}$, such that:*

(a) $\ker \theta_P = P$ *for all $P \in \mathfrak{Q}$; and*
(b)   *the set $\{\theta_P(a) \mid P \in \mathfrak{Q}\}$ is finite for all $a \in A$.*

Note that Ω must be noncompact and A must be nonunital. Since each $P \in \mathfrak{Q}$ is a nonmodular prime ideal in A, it is a prime ideal in $A^\#$. For all $Q \in \mathfrak{Q}$, set

$$\mathfrak{Q}_Q = \{P \in \mathfrak{Q} \mid P \subseteq Q\},$$

and set $I_Q = \bigcap_{P \in \mathfrak{Q}_Q} P$. We begin the proof of Proposition 6.2 with some lemmas.

LEMMA 6.3. *Suppose that $Q_*, Q^* \in \mathfrak{Q}$ and A_* and A^* are subalgebras of A, such that $Q_* \subseteq Q^*$ and $A_* \supseteq A^*$. Suppose further that $A^* \cap Q^* \subseteq I_{Q^*}$, $A_* \cap Q_* = I_{Q_*}$, and $A_* + Q_* = A$. Let $\mathfrak{C}$ be a chain in $\mathfrak{Q}_{Q^*}$ where each ideal in $\mathfrak{C}$ contains Q_*. Then we can find subalgebras $A_Q \subseteq A$ where $Q \in \mathfrak{C}$ satisfying the following conditions.*

(i) $A_Q \cap Q = I_Q$ *and* $A = A_Q + Q$ *for all $Q \in \mathfrak{C}$.*
(ii)   *Whenever $Q_1, Q_2 \in \mathfrak{C}$ and $Q_1 \subseteq Q_2$, then $A_* \supseteq A_{Q_1} \supseteq A_{Q_2} \supseteq A^*$.*

PROOF. Assume towards a contradiction that the lemma fails for some instance of Q_*, Q^*, A_*, A^*, and $\mathfrak{C}$; we can further require that $\mathrm{o}(\mathfrak{C})$ is smallest among such instances (note that $\mathfrak{C}$ is well ordered by Proposition 5.5(iii)).

Now $A_* \cap Q_* = \bigcap_{P \in \mathfrak{Q}_{Q_*}} P$ is the intersection of a subfamily of $\mathfrak{Q}_Q$, and

$$A^* \cap Q \subseteq A^* \cap Q^* \subseteq I_{Q^*} \subseteq I_Q,$$

for all $Q \in \mathfrak{C}$. Thus we are in a situation where Proposition 5.10 applies.

Proposition 5.10 implies that $\mathfrak{C}$ is infinite; otherwise, we can use finite induction to construct $(A_Q : Q \in \mathfrak{C})$. If $\mathfrak{C}$ is order isomorphic to ω, the first infinite ordinal, say

$$\mathfrak{C} = \{Q_1 \subset Q_2 \subset \cdots\},$$

then Proposition 5.10 also enables us to construct (A_{Q_n}) inductively.

In general, there exists a sequence (Q_n) in $\mathfrak{C}$ converging to $Q_\infty = \bigcup_{Q \in \mathfrak{C}} Q$ in the order topology (this is where our proof needs the countability condition in Proposition 6.2). If $Q_\infty \in \mathfrak{C}$, we can construct A_{Q_∞} by Proposition 5.10. As in the previous paragraph, we can then find A_{Q_n} for all $n \in \mathbb{N}$ satisfying both conditions (i) and (ii). The ideals Q_n divide $\mathfrak{C} \setminus \{Q_\infty\}$ into chains that are order isomorphic to ordinals strictly smaller than $\mathrm{o}(\mathfrak{C})$. The minimality of $\mathrm{o}(\mathfrak{C})$ (and the assumption) then imply that we can extend the present collection $\{A_Q\}$ to all $Q \in \mathfrak{C}$ that satisfy conditions (i) and (ii).

Thus we have a contradiction in either case. $\square$

LEMMA 6.4. *We can find subalgebras $A_Q \subseteq A$ for all $Q \in \mathfrak{Q}$ satisfying the following conditions.*

(i) $A_Q \cap Q = I_Q$ and $A = A_Q + Q$ for all $Q \in \mathfrak{Q}$.
(ii) $A_{Q_1} \supseteq A_{Q_2}$ whenever $Q_1, Q_2 \in \mathfrak{Q}$ and $Q_1 \subseteq Q_2$.

PROOF. This follows from the previous lemma, Zorn's lemma, and the fact that all prime ideals containing a given prime ideal in $A = \mathcal{C}_0(\Omega)$ form a chain.   □

Let κ be the set of all tuples of the form $(\delta; \mathfrak{G}; a_1, \ldots, a_m)$, where $\delta > 0$, $\mathfrak{G}$ is a nonempty finite subset of $\mathfrak{Q}$, and $(a_1, \ldots, a_m)$ is a nonempty finite sequence of distinct elements in A. Define a partial order $\prec$ on κ by setting

$$(\delta; \mathfrak{G}; a_1, a_2, \ldots, a_m) \prec (\delta'; \mathfrak{G}'; a_1', a_2', \ldots, a_{m'}')$$

if $\delta > \delta'$, $\mathfrak{G} \subseteq \mathfrak{G}'$, $\{a_1, \ldots, a_m\}$ is a subset of $\{a_1', a_2', \ldots, a_{m'}'\}$. Then $(\kappa, \prec)$ is a net. Fix an ultrafilter $\mathcal{U}$ on κ majorizing this net.

LEMMA 6.5. *Let $(A_Q : Q \in \mathfrak{Q})$ be as in Lemma 4.6. Let $w = (\delta; \mathfrak{G}; a_1, \ldots, a_m) \in \kappa$. Then, for each $P \in \mathfrak{G}$, we can find a tuple*

$$\tau_P(w) = \boldsymbol{x}^{(P)} = (x_1^{(P)}, \ldots, x_m^{(P)}) \quad in \ \mathbb{C}^m$$

satisfying all the following conditions.

(i) $p(x_1^{(P)}, \ldots, x_m^{(P)}) = 0$ for all $p \in \mathbb{C}[X_1, \ldots, X_m]$ with $p(a_1, \ldots, a_m) \in P$.
(ii) $x_k^{(P)} \neq 0$ for all $1 \leq k \leq m$ with $a_k \notin P$.
(iii) $|x_k^{(P)}| \leq \delta$ for all $1 \leq k \leq m$.
(iv) $x_k^{(P)} = x_k^{(Q)}$ for all $1 \leq k \leq m$ and all $P, Q \in \mathfrak{G}$ such that $P \subseteq Q$ and $a_k \in A_Q$.

