ON THE PRODUCT OF $L(1, \chi)$

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Let $k(\geqq 3)$ be a positive integer and $\varphi(k)$ be the Euler function. We denote by χ one of the $\varphi(k)$ characters formed with modulus k, and by χ_0 the principal character. Let $L(s, \chi)$ be the L-series corresponding to χ. Throughout the paper we use c and $c(\varepsilon)$ to denote respectively an absolute positive constant and a positive constant depending on parameter $\varepsilon(>0)$ alone, not necessarily the same at their various occurrences. We use the symbol $Y = O(X)$ for positive X when there exists a c satisfying $|Y| \leqq cX$ in the full domain of existence of X and Y.

It is well known that

$$L(1, \chi) = O(\log k), \quad \text{for} \quad \chi \neq \chi_0. \tag{1}$$

On the other hand. we know from [7] and [5] (numbers in square brackets refer to the references at the end of the paper) that

$$L(1, \chi)^{-1} = O(\log k), \quad \text{for} \quad \chi \neq \chi_0, \tag{2}$$

with one possible exception, and if such a exceptional character exists, it is a real one. Let us denote it by χ_1. If there exists no exceptional character, we take any non-principal character as χ_1. Then, by Siegel's theorem (see [3] and [8]),

$$c(\varepsilon)k^{-\varepsilon} < |L(1, \chi_1)| \tag{3}$$

for any positive ε.

The object of this paper is to estimate $\prod_{\chi \neq \chi_0} L(1, \chi)$ and $\prod_{\chi(-1)=-1} L(1, \chi)$ as precisely as possible, and make some additions to the results of R. Brauer [2], Ankeny and Chowla [1].

1. In what follows, we denote by p and p_ν the primes.

LEMMA 1. $\qquad \sum_{p \leqq x} p^{-1} = O(\log \log x), \quad \text{for} \quad x \geqq 3.$

LEMMA 2. $\qquad c(\log x)^{-1} < \prod_{7 \leqq p \leqq x} (1 - p^{-1}) < c(\log x)^{-1}.$

These are obtained by Theorem 7 of [4].

Let $7 \leqq p_1 < p_2 < \ldots < p_m$. We denote by $F(x; k, l; p_1, p_2, \ldots, p_m)$ the

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number of z satisfying

$$0 < z \leqq x, \quad z \equiv l \pmod{k}, \quad \text{and} \quad p_\nu \nmid z \quad \text{for} \quad \nu = 1, 2, \ldots, m.$$

If we take positive integers $m = m_0, m_1, \ldots, m_{h-1}$ such that

$$m = m_0 > m_1 > m_2 > \ldots > m_{h-1} > m_h = 0,$$

$$L_\nu = \prod_{m_{\nu-1} \geqq r > m_\nu} (1 - p_r^{-1}) \geqq \frac{5}{4}, \quad \text{for} \quad \nu = 1, 2, \ldots, h,$$

$$(1 - p_{m_\nu}^{-1}) L_\nu < \frac{4}{5}, \quad \text{for} \quad \nu = 1, 2, \ldots, h-1, \tag{4}$$

then we have

LEMMA 3. $F(x; k, l; p_1, p_2, \ldots, p_m) < 2xk^{-1} \prod_{\nu=1}^{h} L_\nu + \prod_{\nu=0}^{h-1} (2m_\nu)^2$, for $(k, l) = 1$.

The proof is similar to that of Theorem 79 and Theorem 86 of [6].

LEMMA 4. If $x = 2\varphi(k)$, then

$$\pi(x; k, l) < c \frac{x}{\varphi(k) \log (x/\varphi(k))},$$

for $(k, l) = 1$, where $\pi(x; k, l)$ denotes the number of primes p satisfying $p \leqq x$ and $p \equiv l \pmod{k}$.

