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The Characteristic Numbers of Quartic Plane Curves

Ravi Vakil

Abstract. The characteristic numbers of smooth plane quartics are computed using intersection theory on a component of the moduli space of stable maps. This completes the verification of Zeuthen's prediction of characteristic numbers of smooth plane curves. A short sketch of a computation of the characteristic numbers of plane cubics is also given as an illustration.

Contents

1 Introduction

> The nineteenth century work on finding the characteristic numbers of families of curves of higher degree is rich and lovely. Understanding it well enough to vindicate it and continue it, is possibly the most important part of Hilbert's fifteenth problem remaining open. — [FKM, p. 193]

1.1

The classical *characteristic number problem* for smooth plane curves (studied by Chasles, Zeuthen, Schubert, and other geometers of the nineteenth century) is: how many smooth degree d plane curves curves are there through a fixed general points, and tangent to b fixed general lines (if $a+b=\binom{d+2}{2}-1$)? The success of earlier geometers at correctly computing such numbers (and others from enumerative geometry), despite the lack of a firm theoretical foundation, led Hilbert to include the justification of these methods as one of his famous problems. (For a more complete introduction to the history of such problems, see [K] and S. Kleiman's introduction to [S].)

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H. G. Zeuthen predicted the characteristic numbers of curves of degree at most 4. Only with the advent of Fulton-Macpherson intersection theory have these numbers begun to be verified. The characteristic numbers of the complete cubics were rigorously calculated by P. Aluffi [A1] and S. Kleiman and R. Speiser [KSp], and the first ten characteristic numbers of the smooth quartics were computed by Aluffi [A2] and van Gastel [vG].

It is interesting to compare the results and calculations with those of Zeuthen (Section 9). Although we use quite a different compactification, unlike many other modern solutions of classical enumerative geometry problems (such as the charactersitic numbers for twisted cubics), the calculations are "similar". They give a modern verification, not only of these classical numbers, but, at least to some extent, also of a classical approach.

1.2 Sketch of Method

The classical approach is to interpret the problem as the intersection of divisors (corresponding to the incidence and tangency conditions) on the parameter space of smooth curves, an open subvariety of a projective space. A "good" compactification must be given (hopefully smooth, *e.g.* [A1], at least in codimension 1), and it must be checked that there are natural divisors on the compactification that intersect (transversely) only in the open set corresponding to smooth curves.

The method used here is as follows. Kontsevich's moduli space of stable maps gives us a compactification of the space of smooth quartics. (Explicitly: take the normalization of the component of the moduli stack corresponding generically to closed immersions of smooth curves.) Morally speaking, this compactification is "good" in the sense described above because in a concrete sense, each stable map to $\mathbb{P}^2$ is tangent to only a one-parameter family of lines, so the excess intersection problems of the Hilbert scheme approach do not arise.

This compactification is birational to the parameter space $\mathbb{P}^{14}$ of smooth quartics, on which we have divisors α' (corresponding to curves through a fixed point), β' (corresponding to curves tangent to a fixed line), and Δ (corresponding to nodal curves), and

$$\beta' = 6\alpha', \quad \Delta = 27\alpha'. \tag{1}$$

There are analogous divisors α, β, Δ_0 on the compactification, and equations (1) remain true when "lifted" to the compactification, modulo "boundary divisors". The relevant boundary divisors are determined (Sections 4 and 5), and many of their co-efficients in the "lifts" of (1) are found using one-parameter test families (Section 6). The intersections of the boundary divisors with cycles of the form $\alpha^a\beta^{13-a}$ $(0 \leq a \leq 13)$ are calculated (up to two unknowns, Section 7). Then (the "lifts" of) the equations (1) are intersected with $\alpha^a\beta^{13-a}$, giving a large number of linear equations in the unknowns (including the characteristic numbers), which can be solved (Section 8). The characteristic numbers agree with Zeuthen's predictions. For example, there are 23,011,191,144 smooth plane quartics tangent to 14 general lines.