PROOF. Set $X = (X_1, \ldots, X_m)$. Further, set $\boldsymbol{a}^P = (a_1, \ldots, a_m) \cap A_P$ and $X^P = (X_i : a_i \in A_P)$ for all $P \in \mathfrak{G}$. Without loss of generality, we can assume that none of these is empty. Note that, if $P, Q \in \mathfrak{G}$ and $P \subseteq Q$, then $\boldsymbol{a}^Q \subseteq \boldsymbol{a}^P$ and $X^Q \subseteq X^P$.

Denote by π_P the projection from $\mathbb{C}_X$ onto $\mathbb{C}_{X^P}$, and by π_{PQ} the projection from $\mathbb{C}_{X^P}$ onto $\mathbb{C}_{X^Q}$ whenever $P, Q \in \mathfrak{G}$ and $P \subseteq Q$. We also conveniently consider $\mathbb{C}_{X^P}$ as a subspace of $\mathbb{C}_X$.

For each $Q \in \mathfrak{G}$, define

$$\widetilde{Q} = \{p \in \mathbb{C}[X] \mid p(\boldsymbol{a}) \in Q\},$$
$$\widehat{Q} = \{p \in \mathbb{C}[X^Q] \mid p(\boldsymbol{a}^Q) \in Q\}.$$

We see that if $P, Q \in \mathfrak{G}$ and $P \subseteq Q$, then $\widetilde{P} \subseteq \widetilde{Q}$ are prime ideals in $\mathbb{C}[X]$, and $\widehat{Q} = \widetilde{Q} \cap \mathbb{C}[X^Q]$ is a prime ideal in $\mathbb{C}[X^Q]$. Since $A_Q \cap Q = A_Q \cap P$ in this case,

$$\widehat{Q} = \{p \in \mathbb{C}[X^Q] \mid p(\boldsymbol{a}^Q) \in P\} = \widetilde{P} \cap \mathbb{C}[X^Q];$$

and thus $\widehat{Q} = \mathbb{C}[X^Q] \cap \widehat{P}$.

It follows from Lemma 6.1 that there exist, for all $Q \in \mathfrak{G}$, dense open subsets U_Q of $\mathcal{V}(\widetilde{Q})$ and W_Q of $\mathcal{V}(\widehat{Q})$, such that:

(1) $\pi_Q \colon U_Q \to W_Q$ is an open map;

(2) $\pi_{PQ} \colon W_P \to W_Q$ is an open map whenever $P, Q \in \mathfrak{G}$ and $P \subseteq Q$.

(There are only finitely many varieties here.)

For each $Q \in \mathfrak{G}$, set

$$V_Q = \bigcup_{\substack{1 \le r \le m \\ a_r \notin Q}} \{\boldsymbol{x} = (x_1, \ldots, x_m) \mid x_r = 0\}.$$

Then V_Q is a variety that does not contain $\mathcal{V}(\widetilde{Q})$, and so $\mathcal{V}(\widetilde{Q}) \setminus V_Q$ is also a dense open subset of $\mathcal{V}(\widetilde{Q})$ because $\mathcal{V}(\widetilde{Q})$ is irreducible. Therefore, $U_Q \setminus V_Q$ is again a dense open subset of $\mathcal{V}(\widetilde{Q})$.

Set

$$\Delta = \{\boldsymbol{x} = (x_1, \ldots, x_m) : |x_r| < \delta \text{ when } 1 \le r \le m\}.$$

Finally, set $U_Q' = (U_Q \setminus V_Q) \cap \Delta$ and $W_Q' = W_Q \cap \Delta$. Note that the origin $\boldsymbol{0}$ is in $\mathcal{V}(\widetilde{Q})$ and $\mathcal{V}(\widehat{Q})$ for all $Q \in \mathfrak{G}$. So U_Q' and W_Q' are nonempty open subsets of $\mathcal{V}(\widetilde{Q})$ and $\mathcal{V}(\widehat{Q})$ respectively.

Conditions (i), (ii), and (iii) will be satisfied as long as we choose $\boldsymbol{x}^{(Q)} \in U_Q'$ for all $Q \in \mathfrak{G}$. We now turn our attention to condition (iv).

Fix $P, Q \in \mathfrak{G}$ such that $P \subseteq Q$. Now $U_P \setminus V_P$ is dense in $\mathcal{V}(\widetilde{P})$ and $U_Q' \subseteq \mathcal{V}(\widetilde{Q}) \subseteq \mathcal{V}(\widetilde{P})$, so $\pi_Q(U_P') \cap \pi_Q(U_Q')$ is dense (and open) in $\pi_Q(U_Q')$ (which is open in W_Q). Note that $\pi_Q(U_P') = \pi_{PQ}[\pi_P(U_P')]$. In fact, we see that $\pi_{PQ}(D) \cap \pi_Q(U_Q')$ is dense and open in $\pi_Q(U_Q')$ for every dense open subset D of $\pi_P(U_P')$.

Thus we can define a dense open subset D_Q of $\pi_Q(U_Q')$ (in $\mathbb{C}_{X_Q}$) for each $Q \in \mathfrak{G}$ as follows. For all P minimal in $\mathfrak{G}$, set $D_P = \pi_P(U_P')$. Then define inductively

$$D_Q = \bigcap_{\substack{P \in \mathfrak{G} \\ P \subset Q}} \pi_{PQ}(D_P) \cap \pi_Q(U_Q').$$

The nonemptiness of the sets D_P then allows us to pick tuples $\boldsymbol{\alpha}^Q \in D_Q$ for all $Q \in \mathfrak{G}$, such that $\boldsymbol{\alpha}^Q = \pi_{PQ}(\boldsymbol{\alpha}^P)$ when $P, Q \in \mathfrak{G}$ and $P \subseteq Q$. (This can be done inductively, starting from the maximal elements in $\mathfrak{G}$; note that $\mathfrak{G}$ is finite and that the prime ideals containing a given prime ideal form a chain.) Finally, for all $Q \in \mathfrak{G}$, choose $\boldsymbol{x}^{(Q)} \in U_Q'$ (in $\mathbb{C}^m = \mathbb{C}_X$) such that $\pi_Q(\boldsymbol{x}^{(Q)}) = \boldsymbol{\alpha}^Q$. It can be checked that these are the desired tuples. $\square$

PROOF OF PROPOSITION 6.2. We retain the notation of the previous lemmas. Define $\xi_P \colon A \to \mathbb{C}^\kappa$ for all $P \in \mathfrak{Q}$ as follows. For each $a \in A$ and $w = (\delta; \mathfrak{G}; a_1, \ldots, a_m) \in \kappa$, if $P \in \mathfrak{G}$ and if a is in $(a_1, \ldots, a_m)$, say $a = a_k$ (there is at most one such k),

then set $\xi_P(a)(w) = \tau_P(w)(k)$; otherwise, set $\xi_P(a)(w) = 0$. Define $\theta_P(a)$ to be the equivalence class in $\mathbb{C}^\kappa/\mathcal{U}$ containing $\xi_P(a)$.