Suppose that $x \geqq 7^a$, $a (> 1)$ being a positive number to be determined later. We arrange all primes between 7 and $\sqrt[a]{x}$ except the prime factors of k such that

$$7 \leqq p_1 < p_2 < \ldots < p_m \leqq \sqrt[a]{x}.$$

If we write

$$D(w, k) = \prod_{\substack{7 \leqq p \leqq w \\ p \nmid k}} (1 - p^{-1})^{-1},$$

then

$$\prod_{\nu=1}^{h} L_\nu = \prod_{r=1}^{m} (1 - p_r^{-1}) = \prod_{7 \leqq p \leqq \sqrt[a]{x}} (1 - p^{-1}) D(\sqrt[a]{x}, k). \tag{5}$$

By Lemma 2,

$$\prod_{r=1}^{s} (1 - p_r^{-1}) < c (\log p_s)^{-1} D(p_s, k) < c (\log 2s)^{-1} D(\sqrt[a]{x}, k)$$

for $s = 1, 2, \ldots, m$, it follows therefore that

$$\log (2m_\nu) < cD(\sqrt[a]{x}, k) \prod_{r=1}^{m_\nu} (1 - p_r^{-1})^{-1}$$

$$= cD(\sqrt[a]{x}, k)\prod_{r=1}^{m}(1-p_r^{-1})^{-1}L_1L_2\ldots L_\nu$$

$$= c\prod_{7\leqq p\leqq \sqrt[a]{x}}(1-p^{-1})^{-1}L_1L_2\ldots L_\nu$$

$$(6)\qquad < (c/a)(\log x)L_1L_2\ldots L_\nu < (c/a)(14/15)^\nu \log x$$

by (4), whence follows

$$(7)\qquad \prod_{\nu=1}^{h-1}(2m_\nu)^2 < x^{c/a}.$$

On the other hand

$$(8)\qquad \prod_{\nu=1}^{h}L_\nu < ca(\log x)^{-1}D(\sqrt[a]{x}, k) \leqq ca(\log x)^{-1}\prod_{p\mid k}(1-p^{-1})^{-1}$$

$$= ca(\log x)^{-1}k\varphi(k)^{-1},$$

by (5) and Lemma 2.

Inserting (7) and (8) for the right of Lemma 3,
we obtain

$$F(x\,;\,k,\,l\,;\,p_1,\,p_2,\,\ldots,\,p_m) < ca\frac{x}{\varphi(k)\log x} + x^{c/a}$$

for $(k, l)=1$, provided that $x\geqq 7^a$. This, together with the inequality

$$\pi(x\,;\,k,\,l) \leqq F(x\,;\,k,\,l\,;\,p_1,\,p_2,\,\ldots,\,p_m) + m,$$

gives

$$(9)\qquad \pi(x\,;\,k,\,l) < ca\frac{x}{\varphi(k)\log x} + 2x^{c/a}$$

by (6), where we may suppose that the constants c in both terms of the right are the same and $c>1$.

Suppose first that $x\geqq 7^{2c}\varphi(k)$. Then we have

$$(10)\qquad \Delta = \frac{x}{\varphi(k)}\Big/\log\left(\frac{x}{\varphi(k)}\right) \geqq \sqrt{\frac{x}{\varphi(k)}} \geqq 7^c.$$

we can easily verify from (10) that the restriction $x\geqq 7^a$ is satisfied, if we put

$$a = \frac{c\log x}{\log \Delta}.$$

Inserting this in (9), and using the first inequality of (10), we get

$$\pi(x\,;\,k,\,l) < c^2\frac{x}{\varphi(k)\log \Delta} + 2\Delta < 2(c^2+1)\Delta,$$

which is just what the Lemma claims.

Next we consider the trivial case, $2\varphi(k) \leqq x < 7^{2c}\varphi(k)$. Then

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$$\pi(x;\, k,\, l) < \left[\frac{x}{k}\right] + 1 \leqq \frac{x}{k} + 1 < \frac{2x}{\varphi(k)} < 4c(\log 7)\Delta.$$

Thus the Lemma is completely proved.