Section 3 is a self-contained example of this approach, giving a sketch of a quick calculation of the characteristic numbers of smooth plane cubics.

1.3

In summary, this paper resolves a problem of long-standing interest by a classical approach, but using beautiful modern machinery, the theory of stable maps. If the measure of a new idea is its ability to shed light on areas of previous interest, then this is yet another example of the power of Kontsevich's moduli space of stable maps.

1.4 Acknowledgements

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2 Conventions and Background Results

2.1

We follow the same conventions as in [V2]. We work over a fixed algebraically closed field k of characteristic 0. By *scheme*, we mean scheme of finite type over k. By *variety*, we mean a separated integral scheme. By *stack* we mean Deligne-Mumford stack of finite type over k. All morphisms of schemes and stacks are assumed to be defined over k, and fibre products are over k unless otherwise specified.

If S is a Deligne-Mumford stack, then a *family of nodal curves* (or a *nodal curve*) over S is defined as usual (see [V2, 2.2] for example; see [DM] for the canonical treatment).

For basic definitions and results about *maps of nodal curves* and *stable maps*, see [FP] (or the brief summary in [V2, 2.6]). Let $\overline{\mathcal{M}}_g(\mathbb{P}^2, d)$ be the stack whose category of sections of a scheme S is the category of families of stable maps to $\mathbb{P}^2$ over S of degree d and arithmetic genus g. For definitions and basic results, see [FP]. It is a fine moduli stack of Deligne-Mumford type. There is a "universal map" over $\overline{\mathcal{M}}_g(\mathbb{P}^2, d)$ that is a family of maps of nodal curves. There is an open substack $\mathcal{M}_g(\mathbb{P}^2, d)$ that is a fine moduli stack of maps of *smooth* curves. There is a unique component of $\overline{\mathcal{M}}_g(\mathbb{P}^2, d)$ that is the closure of such maps (of dimension $3d + g - 1$); call this component $\overline{\mathcal{M}}_g(\mathbb{P}^2, d)^+$.

If $\rho\colon C \to \mathbb{P}^2 \times S$ is a family of maps of nodal curves to $\mathbb{P}^2$ over S, where S is a Deligne-Mumford stack of pure dimension e, then two classes α and β in $A^1 S$ (the operational Chow ring of S), functorial in S, were defined in [V2, Section 3]. The divisor α corresponds to maps through a fixed general point, and β corresponds to maps tangent to a fixed general line. We say that $\alpha^a \beta^b [S]$ $(a + b = e)$ are the *characteristic numbers* of the family of maps. If all the characteristic numbers of the family are 0, we say the family is *enumeratively irrelevant*. Recall conditions (*) and (**) on families of maps of nodal curves, from [V2, Section 2.4]:

(*) Over a dense open substack of S, the curve C is smooth, and ρ factors $C \xrightarrow{\alpha} C' \xrightarrow{\rho'} \mathbb{P}^2 \times S$ where ρ' is unramified and gives a birational map from C' to its image; α is a

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degree d_α map with only simple ramification (*i.e.*, reduced ramification divisor); and the images of the simple ramifications are distinct in $\mathbb{P}^2$.

(**) Over the normal locus (a dense open substack) of S, each component of the normalization of C (which is a family of maps of nodal curves) satisfies (*).

If the family satisfies condition (**), then the characteristic numbers can be interpreted enumeratively using [V2, Theorem 3.15], as counting maps (with multiplicity).

The classical *characteristic number problem* for curves in $\mathbb{P}^2$ studied by geometers of the last century is: *how many irreducible nodal degree d geometric genus g maps are there through a general points, and tangent to b general lines (if $a+b = 3d+g-1$)?* By [V2, Theorem 3.15], this number is the degree of $\alpha^a \beta^b [\overline{\mathcal{M}}_g(\mathbb{P}^2, d)^+]$.

2.2 Genus 3 Curves

Let $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ be the normalization of $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^+$.