It follows from Lemma 6.5 that θ_P is an algebra homomorphism from A into $(\mathbb{C}^\kappa/\mathcal{U})^\circ$ with $\ker\theta_P = P$, for all $P \in \mathfrak{Q}$. Moreover, whenever $P, Q \in \mathfrak{G}$, $P \subseteq Q$ and $a \in A_Q$, we have $\theta_P(a) = \theta_Q(a)$.

Let (P_n) be any pseudofinite sequence in $\mathfrak{Q}$, and let $Q = \bigcup_{n=1}^\infty P_n$. Let $a \in A$. Write $a = b + x$, where $b \in A_Q$ and $x \in Q$. Then $x \in P_n$ for all $n > n_0$ for some $n_0 \in \mathbb{N}$. Therefore $\theta_{P_n}(a) = \theta_{P_n}(b) = \theta_Q(b)$ for all $n > n_0$. Thus the sequence $(\theta_{P_n}(a))$ is eventually constant.

Finally, assume towards a contradiction that the set $\{\theta_P(a) \mid P \in \mathfrak{Q}\}$ is infinite for some $a \in A$. Let (P_n) be a sequence in $\mathfrak{Q}$ such that all $\theta_{P_n}(a)$ are distinct. By the compactness of $\mathfrak{Q}$, we can assume that (P_n) is pseudofinite. However, the previous paragraph shows that the set $\{\theta_{P_n}(a) \mid n \in \mathbb{N}\}$ is finite. This contradiction concludes the proof of Proposition 6.2. □

REMARK 6.6. The idea of mapping into an ultrapower in the above proposition generalizes the approach in [5] of the theorem of Esterle [9] on embedding integral domains into radical Banach algebras.

We are now ready to prove our main results.

THEOREM 6.7. *Suppose that Ω is a locally compact space.*

(i)   *Let θ be a homomorphism from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra R. Then $\ker\theta$ is the intersection of a (relatively) compact family of nonmodular prime ideals in $\mathcal{C}_0(\Omega)$.*

(ii)   *(CH) Let I be the intersection of a relatively compact family $\mathfrak{P}$ of nonmodular prime ideals in $\mathcal{C}_0(\Omega)$ such that every chain in the closure of $\mathfrak{P}$ is countable. Suppose that*

$$|\mathcal{C}_0(\Omega)/I| = \mathfrak{c}.$$

Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra such that $\ker\theta = I$.

PROOF. (i) Since θ maps into a radical algebra, we see that $\ker\theta : f$ is nonmodular for each $f \in \mathcal{C}_0(\Omega)$. Theorem 1.1 shows that $\ker\theta = \mathcal{I}(\theta)$, and so it is an abstract continuity ideal in $\mathcal{C}_0(\Omega)$. The result then follows from Corollary 4.15(i). Proposition 5.5 then allows us to pass from a relatively compact family of prime ideals to a compact family of prime ideals with the same intersection.

(ii) Let $\mathfrak{Q}$ be the closure of $\mathfrak{P}$. Then every chain in $\mathfrak{Q}$ is countable, every ideal in $\mathfrak{Q}$ is nonmodular, and $I = \bigcap_{P \in \mathfrak{Q}} P$.

Let $\theta_P \colon \mathcal{C}_0(\Omega) \to (\mathbb{C}^\kappa/\mathcal{U})^\circ$, where $P \in \mathfrak{Q}$, be the homomorphisms defined in Proposition 6.2. Let B be the subalgebra of $(\mathbb{C}^\kappa/\mathcal{U})^\circ$ generated by all the images of θ_P for all $P \in \mathfrak{Q}$. Then B is a nonunital integral domain. Also

$$|B| = \left| \bigcup_{P \in \mathfrak{Q}} \theta_P(\mathcal{C}_0(\Omega)) \right| = \left| \bigcup_{a \in \mathcal{C}_0(\Omega)} \{\theta_P(a) \mid P \in \mathfrak{Q}\} \right| = \mathfrak{c},$$

since, for each $b \in a + I$,

$$\{\theta_P(b) \mid P \in \mathfrak{Q}\} = \{\theta_P(a) \mid P \in \mathfrak{Q}\},$$

which is finite. Thus [9] there exists an embedding $\psi \colon B \to R_0$ where R_0 is any universal radical Banach algebra. Then the map

$$\theta \colon \mathcal{C}_0(\Omega) \to \prod_{P \in \mathfrak{Q}} R_0, \quad a \mapsto ((\psi \circ \theta_P)(a) : P \in \mathfrak{Q})$$

is a homomorphism with kernel $\bigcap_{P \in \mathfrak{Q}} P = I$. It follows from Proposition 6.2 that the image of θ is contained in the radical of $\ell^\infty(\mathfrak{Q}, R_0)$. □

THEOREM 6.8. *Suppose that Ω is a locally compact space.*

(i) *Let θ be a homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra B. Then $\mathcal{I}(\theta)$ is the intersection of a (relatively) compact family of prime ideals in $\mathcal{C}_0(\Omega)$.*

(ii) *(CH) Let I be the intersection of a relatively compact family $\mathfrak{P}$ of prime ideals in $\mathcal{C}_0(\Omega)$ such that every chain in the closure of $\mathfrak{P}$ is countable. Suppose that*

$$|\mathcal{C}_0(\Omega)/I| = \mathfrak{c}.$$

Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a Banach algebra such that $\mathcal{I}(\theta) = I$.

PROOF. (i) The continuity ideal $\mathcal{I}(\theta)$ is an abstract continuity ideal in $\mathcal{C}_0(\Omega)$. The result then follows from Corollary 4.15(i) and Proposition 5.5.