2. We write for simplicity

$$Q(x) = \sum_{\chi \neq \chi_0,\, \chi_1} \sum_{n \leqq x} \chi(n)\Lambda(n),$$

where $\Lambda(n)$ is $\log p$ if n is a positive power of a prime p and is 0 otherwise.

LEMMA 5. If $k \leqq \exp(\sqrt{\log x})$ and $\chi \neq \chi_0$, χ_1, then

$$\sum_{n \leqq x} \chi(n)\Lambda(n) = O(x \exp(-c\sqrt{\log x})).$$

This is the result of Page [7].

LEMMA 6. If $x \geqq \exp(\log k)^3$, then $Q(x) = O(x(\log x)^{-1})$.

By Lemma 5,

$$Q(x) = O(kx \exp(-c\sqrt{\log x})) = O(x \exp(\log k - c\sqrt{\log x}))$$
$$= O(x \exp((\log x)^{\frac{1}{3}} - c(\log x)^{\frac{1}{2}})) = O(x(\log x)^{-1}),$$

since $\log k \leqq \sqrt[3]{(\log x)}$.

3. Let a, b and n be positive integers.

LEMMA 7. If $(a, k) = 1$, then the number of solutions of

$$x^n \equiv a \pmod{k}$$

is at most $n^{\omega(k)+1}$ where $\omega(k)$ denotes the number of prime factors of k.

This follows from the fact that the number of solutions of $x^n \equiv a \pmod{p^b}$, $(a, p) = 1$, is at most n if p is an odd prime, and n^2 if $p = 2$.

LEMMA 8. $$\omega(k) = O(\log k(\log\log k)^{-1}).$$

Suppose that

$$\omega(k) = r,\ k = p_1^{b_1} p_2^{b_2} \ldots p_r^{b_r} \quad \text{and} \quad p_1 < p_2 < \ldots < p_r.$$

By the prime number theorem,

$$cr \log p'_r < \sum_{\nu=1}^{r} \log p'_\nu \leqq \sum_{\nu=1}^{r} \log p_\nu \leqq \log k, \tag{11}$$

where p'_r denotes the r-th prime. If $r > \log k(\log\log k)^{-1}$, then

$$p'_r > cr \log r > c \log k.$$

This combined with (11) gives $r < c(\log k)(\log\log k)^{-1}$.

LEMMA 9. $$\sum_{n \geqq 2} n^{-1} \sum_{\substack{p \leqq k \\ p^n \equiv 1 \pmod{k}}} p^{-n} = O(\omega(k)k^{-1}).$$

By Lemma 7

$$(12) \qquad \sum_{n\geqq 2} n^{-1} \sum_{\substack{p\leqq k \\ p^n\equiv 1 \pmod k}} p^{-n} = O\Big(\sum_{n\geqq 2} n^{-1} \sum_{(n^{\omega(k)+1})} (km+1)^{-1}\Big),$$

where $\sum_{(*)}$ means that the number of terms of the sum is $\leqq *$. If we put

$$a_0 = 0, \quad a_n = \sum_{\nu=1}^{n} \nu^{\omega(k)+1}$$

for $n = 1, 2, \ldots$, then

$$\sum_{n\geqq 2} n^{-1} \sum_{(n^{\omega(k)+1})} m^{-1} = O\Big(\sum_{n\geqq 1} n^{-1} \sum_{a_{n-1}<m\leqq a_n} m^{-1}\Big).$$

Since

$$\begin{aligned}\sum_{a_{n-1}<m\leqq a_n} m^{-1} &< \log(a_n/a_{n-1}) \\ &< \log\Big(\int_0^{n+1} x^{\omega(k)+1}dx \Big/ \int_0^{n-1} x^{\omega(k)+1}dx\Big) \\ &= (\omega(k)+2)\log((n+1)/(n-1)) = O(\omega(k)n^{-1})\end{aligned}$$

for $n \geqq 2$, it follows from (12) that

$$\sum_{n\geqq 2} n^{-1} \sum_{\substack{p\leqq k \\ p^n\equiv 1 \pmod k}} p^{-n} = O(\omega(k)k^{-1}\sum_{n\geqq 1} n^{-2}) = O(\omega(k)k^{-1}).$$