On the Deligne-Mumford stack $\overline{\mathcal{M}}_3$, let h be the divisor that is the class of (the closure of the locus corresponding to) smooth hyperelliptic curves. Let δ_0 the the divisor corresponding to irreducible nodal curves. Let δ_1 be the divisor corresponding to nodal curves with a component of arithmetic genus 1.

3 Aside: The Complete Cubics Revisited

3.1

As an example of the method, we sketch a derivation of the characteristic numbers of smooth plane cubics. The characteristic numbers of smooth plane cubics were predicted by Zeuthen in the last century. They have since been calculated rigorously in the 1980's by Aluffi ([A1], using a smooth compactification, the complete cubics) and Kleiman and Speiser ([KSp], using codimension 1 degenerations), and more recently by the author [V2] and Graber and Pandharipande (using the theory of gravitational descendants [GP]). The numbers have also been computed (although not rigorously proved) by degeneration of the point and tangency conditions [V4].

3.2

Many verifications will be left to the reader. As an exercise, the reader may enjoy using the same method to quickly calculate characteristic numbers of smooth plane conics. (In this case, the method turns out to be identical in substance to the method of complete conics.)

Let $\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)^*$ be the normalization of the component of the moduli stack $\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)$ that is the closure of the locus of immersions of smooth curves. Then there are three enumeratively relevant boundary divisors:

(i) Δ_0 is the closure of the locus of immersions of nodal cubics,

(ii) I is the closure of the locus of 3-to-1 maps from a smooth elliptic curve onto a line in $\mathbb{P}^2$ (ramifying at 6 points), and

(iii) T is the closure of the locus of maps from curves $C_0 \cup C_1$ where C_i is smooth of genus i, the two curves meet at a node, C_0 maps to a line, and C_1 maps 2-to-1 onto a line.

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(There are three other, enumeratively irrelevant, divisors; see [V1, Lemma 5.9] for their description.) The divisor Δ_0 won't concern us in this example, but its analogue will be necessary for the quartic case.

As described in Section 2 and [V2, Section 3.16], there are also two divisors α and β such that the characteristic number of cubics through a points and tangent to b lines is the degree of $\alpha^a\beta^b[\overline{\mathcal{M}}_1(\mathbb{P}^2,3)^*]$.

On the $\mathbb{P}^9$ parametrizing plane cubics, there are analogous divisors α' and β', and $\beta' = 4\alpha'$. The two spaces are birational, with isomorphic open subschemes parametrizing closed immersions. Hence, in $A_8(\overline{\mathcal{M}}_1(\mathbb{P}^2,3)^*)$, modulo enumeratively irrelevant divisors,

$$4\alpha = \beta + tT + iI. \tag{2}$$

for some rational t and i. We can find t and i by intersecting this with suitable one-parameter families.

Consider a pencil joining a general cubic curve and a triple line. In other words, if $p(x,y,z) = 0$ describes a general cubic, consider the pencil $\lambda p(x,y,z) + \mu x^3 = 0$ with $[\lambda,\mu] \in \mathbb{P}^1$. This describes a family of nodal curves except at the point corresponding to the triple line; perform stable reduction (for maps) to complete the family, and get a map $\mathbb{P}^1 \to \overline{\mathcal{M}}_1(\mathbb{P}^2,3)^*$. On this family, compute that $T = 0$ (as the family misses T), $\alpha = 1$ (because it's a pencil), and $\beta = 2$. Check that the family intersects the Weil divisor I transversely at one point, and hence $I = 1/3$. (It is essential to work with stacks rather than schemes! The 1/3 comes from the fact that the limit stable map has an automorphism group of order 3.) Hence $i = 6$.