(ii) Denote by $\mathfrak{P}_0$ the set of nonmodular ideals in $\mathfrak{P}$. Let $\mathfrak{Q}'$ be the closure of $\mathfrak{P} \setminus \mathfrak{P}_0$. By Lemma 5.7, $\mathfrak{Q}'$ has only finitely many roofs. Denote by $Q_1, \ldots, Q_n$ the roofs of $\mathfrak{Q}'$, and set $\mathfrak{P}_i = \{P \in \mathfrak{P} \setminus \mathfrak{P}_0 \mid P \subseteq Q_i\}$.

Let $1 \le k \le n$. First, it is easy to see that Q_k is a modular prime ideal. Pick a modular identity u for Q_k, and pick $a \notin Q_k$. Then $a - au \in Q_k$, and so, by Lemma 5.9, there exists $v \in Q_k$ such that $a - au - av \in \bigcap_{P \in \mathfrak{P}_k} P$. It follows easily that $u + v$ is a modular identity for $\bigcap_{P \in \mathfrak{P}_k} P$; denote it by u_k.

Theorem 6.7 shows that there exists a homomorphism θ_0 from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra R_0 such that $\ker \theta_0 = \bigcap_{P \in \mathfrak{P}_0} P$. Similarly, when $1 \le k \le n$, there exists a homomorphism θ_k from M_k into R_k such that $\ker \theta_k = \bigcap_{P \in \mathfrak{P}_k} P$, where M_k is the maximal modular ideal containing Q_k. We extend θ_k to a homomorphism from $\mathcal{C}_0(\Omega)$ into $R^\#$, setting $\theta_k(u_k) = \mathbf{e}_{R_k}$. It still remains true that $\ker \theta_k = \bigcap_{P \in \mathfrak{P}_k} P$.

It follows from the result of Bade and Curtis that $\mathcal{I}(\theta_k) = \ker \theta_k$ when $0 \le k \le n$. Thus the homomorphism $\theta \colon \mathcal{C}_0(\Omega) \to \prod_{k=0}^n R_k^\#$ defined by $\theta(a) = (\theta_0(a), \ldots, \theta_n(a))$ satisfies

$$\mathcal{I}(\theta) = \bigcap_{k=0}^n \mathcal{I}(\theta_k) = \bigcap_{P \in \mathfrak{P}} P = I,$$

as required. □

REMARK 6.9. Let I be the intersection of a relatively compact family $\mathfrak{P}$ of prime ideals in $\mathcal{C}_0(\Omega)$. Suppose that I is also the intersection of a countable family of prime ideals. Denote by $\mathfrak{P}_*$ the set of minimal elements in $\mathfrak{P}$. Then $\mathfrak{P}_*$ is also relatively compact, and $I = \bigcap_{P \in \mathfrak{P}_*} P$, by Proposition 5.5(iii). Since I is the intersection of a countable family of prime ideals, we see that $\mathfrak{P}_*$ is countable (using Lemma 5.6). Since $\mathfrak{P}_*$ is countable, it is obvious that every chain in the closure of $\mathfrak{P}_*$ is countable.

7. Examples on metrizable locally compact spaces

For examples of pseudofinite sequences of prime ideals in $\mathcal{C}_0(\Omega)$, see [15]. In this section, we shall construct relatively compact families of prime ideals that are not unions of finitely many pseudofinite families.

For the entire section, fix a well-ordered set κ. Set $\kappa^{(0)} = \kappa$. For all $n \in \mathbb{N}$, define $\kappa^{(n)}$ inductively as the set of limiting elements in $\kappa^{(n-1)}$. We shall only consider those κ for which $\kappa^{(n)} = \emptyset$ for some $n \in \mathbb{N}$. This condition forces κ to be countable. Let d be the largest integer for which $\kappa^{(d)} \neq \emptyset$; we call it the *depth* of κ. For simplicity, we also suppose that κ has the largest element, $\max \kappa$, and that $\kappa^{(d)} = \{\max \kappa\}$. Otherwise, we can always replace κ by a bigger well-ordered set.

For all $\alpha \in \kappa$ define $l(\alpha)$ to be the largest integer l for which $\alpha \in \kappa^{(l)}$. We define a relation $\prec$ on κ as follows: for all $\alpha, \beta \in \kappa$, we write $\alpha \prec \beta$ if β is the smallest element in κ such that $\beta \geq \alpha$ and $l(\beta) = l(\alpha) + 1$. We then define a partial order $\ll$ on κ as follows: for all $\alpha, \beta \in \kappa$, we write $\alpha \ll \beta$ if there exists a finite sequence $\alpha = \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_n = \beta$. Note that, for each $\alpha \in \kappa$, there is exactly one chain $\alpha = \gamma_1 \prec \gamma_2 \prec \cdots$, which must be finite, and the ultimate end point of this chain is $\max \kappa$. We also set

$$\kappa_0 = \{\beta \in \kappa \mid l(\beta) = 0\}.$$

LEMMA 7.1. *The relation $\ll$ on the well-ordered set κ has the following properties.*

(i)   *If $\beta \ll \alpha$ and $\alpha \in \kappa_0$ then $\beta = \alpha$.*
(ii) *If $\gamma \ll \alpha$ and $\gamma \ll \beta$ then either $\alpha \ll \beta$ or $\beta \ll \alpha$.*
(iii) *If $\beta \ll \alpha$ and $\beta \leq \gamma \leq \alpha$ then $\gamma \ll \alpha$.*
(v)   *If $\beta \ll \alpha$ and $\beta \neq \alpha$ then there exists $\gamma \in \kappa_0$ such that $\beta < \gamma < \alpha$ (and so $\gamma \ll \alpha$).*

PROOF. The proof is routine. $\qquad \square$

We start our construction with the following general lemma.

LEMMA 7.2. *Let A be a commutative algebra and Q be an ideal that either is prime in A or is A itself. Suppose that we have $(f_\alpha : \alpha \in \kappa) \subseteq Q$ and a semiprime ideal $I \subseteq Q$ with the following properties.*

(i) *$f_\alpha \notin I$ and $I : f_\alpha \subseteq Q$ for all $\alpha \in \kappa$.*
(ii) *$f_\alpha f_\beta \in I$ whenever both $\alpha \not\ll \beta$ and $\beta \not\ll \alpha$ in κ.*

(iii) If $gf_\alpha \in I$ for some $\alpha \in \kappa$ then $gf_\beta \in I$ for all $\beta \ll \alpha$.

(iv) If $gf_{\beta_0} \in I$ for some $\beta_0 \in \kappa_0$, then, whenever $\alpha \gg \beta_0$, there exists $\beta_1 \in \kappa_0$ with $\beta_1 \ll \alpha$ such that $gf_\beta \in I$ for all $\beta \in \kappa_0$ with $\beta_1 \le \beta \le \alpha$.