4. We write

$$\begin{aligned}\prod_{\chi\neq\chi_0,\chi_1} L(1,\chi) &= \exp\Big(\sum_{\chi\neq\chi_0,\chi_1} \sum_{n\geqq 2} \chi(n)\Lambda(n)(n\log n)^{-1}\Big) \\ &= \exp\Big(\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4 + \sum\nolimits_5 + \sum\nolimits_6\Big), \text{ say,}\end{aligned}$$

where

$$\sum\nolimits_1 = \sum_{p<A} p^{-1}\Big(\sum_{\chi}\chi(p)\Big), \quad \sum\nolimits_2 = \sum_{A\leqq p<B} p^{-1}\Big(\sum_{\chi}\chi(p)\Big),$$

$$\sum\nolimits_3 = -\sum_{p<B} p^{-1}(\chi_0(p)+\chi_1(p)),$$

$$\sum\nolimits_4 = \sum_{p^n<B}\sum_{n\geqq 2} (np^n)^{-1}\Big(\sum_{\chi}\chi(p^n)\Big),$$

$$\sum\nolimits_5 = -\sum_{p^n<B}\sum_{n\geqq 2} (np^n)^{-1}(\chi_0(p^n)+\chi_1(p^n)),$$

$$\sum\nolimits_6 = \sum_{B\leqq n}\sum_{\chi\neq\chi_0,\chi_1} \chi(n)\Lambda(n)(n\log n)^{-1}$$

and

$$A = 2k, \quad B = [\exp(\log k)^3].$$

Now we shall estimate $\sum_\nu$ using the above lemmas.

$$\sum\nolimits_1 = \varphi(k)\sum_{\substack{p<A \\ p\equiv 1 \pmod k}} p^{-1} \leqq \varphi(k) \sum_{mk+1<A} (mk+1)^{-1} = O(1).$$

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$$\sum_2=\varphi(k)\sum_{\substack{A\leqq p<B\\ p\equiv 1\ (\mathrm{mod}\ k)}}p^{-1}=\varphi(k)\sum_{A\leqq n<B}(\pi(n;\ k,1)-\pi(n-1;\ k,1))n^{-1}$$

$$=\varphi(k)\Big(\sum_{A\leqq n<B}\pi(n;\ k,1)(n^{-1}-(n+1)^{-1})-\pi(A-1;\ k,1,)A^{-1}$$

$$+\pi(B-1;\ k,1)B^{-1}\Big)$$

$$=\mathrm{O}\Big(\sum_{A\leqq n<B}(n+1)^{-1}\Big(\log\frac{n}{\varphi(k)}\Big)^{-1}+\mathrm{O}(1)\Big)\quad\text{(by Lemma 4)}$$

$$=\mathrm{O}\Big(\int_A^B\frac{du}{u\log(u/\varphi(k))}\Big)+\mathrm{O}(1)=\mathrm{O}(\log\log k).$$

$$\sum_3=\mathrm{O}(\sum_{p<B}p^{-1})=\mathrm{O}(\log\log B)=\mathrm{O}(\log\log k).\quad\text{(by Lemma 1)}$$

$$\sum_4=\varphi(k)\sum_{\substack{p^n<B\\ p^n\equiv 1\ (\mathrm{mod}\ k)}}\sum_{n\geqq 2}(np^n)^{-1}\leqq\varphi(k)\sum_{\substack{p\\ p^n\equiv 1\ (\mathrm{mod}\ k)}}\sum_{n\geqq 2}(np^n)^{-1}$$

$$=\varphi(k)\sum_{\substack{p>k\\ p^n\equiv 1\ (\mathrm{mod}\ k)}}\sum_{n\geqq 2}(np^n)^{-1}+\varphi(k)\sum_{n\geqq 2}n^{-1}\sum_{\substack{p\leqq k\\ p^n\equiv 1\ (\mathrm{mod}\ k)}}p^{-n}$$