Next, take a pencil joining a general cubic and $x^2y = 0$. On this family, $I = 0$, $\alpha = 1$, $\beta = 3$, and $T = 1/2$. Hence $t = 2$, and (2) can be rewritten

$$4\alpha = \beta + 2T + 6I. \tag{3}$$

3.3

We can easily compute the characteristic numbers of T and I. For example, the degree of $\beta^8[I]$ (the number of maps in I tangent to 8 fixed general lines) can be computed as follows. For a map in I to be tangent to 8 general lines, the image line ℓ of the map must pass through the intersections of two pairs of these lines (there are $3\binom{8}{4} = 210$ choices of two pairs). Then the map must be a triple cover of ℓ, branched over the intersection of ℓ with the 8 lines (which are 6 points). The number of connected triple covers of ℓ with 6 given branch points is $(\frac{3^5-3}{3!}) = 40$. (Proof: Rigidify the combinatorial problem by fixing some other point in $\mathbb{P}^1$, and labeling the 3 points mapping to it. Monodromy about the six branch points gives transpositions in S_3, and the product of these transpositions must be the identity. Conversely, six such transpositions uniquely determine a cover, by the Riemann existence theorem. Thus five of the transpositions can be chosen arbitrarily (3^5 choices), and the sixth is then determined. However, the five cannot all be the same transposition, as then the cover would be disconnected (leaving $3^5 - 3$ choices). Finally, divide by 3! to account for the labeling of the 3 points.) Hence the degree of $\beta^8[I] = 210 \times 40 = 8400$. (This is actually a special case of a formula of Hurwitz [Hu].) The other non-zero characteristic numbers of I are $\alpha^2\beta^6[I] = 360$ and $\alpha\beta^7[I] = 2520$.

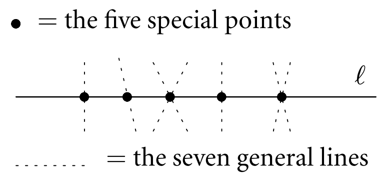

Figure 1: The five special points on ℓ

3.4

As another example, we compute the degree of $\alpha\beta^7[T]$ (the number of maps in T through a fixed point and tangent to 7 general lines, with appropriate multiplicity). For such maps, C_0 must go through the fixed point. Let ℓ be the image of C_1, and m be the image of C_0. Then ℓ must pass through the intersection of two pairs of the seven lines, so ℓ meets the seven lines at a total of five "special" points. (See Figure 1 for a pictorial representation.) Through each of three of special points one of the seven lines passes. Through each of the other two special points, two of the seven lines pass.

The cover $C_1 \to \ell$ ramifies at 4 of these 5 special points, and m meets ℓ at the fifth. Each such map is counted with multiplicity 2^a, where a is the number of the 7 lines passing through the intersection of m and ℓ (*i.e.*, the image of the node of the source curve). There are $\binom{7}{2,2,3}/2 = 105$ ways of choosing ℓ. Then m can pass through one of the three special points through which one of the seven lines pass (there are 3 ways of choosing this point, and the multiplicity is 2^1), or m can pass through one of the two special points through which two of the seven lines pass (there are 2 ways of choosing this point, and the multiplicity is 2^2).

Hence the degree of $\alpha\beta^7[T]$ is $105(3 \times 2 + 2 \times 4) = 1470$. (The other non-zero characteristic numbers of T are $\alpha^4\beta^4[T] = 24$, $\alpha^3\beta^5[T] = 240$, $\alpha^2\beta^6[T] = 885$.)

Thus the characteristic numbers of I and T can really be computed by hand.

3.5

If we intersect (3) with $\alpha^a\beta^{8-a}$ $(0 \leq a \leq 8)$, we have an equation relating two "adjacent" characteristic numbers of smooth cubics, and characteristic numbers of T and I. As the degree of $\alpha^9[\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)^*]$ is 1 (there is one smooth cubic through 9 general points), we can compute all the characteristic numbers inductively.