Then there exist prime ideals P_α, where $\alpha \in \kappa$, such that:

(a) $f_\alpha \notin P_\alpha$ and $I : f_\alpha \subseteq P_\alpha \subseteq Q$ for all $\alpha \in \kappa$;

(b) $P_\alpha = \bigcup_{\beta \ll \alpha, \beta \ne \alpha} P_\beta$ for all $\alpha \in \kappa$;

(c) if $g \in P_\alpha$ for some $\alpha \in \kappa$ then there exists $\beta_1 \in \kappa_0$ with $\beta_1 \ll \alpha$ such that $g \in P_\beta$ for all $\beta \in \kappa_0$ with $\beta_1 \le \beta \le \alpha$.

PROOF. We prove the lemma by induction on the depth d of κ.

If $d = 0$, then $\kappa = \{0\}$. Conditions (i) to (iv) reduce to I being semiprime, $f_0 \notin I$ and $I : f_0 \subseteq Q$. It follows that $I \cap S = \emptyset$, where $S = \{f_0^k, f_0^k f \mid k \ge 1, f \in A \setminus Q\}$ is closed under multiplication. Therefore, we can find a prime ideal P_0 such that $I_0 \subseteq P_0$ and $P_0 \cap S = \emptyset$, and so $P_0 \subseteq Q$ and $f_0 \notin P_0$. We see that P_0 is the required prime ideal.

Now, suppose that the result holds whenever the depth is less than d. By Zorn's lemma, we can find a semiprime ideal J containing I such that J is maximal with respect to conditions (i) to (iv).

Claim 1: If $f \notin Q$ then $J : f = J$. Indeed, it is clear that $J : f$ is semiprime and satisfies conditions (i) to (iv). So the maximality of J implies $J : f = J$.

Claim 2: If $f \notin J$ then $J : f \subseteq Q$. For otherwise, there would exist $g \in J : f \setminus Q$, and so $f \in J : g = J$, by Claim 1; this is a contradiction.

Set $P = \bigcup_{\alpha \in \kappa} J : f_\alpha$. Then $P \subseteq Q$. Condition (iii) implies that

$$P = \bigcup_{\alpha \in \kappa_0} J : f_\alpha$$

and condition (iv) implies that P is an ideal, by choosing $\alpha = \max \kappa$.

Claim 3: If $f \notin P$ then $J : f = J$. Indeed, it is easy to see that $J : f$ is semiprime and satisfies conditions (i) to (iv) (the least obvious one is (i), however, since $f \notin P$, $ff_\alpha \notin J$, and so $f_\alpha \notin J : f$ and $(J : f) : f_\alpha = J : ff_\alpha \subseteq Q$ by Claim 2). So the maximality of J implies $J : f = J$.

Claim 4: P is either prime in A or A itself. Indeed, if $f, g \notin P$, then, by Claim 3,

$$g \notin \bigcup_{\alpha \in \kappa} J : f_\alpha = \bigcup_{\alpha \in \kappa} (J : f) : f_\alpha = P : f.$$

Thus $fg \notin P$.

Let $\alpha_1 < \alpha_2 < \cdots$ be the nonlimiting elements in $\kappa^{(d-1)}$; their limit is $\max \kappa$. Set $\kappa_1 = \{\alpha \in \kappa \mid \alpha \le \alpha_1\}$, and $\kappa_n = \{\alpha \in \kappa \mid \alpha_{n-1} < \alpha \le \alpha_n\}$ for all $n \ge 2$. Each κ_n has depth $d - 1$, and $\kappa = \bigcup_{n=1}^\infty \kappa_n \cup \{\max \kappa\}$.

For all $n \in \mathbb{N}$, we see that $(f_\alpha : \alpha \in \kappa_n)$, J, and P are such that $(f_\alpha : \alpha \in \kappa_n) \subseteq P$ and $J \subseteq P$, and conditions (i) to (iv) hold, with κ_n replacing κ, J replacing I, and P

replacing Q. So, by induction, there exist prime ideals P_α, where $\alpha \in \kappa_n$, satisfying conditions (a) to (c) (with obvious modifications). Set $P_{\max \kappa} = P$.

Note that if $\beta \ll \alpha < \max \kappa$ then both α and β belong to the same κ_n for some $n \in \mathbb{N}$. We see that the combined family $(P_\alpha : \alpha \in \kappa)$ obviously satisfies conditions (a) to (c) (with J replacing I); the only one that we really need to check is condition (c) when $\alpha = \max \kappa$, but this follows from the facts that $J : \beta \subseteq P_\beta \subseteq P_{\max \kappa}$, that

$$P_{\max \kappa} = \bigcup_{\beta \in \kappa_0} J : f_\beta,$$

and that J satisfies condition (iv). $\qquad \square$

Now, let Ω be a metrizable locally compact space. We define a nonincreasing sequence $(\partial^{(n)} \Omega^\flat : n \in \mathbb{Z}^+)$ of compact subsets of $\Omega^\flat$ as follows:

(i)   put $\partial^{(0)} \Omega^\flat = \Omega^\flat$;

(ii)   for all $n \in \mathbb{Z}^+$, define $\partial^{(n+1)} \Omega^\flat$ to be the set of all limit points of $\partial^{(n)} \Omega^\flat$.

Define $\partial^{(\infty)} \Omega^\flat = \bigcap_{n \in \mathbb{Z}^+} \partial^{(n)} \Omega^\flat$. By the compactness, either $\partial^{(\infty)} \Omega^\flat$ is nonempty or $\partial^{(l)} \Omega^\flat$ is empty for some $l \in \mathbb{Z}^+$.

To construct nontrivial pseudofinite sequences of prime ideals in $\mathcal{C}_0(\Omega)$ it is necessary that there exists $p \in \partial^{(\infty)} \Omega^\flat$; this follows from [15, Proposition 8.7]. Remark that all uncountable Polish spaces possess such a point p, and there are even countable compact subspaces of $\mathbb{R}$ satisfying this condition.