$$=\mathrm{O}(\varphi(k)\sum_{p>k}(p(p-1))^{-1})+\mathrm{O}(\omega(k))\quad\text{(by Lemma 9)}$$

$$=\mathrm{O}(\omega(k)).$$

$$\sum_5=\mathrm{O}(\sum_p\sum_{n\geqq 2}(np^n)^{-1})=\mathrm{O}(\sum_p(p(p-1))^{-1})=\mathrm{O}(1).$$

$$\sum_6=\sum_{B\leqq n}(Q(n)-Q(n-1))(n\log n)^{-1}$$

$$=\sum_{B\leqq n}Q(n)((n\log n)^{-1}-((n+1)\log(n+1))^{-1})$$

$$-Q(B-1)(B\log B)^{-1}$$

$$=\mathrm{O}\Big(\sum_{B\leqq n}\frac{n}{\log n}\int_n^{n+1}\frac{\log x+1}{x^2\log^2 x}dx+\frac{1}{\log^2 B}\Big)\quad\text{(by Lemma 6)}$$

$$=\mathrm{O}\Big(\int_B^\infty\frac{dx}{x\log^2 x}\Big)=\mathrm{O}(1).$$

Collecting these results, we obtain

$$\exp(-c(\log\log k+\omega(k)))<\prod_{\chi\neq\chi_0,\chi_1}L(1,\chi)<\exp(c(\log\log k+\omega(k))).$$

This, combined with (1), (3) and Lemma 8, gives the following

THEOREM 1. $c(\varepsilon)k^{-\varepsilon}<|\prod_{\chi\neq\chi_0}L(1,\chi)|<\exp(c(\log\log k+\omega(k)))$.

In a similar way, we have

THEOREM 2. $c(\varepsilon)k^{-\varepsilon}<|\prod_{\chi(-1)=-1}L(1,\chi)|<\exp(c(\log\log k+\omega(k)))$.

5. For the cyclotomic field $\Omega=P(\zeta)$, $\Omega_0=P(\zeta+\zeta^{-1})$, where ζ is a l-th root of unity, l being an odd prime, it is well known that

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$$(13)\qquad \prod_{\chi \neq \chi_0} L(1, \chi) = \frac{(2\pi)^m Rh}{2\, l\sqrt{|d|}}$$

$$(14)\qquad \prod_{\chi \neq \chi_0,\ \chi(-1)=1} L(1, \chi) = \frac{2^m R_0 h_0}{2\sqrt{|d_0|}}$$

where h, h_0, are respectively the class numbers of Ω and Ω_0 and R, R_0 the regulators of them, and further $m = \frac{l-1}{2}$, $|d| = l^{l-2}$, $|d_0| = l^{m-1}$ and $R = R_0 2^{m-1}$.

Combined (13) with Theorem 1, we can infer the following

THEOREM 3. For any positive ε,

$$2\left(\frac{l}{\sqrt{2}\,\pi}\right)^{l-1} \frac{c(\varepsilon)}{l^{\varepsilon}} < Rh < 2\left(\frac{l}{\sqrt{2}\,\pi}\right)^{l-1} (\log l)^c.$$

In this special case, the result is sharper than the one obtained by R. Brauer (see [2]) for the general finite algebraic extension.

Let h_1 be the so-called first factor of the class number of Ω. Then

$$h_1 = h/h_0 = (l\sqrt{l}^{\,m}/2^{m-1}\pi^m) \prod_{\chi(-1)=-1} L(1, \chi).$$

Combined this with Theorem 2, we can infer the following

THEOREM 4. For any positive ε,

$$2\,l\left(\frac{\sqrt{l}}{2\,\pi}\right)^{\frac{l-1}{2}} \frac{c(\varepsilon)}{l^{\varepsilon}} < h_1 < 2\,l\left(\frac{\sqrt{l}}{2\,\pi}\right)^{\frac{l-1}{2}} (\log l)^c.$$

The second inequality is better than the result obtained by Ankeny and Chowla (see [1]) on the extended Riemann hypothesis.

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