As an example, the degree of $\alpha\beta^8[\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)^*]$ is 21004; from this we will calculate the degree of $\beta^9[\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)^*]$. Intersecting (3) with β^8, we get

$$\deg(\beta^9[\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)^*]) = 4 \times 21004 - 2 \deg \beta^8[T] - 6 \deg \beta^8[I].$$

As $\beta^8[T] = 0$ (exercise) and $\deg \beta^8[I] = 8400$ from above,

$$\deg(\beta^9[\overline{\mathcal{M}}_1(\mathbb{P}^2, 3)^*]) = 4 \times 21004 - 6 \times 8400 = 33616.$$

Thus the characteristic numbers of plane cubics can be really be computed by hand. The moduli space of stable maps, by providing an excellent compactification of the space of smooth cubics, makes this classical problem much easier.

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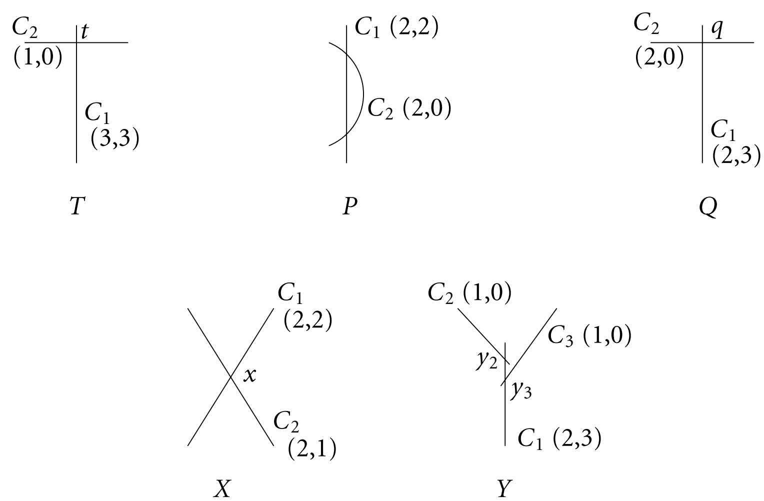

Figure 2: Source curves corresponding to general points of the boundary divisors T, P, Q, X, Y (components labeled (degree, genus))

4 Boundary Divisors

4.1

We next describe the divisors on $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ that are pertinent to the argument. Recall that a divisor B is *enumeratively irrelevant* if $\alpha^a\beta^b[B] = 0$ for all $a + b = 13$. We will see that the following families are the enumeratively relevant boundary divisors on $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$. Each locus is clearly irreducible of dimension 13. It will not be immediate that these loci lie on $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$, but that will follow from Theorem 8.2.

Let Δ_0 be the closure of the points in $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ parametrizing immersions of nodal curves. Let H be the closure of points parametrizing smooth hyperelliptic curves mapping canonically to the plane (and hence two-to-one onto a conic). Let I be the closure of points parametrizing smooth genus 3 curves mapping canonically to a line in the plane (*i.e.*, $\rho^*\mathcal{O}_{\mathbb{P}^2(1)} \cong \mathcal{K}_C$).

The boundary divisors T, P, Q, X, and Y are described in Figure 2. The source curve is given (diagrammatically), where the components are labeled C_i, and each component is labeled with an ordered pair of the degree and genus of the map (restricted to that component). For T, the component C_1 is mapped to $\mathbb{P}^2$ by the line bundle $\mathcal{K}_{C_1}(-t)$. (Equivalently, if ρ is the morphism from C_1 to $\mathbb{P}^2$, $\rho^{-1}\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{K}_{C_1}(-t)$.) The component C_1 triple-covers a line. For P, Q, X, and Y, the image of C_1 is necessarily a double-line. For Q, the point q is required to be a Weierstrass point of C_1, and the image of C_2 (a smooth plane conic) is required to be tangent to the image of C_1. For X, the map from C_2 to $\mathbb{P}^2$ (a double cover of a line) is required to ramify at the point x. For Y, the points y_2 and y_3 are required to be hyperelliptically conjugate.

Figure 3 depicts images of the maps corresponding to the general points of each of the divisors described (with ramifications of the maps indicated suggestively).