We need some further preparations. For each $\beta \in \kappa \setminus \{\max \kappa\}$, there exists a unique $\alpha \in \kappa$ such that $\beta \prec \alpha$. We define $t(\beta)$ to be the number of $\gamma \in \kappa$ such that $\gamma \prec \alpha$ and $\gamma \leq \beta$; there are only finitely many such γ. For each $\beta \in \kappa_0$, there exists a unique $(\alpha_1, \ldots, \alpha_{d-1}) \in \kappa$ such that

$$\beta \prec \alpha_1 \prec \cdots \prec \alpha_{d-1} \prec \max \kappa;$$

set $w(\beta) = \max\{t(\beta), t(\alpha_1), \ldots, t(\alpha_{d-1})\}$. For all $k \in \mathbb{N}$,

$$|\{\alpha \in \kappa_0 \mid w(\alpha) \leq k\}| = k^d.$$

Adjoin ∞ to $\mathbb{N}$ to obtain its one-point compactification $\mathbb{N}^\flat$; the convention is that $\infty > n$ for all $n \in \mathbb{N}$ and $2^{-\infty} = 0$. Define Ξ to be the subset of the product space $(\mathbb{N}^\flat)^{\kappa_0}$ consisting of all elements $(n_\alpha : \alpha \in \kappa_0)$ with the property that there exists a finite set $F \subseteq \kappa_0$ such that $n_\alpha = \infty$ when $\alpha \in \kappa_0 \setminus F$ and such that $n_\alpha \geq \max\{w(\beta) \mid \beta \in F\}$ when $\alpha \in F$. It is easy to see that Ξ is a closed subset of $(\mathbb{N}^\flat)^{\kappa_0}$.

LEMMA 7.3. *Let Ω be a metrizable locally compact space and $p \in \partial^{(\infty)} \Omega^\flat$. Then the space Ξ can be continuously embedded into $\Omega^\flat$ such that the point $\infty = (\infty, \infty, \ldots)$ is mapped into p.*

PROOF. As shown in [15, Lemmas 9.1 and 9.2], the compact subset

$$\Delta = \{0\} \cup \left\{ \sum_{i=1}^k 2^{-n_i} \, \middle| \, k, n_1, n_2, \ldots, n_k \in \mathbb{N} \text{ and } k \leq n_1 < \cdots < n_k \right\}$$

of $\mathbb{R}$ is continuously embedded in $\Omega^\flat$ such that 0 is mapped into p.

Choose an injective map $r\colon \kappa_0 \times \mathbb{N} \to \mathbb{N}$ such that $r(\alpha, j) \geq j^d$ for all $\alpha \in \kappa_0$ and $j \in \mathbb{N}$; this is possible since κ_0 is countable. For convenience, for all $\alpha \in \kappa_0$, we also set $r(\alpha, \infty) = \infty$. Define a map τ from Ξ into $[0, 1]$ as follows: for all $(n_\alpha)_{\alpha \in \kappa_0} \in \Xi$, set

$$\tau\colon (n_\alpha)_{\alpha \in \kappa_0} \mapsto \sum_{\alpha \in \kappa_0} 2^{-r(\alpha, n_\alpha)}.$$

Then we see that τ is well defined, injective and continuous.

Let $(n_\alpha)_{\alpha \in \kappa_0} \in \Xi$ be arbitrary. Set $F = \{\alpha \in \kappa_0 \mid n_\alpha < \infty\}$. Then F is a finite subset of κ_0, and $n_\alpha \geq k = \max\{w(\beta) \mid \beta \in F\}$ for all $\alpha \in F$. So we see that, for $\alpha \in F$,

$$r(\alpha, n_\alpha) \geq n_\alpha^d \geq k^d = |\{\beta \in \kappa_0 \mid w(\beta) \leq k\}| \geq |F|.$$

From this, we deduce that τ maps Ξ into Δ, and the lemma follows. $\qquad\square$

LEMMA 7.4. *Let Ω be a metrizable locally compact space, and let $p \in \partial^{(\infty)}\Omega^\flat$. Then there exist a family of prime ideals $(P_\alpha : \alpha \in \kappa)$ in $\mathcal{C}_0(\Omega)$, where each ideal is supported at p, and a family of functions $(f_\alpha : \alpha \in \kappa)$ in $\mathcal{C}_0(\Omega)$ such that:*

(a) *$f_\alpha \notin P_\alpha$ and $f_\beta \in P_\alpha$ whenever both $\beta \not\ll \alpha$ and $\alpha \not\ll \beta$;*
(b) *$P_\alpha = \bigcup_{\beta \ll \alpha, \beta \neq \alpha} P_\beta$ for all $\alpha \in \kappa$;*
(c) *for all $\alpha \in \kappa$, if $g \in P_\alpha$ then there exists $\beta_1 \in \kappa_0$ with $\beta_1 \ll \alpha$ such that $g \in P_\beta$ for all $\beta \in \kappa_0$ with $\beta_1 \leq \beta \leq \alpha$.*

PROOF. It follows from the previous lemma that we only need to consider the case where $\Omega = \Xi$ and $p = \infty$. Thus suppose that $\Omega = \Xi$ and $p = \infty$.

For each $\alpha \in \kappa_0$ define

$$Z_\alpha = \{(n_\beta)_{\beta \in \kappa_0} \in \Xi \mid n_\alpha = \infty\},$$

and for all $\alpha \in \kappa \setminus \kappa_0$ define

$$Z_\alpha = \bigcap_{\substack{\beta \in \kappa_0 \\ \beta \ll \alpha}} Z_\beta.$$

Then choose $f_\alpha \in \mathcal{C}(\Xi)$ such that $Z_\alpha = \mathbf{Z}(f_\alpha)$. Let $\mathcal{F}$ be the z-filter generated by all $Z_\alpha \cup Z_\beta$ where $\alpha, \beta \in \kappa$, $\alpha \not\ll \beta$ and $\beta \not\ll \alpha$. Then define $I = \mathbf{Z}^{-1}[\mathcal{F}]$. Obviously I is a semiprime ideal, $I \subseteq M_\infty$, and $(f_\alpha : \alpha \in \kappa) \subseteq M_\infty$.

It is sufficient to prove that (f_α), I, and M_∞ satisfy conditions (i) to (iv) of Lemma 7.2.

First, for all $\gamma \in \kappa$, we see that $f \in I : f_\gamma$ if and only if

$$\mathbf{Z}(f) \cup Z_\gamma \supseteq \bigcap_{k=1}^{n} (Z_{\alpha_k} \cup Z_{\beta_k}),$$

where $\alpha_k \not\ll \beta_k$ and $\beta_k \not\ll \alpha_k$ for each k. We see from Lemma 7.1 that, for all k, one of the following three cases must happen:

(1) $\alpha_k \not\ll \gamma$ and $\gamma \not\ll \alpha_k$;
(2) $\gamma \not\ll \beta_k$ and $\beta_k \not\ll \gamma$;
(3) $\alpha_k \ll \gamma$, $\beta_k \ll \gamma$, $\alpha_k \not\ll \beta_k$ and $\beta_k \not\ll \alpha_k$.

Thus we see that $f \in I : f_\gamma$ implies that

$$\mathbf{Z}(f) \cup Z_\gamma \supseteq \bigcap_{i=1}^{r} Z_{\varrho_i} \cap \bigcap_{j=1}^{s} (Z_{\sigma_j} \cup Z_{\zeta_j}),$$

where $\varrho_i \not\ll \gamma$ and $\gamma \not\ll \varrho_i$ for all i, and $\sigma_j \ll \gamma$, $\zeta_j \ll \gamma$, $\sigma_j \not\ll \zeta_j$ and $\zeta_j \not\ll \sigma_j$ for all j. In particular, we see that $\sigma_j < \gamma$ whenever $1 \leq j \leq s$, and so, by Lemma 7.1(iv), there exists $\beta \in \kappa_0$ such that

$$\beta > \max\{\sigma_j \mid 1 \leq j \leq s\} \quad \text{and} \quad \beta \ll \gamma.$$

These also imply that $\beta \not\ll \sigma_j$ if $1 \leq j \leq s$ and $\beta \not\ll \varrho_i$ if $1 \leq i \leq r$. Thus for all $k \geq w(\beta)$, we set $n_\beta^{(k)} = k$ and $n_\alpha^{(k)} = \infty$ when $\alpha \neq \beta$, and then

$$(n_\alpha^{(k)} : \alpha \in \kappa_0) \in \left(\bigcap_{i=1}^{r} Z_{\varrho_i} \cap \bigcap_{j=1}^{s} Z_{\sigma_j} \right) \setminus Z_\gamma \subseteq \mathbf{Z}(f).$$

On the other hand, $\lim_{k \to \infty} (n_\alpha^{(k)} : \alpha \in \kappa_0) = \infty$ in Ξ. It follows that $\infty \in \mathbf{Z}(f)$. Hence, $I : f_\gamma \subseteq M_\infty$ and $f_\gamma \notin I$, so condition (i) of Lemma 7.2 holds.

It is obvious that conditions (ii) and (iii) of Lemma 7.2 are satisfied by the definitions of I and the sets Z_α.

Now, let $\beta_0, \alpha \in \kappa$ and let $g \in \mathcal{C}(\Xi)$ be such that $l(\beta_0) = 0$, $\beta_0 \ll \alpha$, $\beta_0 \neq \alpha$ (these imply that $\alpha \notin \kappa_0$) and $gf_{\beta_0} \in I$. Then, from the previous discussion, noting that $l(\beta_0) = 0$,

$$\mathbf{Z}(g) \cup Z_{\beta_0} \supseteq \bigcap_{i=1}^{r} Z_{\varrho_i} \quad \text{or} \quad \mathbf{Z}(g) \supseteq \overline{\bigcap_{i=1}^{r} Z_{\varrho_i} \setminus Z_{\beta_0}},$$

where $\beta_0 \not\ll \varrho_i$, which implies that $\alpha \not\ll \varrho_i$ for all i. We claim that

$$\mathbf{Z}(g) \supseteq \bigcap_{i=1}^{r} Z_{\varrho_i} \cap \bigcap_{\substack{\gamma \in \kappa_0 \\ w(\gamma) \leq w(\beta_0)}} Z_\gamma.$$

Indeed, let $(n_\gamma : \gamma \in \kappa_0) \neq \infty$ be in the right-hand-side set. Then, $n_\gamma = \infty$ whenever $\gamma \ll \varrho_i$ for some i or $w(\gamma) \leq w(\beta_0)$ for all $\gamma \in \kappa_0$. Set $F = \{\gamma \in \kappa_0 \mid n_\gamma < \infty\}$. Then F is a nonempty finite set and

$$n_\gamma \geq \max\{w(\beta) \mid \beta \in F\} > w(\beta_0) \quad \forall \gamma \in F.$$

Thus by setting $n_{\beta_0}^{(k)} = k$ when $k \geq \max\{w(\beta) \mid \beta \in F\}$ and $n_\gamma^{(k)} = n_\gamma$ when $\gamma \neq \beta_0$, we obtain

$$(n_\gamma^{(k)} : \gamma \in \kappa_0) \in \bigcap_{i=1}^{r} Z_{\varrho_i} \setminus Z_{\beta_0} \subseteq \mathbf{Z}(g).$$

On the other hand, we see that $\lim_{k \to \infty} (n_\gamma^{(k)} : \gamma \in \kappa_0) = (n_\gamma : \gamma \in \kappa_0)$ in Ξ. It follows that $(n_\gamma : \gamma \in \kappa_0) \in \mathbf{Z}(g)$. The claim follows.

Set

$$A = \{\gamma \in \kappa_0 \mid \gamma \ll \alpha \text{ and } w(\gamma) \leq w(\beta_0)\}$$

and

$$B = \{\varrho_i \mid 1 \leq i \leq r, \varrho_i \ll \alpha\}.$$

By Lemma 7.1(iv), we can choose $\beta_1 \in \kappa_0$ such that $\beta_1 \ll \alpha$ and

$$\beta_1 > \max(A \cup B); \tag{7.1}$$

note that $\max(A \cup B) \ll \alpha$ and $\max(A \cup B) \neq \alpha$. Let $\beta \in \kappa_0$ be such that $\beta_1 \leq \beta \leq \alpha$. Then $\beta \ll \alpha$. We see that $\varrho_i \not\ll \beta$ when $1 \leq i \leq r$; otherwise we see that $\varrho_i = \beta \ll \alpha$ but then $\beta \geq \beta_1 > \varrho_i$ by (7.1), contradicting $\varrho_i = \beta$. We also see that $\beta \not\ll \varrho_i$ when $1 \leq i \leq r$; otherwise either $\varrho_i \ll \alpha$ or $\alpha \ll \varrho_i$, and we have already ruled out $\alpha \ll \varrho_i$ from the beginning of the previous paragraph, so $\varrho_i \ll \alpha$, and again, this and (7.1) imply that $\beta \geq \beta_1 > \varrho_i$, contradicting $\beta \ll \varrho_i$. We also see that $\beta \not\ll \gamma$ and $\gamma \not\ll \beta$ for all $\gamma \in \kappa_0$ with $w(\gamma) \leq w(\beta_0)$; otherwise $\gamma = \beta \ll \alpha$. However, this and (7.1) imply that $\beta \geq \beta_1 > \gamma$. Thus

$$\mathbf{Z}(g) \cup Z_\beta \supseteq \bigcap_{i=1}^r (Z_{\varrho_i} \cup Z_\beta) \cap \bigcap_{\substack{\gamma \in \kappa_0 \\ w(\gamma) \leq w(\beta_0)}} Z_\gamma \cup Z_\beta \in \mathcal{F};$$

and so $g f_\beta \in I$. Hence, condition (iv) of Lemma 7.2 is also satisfied. $\square$

We now state the main theorem of this section.