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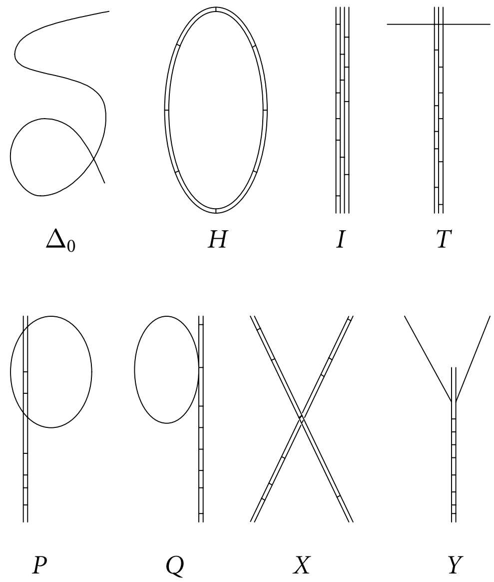

Figure 3: The images of the general maps in the boundary divisors Δ_0, H, I, T, P, Q, X, Y

The fundamental theorem of this section is the following.

Theorem 4.2 *The enumeratively relevant boundary divisors of $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ are Δ_0, H, I, T, P, Q, X, Y.*

The proof is given in Section 5.

Corollary 4.3 *Modulo enumeratively irrelevant divisors, in $A_{13}\big(\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*\big)$,*

$$6\alpha = \beta + hH + iI + tT + pP + qQ + xX + yY \tag{4}$$

$$27\alpha = \Delta_0 + h'H + i'I + t'T + p'P + q'Q + x'X + y'Y. \tag{5}$$

for some rational numbers h, i, ..., x', y'.

Proof Let $\mathcal{U} \subset \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ be the open subscheme corresponding to closed immersions of genus 3 curves. Then $\mathcal{U} \cong \mathbb{P}^{14} \setminus \Xi$ (where $\mathbb{P}^{14}$ is the Hilbert scheme parametrizing plane quartics, and Ξ is a subset of codimension greater than 1), as both sides represent the same functor. Then by standard arguments, $\beta|_{\mathcal{U}} = 6\alpha|_{\mathcal{U}}$ and $\Delta_0|_{\mathcal{U}} = 27\alpha|_{\mathcal{U}}$ in $A_{13}(\mathcal{U})$, so in $A_{13}\big(\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*\big)$, $\beta = 6\alpha$ and $\Delta_0 = 27\alpha$ modulo boundary divisors except Δ_0. ■

4.4 A Criterion for 1-Parameter Families to Intersect Divisors with Multiplicity 1

Let $\mathcal{C} \to M$ be a family of nodal curves over a stack M, such that the curve over the generic point is smooth. Let Δ be an irreducible divisor on M such that the universal curve over

Δ is singular (*i.e.*, has a node). Let $f\colon S \to M$ be a morphism from a smooth curve to M, intersecting Δ at a point $s \in S$, such that the pullback of the universal curve $\mathcal{C}$ to the generic point of S is smooth. Suppose that Δ is locally Cartier at $f(s)$. Recall that if the total space of the pullback of the universal curve $\mathcal{C}$ to S is smooth above s, then $f^*\Delta$ contains s with multiplicity one, *i.e.*, the one-parameter family intersects Δ transversely. (Sketch of proof: the formal deformation space of a node is smooth and one-dimensional; let $(D, 0)$ be this pointed space. The universal curve over D is smooth, and the universal curve pulled back to a cover of D ramified at 0 is singular. Choose any node of the curve above s. Then the map $S \to M$ induces a morphism π from a formal neighborhood of $s \in S$ to D, and as the total space of the universal curve over S is smooth above s, this map must be unramified, so π is étale. But $f^{-1}(\Delta)$ is scheme-theoretically contained in π^*0 which is the reduced point s, so $f^{-1}(\Delta)$ is a reduced point.)