THEOREM 7.5. *Suppose that $p \in \partial^{(\infty)} \Omega^\flat$ where Ω is a metrizable locally compact space, and that κ is a well-ordered set as above. Then there exists a compact family of prime ideals $(P_\alpha : \alpha \in \kappa)$ in $\mathcal{C}_0(\Omega)$, where each ideal is supported at p, satisfying the following conditions.*

(a) *$(P_{\alpha_n} : n \geq n_0)$ is a pseudofinite sequence with union P_α for some $n_0 \in \mathbb{N}$ whenever (α_n) converges to α in the order topology of κ.*

(b) *$P_\alpha \subseteq P_\beta$ if and only if $\alpha \ll \beta$.*

Let $\mathfrak{P}$ be any relatively compact family of prime ideals with the same intersection as $\bigcap_{\alpha \in \kappa} P_\alpha$. Then the closure of $\mathfrak{P}$ contains a chain of length $d + 1$. In particular, $\mathfrak{P}$ is not the union of any finite collection of many pseudofinite subfamilies of prime ideals when $d > 1$.

PROOF. Let $(P_\alpha : \alpha \in \kappa)$ be the family constructed in Lemma 7.4.

(a) Without loss of generality, we assume that $\alpha_n \neq \alpha$ for all $n \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ such that $\alpha_n \ll \alpha$ for all $n \geq n_0$. We see that $P_{\alpha_n} \subseteq P_\alpha$ for all $n \geq n_0$ and

$$P_\alpha = \bigcup_{\substack{\beta \in \kappa_0 \\ \beta \ll \alpha}} P_\beta.$$

So by Theorem 7.5(c), for each $g \in P_\alpha$, there exists $\beta_1 \in \kappa_0$ with $\beta_1 \ll \alpha$ such that $g \in P_\beta$ for all $\beta \in \kappa_0$ with $\beta_1 \leq \beta \leq \alpha$. Choose $n_1 \geq n_0$ such that $\alpha_n \geq \beta_1$ for all $n \geq n_1$. Let $n \geq n_1$ be arbitrary. Pick $\beta' \in \kappa_0$ such that $\beta' \ll \alpha_n$. If $\beta' \leq \beta_1$,

then $\beta_1 \ll \alpha_n$ by Lemma 7.1(iii), and so $g \in P_{\beta_1} \subseteq P_{\alpha_n}$. Otherwise $\beta_1 < \beta'$, then $g \in P_{\beta'} \subseteq P_{\alpha_n}$. Thus $(P_{\alpha_n} : n \geq n_0)$ is a pseudofinite sequence whose union is P_α.

(b) This is a consequence of Lemma 7.4(a) and (b).

In the order topology, κ is a compact metrizable space; each sequence in κ has a convergence subsequence. The compactness of $\{P_\alpha \mid \alpha \in \kappa\}$ thus follows from (a).

Let $\mathfrak{P}$ be a relatively compact family of prime ideals such that

$$\bigcap_{P \in \mathfrak{P}} P = I = \bigcap_{\alpha \in \kappa} P_\alpha.$$

Denote by $\mathfrak{P}_*$ the subfamily consisting of minimal elements of $\mathfrak{P}$. By Proposition 5.5(iii), we see that $\bigcap_{P \in \mathfrak{P}_*} P = I$. Lemma 5.6 shows that, for each $P \in \mathfrak{P}_*$ there exists $f_P \notin P$ but $f_P \in Q$ for all $Q \in \mathfrak{P}_* \setminus \{P\}$; in particular, $P = I : f_P$. This and Lemma 4.8 then imply that $\mathfrak{P}_*$ is exactly the set of prime ideals of the form $I : f$ for some $f \in \mathcal{C}_0(\Omega)$. Similarly, $\{P_\alpha \mid \alpha \in \kappa_0\}$ is also exactly the set of prime ideals of the form $I : f$ for some $f \in \mathcal{C}_0(\Omega)$. Thus

$$\mathfrak{P}_* = \{P_\alpha \mid \alpha \in \kappa_0\} \quad \text{and its closure is } \{P_\alpha \mid \alpha \in \kappa\}.$$

Therefore, whenever $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_d = \max \kappa$, we have a chain of length $d + 1$ in the closure of $\mathfrak{P}_*$, namely $P_{\alpha_0} \subset \cdots \subset P_{\alpha_d}$. Obviously, this shows that $\mathfrak{P}$ cannot be the union of any finite number of pseudofinite families when the depth of κ is larger than 1. $\square$

REMARK 7.6. Since the cardinality of $\mathcal{C}(\Xi)$ is $\mathfrak{c}$, we see that, for $(P_\alpha : \alpha \in \kappa)$ as in the above theorem,

$$\left| \mathcal{C}_0(\Omega) \Big/ \bigcap_{\alpha \in \kappa} P_\alpha \right| = \mathfrak{c}.$$

Let Ω be any metrizable locally compact space with $\partial^{(\infty)}\Omega \neq \emptyset$. Let $\mathfrak{P}$ be a relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$. Since we are mostly interested in $\bigcap_{P \in \mathfrak{P}} P$, by passing to the minimal elements of $\mathfrak{P}$, we can suppose that $P \not\subset Q$ for each distinct $P, Q \in \mathfrak{P}$. Let us call a family satisfying this condition *reduced*.

QUESTION. Is there a reduced relatively compact family of prime ideals $\mathfrak{P}$ in $\mathcal{C}_0(\Omega)$ whose closure contains an uncountable chain or (merely) an infinite chain? If there exists such a reduced family $\mathfrak{P}$ whose closure contains an uncountable chain, is it still possible to remove the countability condition from Theorems 6.7 and 6.8?

We know that every chain in the closure of a relatively compact family of prime ideals must be well ordered (Proposition 5.5(iii)). However, we show in [16] the existence of uncountable well-ordered chains of prime ideals in $\mathcal{C}_0(\Omega)$. We also show in [16] that there is an uncountable, nonredundant, pseudofinite family of prime ideals in $\mathcal{C}_0(\Omega)$, thus, in particular, showing that there exists an uncountable, reduced, relatively compact family of prime ideals (however, every chain in the closure of a pseudofinite family of prime ideals has length at most 2).

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