4.5 Description of H, Δ_0, X in Terms of h, δ_0, δ_1

Let ψ be the natural morphism $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^* \to \overline{\mathcal{M}}_3$. Then let a be the multiplicity of ψ^*h along H, b be the multiplicity of $\psi^*\delta_0$ along Δ_0, and c be the multiplicity of $\psi^*\delta_1$ along X; a, b, and c are integers. We will see later that $a = b = c = 1$, using Criterion 4.4.

4.6 The Enumerative Geometry of I

A dimension count shows that of the 12-dimensional family of quadruple covers of $\mathbb{P}^1$ by smooth genus 3 curves, an 11-dimensional family corresponds to canonical covers. In other words, if 11 general points are fixed on $\mathbb{P}^1$, there are a finite number of quadruple canonical covers branched at those 11 points; call this number ι.

Let M be the space of genus 3 degree 4 admissible covers (with labeled branch points), and D_0 the divisor that is the closure of the locus of canonically mapped smooth curves. If $\pi\colon M \to \overline{M}_{0,12}$ is the natural map remembering only the branch points, let $D = \pi_* D_0$. (Surprisingly, D has multiplicity 120; see Section 9.1.) Let Δ_I be the boundary divisor on $\overline{M}_{0,12}$ whose general point parametrizes a curve with two components, with 2 of the marked points on one of the components, and let S be the set of boundary divisors not supported on Δ_I. Let B be the one-parameter family with 11 of the labeled points of $\mathbb{P}^1$ distinct and fixed and 1 moving, so $B\cdot\Delta_I = 11$ and $B\cdot\Delta = 0$ for any $\Delta \in S$. By symmetry, B meets each of the components of Δ_I with equal multiplicity, and D contains each of the components of Δ_I with equal multiplicity. Then as $B \cdot D = \iota$, $D \equiv \frac{\iota}{11}\Delta_I \pmod S$.

4.7 The Enumerative Geometry of T

Similar to the previous case, consider $\overline{M}_{0,11}$, where the 11 points are labeled u, $p_1, \dots,$ p_{10}. Let Δ_T be the boundary divisor where (generically) the curve has 2 components, one with two points p_i, p_j and one with the rest. Let $\Delta_{T,u}$ be the boundary divisor where (generically) the curve has 2 components, one with two points u, p_i, and one with the rest. Let S be the set of boundary divisors not supported on $\Delta_T \cup \Delta_{T,u}$. Let D be the divisor on $\overline{M}_{0,11}$ that is the closure of the pushforward of the points of the pointed Hurwitz scheme (where the marked points are a point t and the branch points $p_1, \dots, p_{10}$) corresponding to maps induced by the linear system $\mathcal{K}_C(-t)$.

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If 10 general points p_i are fixed, then there are $(3^9 - 3)/3! = 3280$ possible connected triple covers branched there (see Section 3.3 for an explanation of how to count connected triple covers). For each such cover $\pi\colon C \to \mathbb{P}^1$ there is exactly one point $t \in C$ such that π comes from the linear system $\mathcal{K}_C(-t)$: if $|\mathcal{L}|$ is the linear system corresponding to π, then $h^0(C, \mathcal{L}) \geq 2$, so by Riemann-Roch, $h^0(C, \mathcal{K} \otimes \mathcal{L}^{-1}) \geq 1$. But $h^0(C, \mathcal{K} \otimes \mathcal{L}^{-1}) < 2$ as $\mathcal{K} \otimes \mathcal{L}^{-1}$ is a degree 1 line bundle on an irrational curve, so $h^0(C, \mathcal{K} \otimes \mathcal{L}^{-1}) = 1$, and $\mathcal{K} \otimes \mathcal{L}^{-1} \cong \mathcal{O}(t)$ for a unique $t \in C$.

If 9 of the points $p_1, \dots, p_9$ and u are fixed on $\mathbb{P}^1$, then let τ be the number of genus 3 triple covers $\pi\colon C \to \mathbb{P}^1$ branched at the 9 points $p_1, \dots, p_9$ (and one other) with a point $t \in C$ with $\pi(t) = u$, such that π is induced by the linear series $\mathcal{K}_C(-t)$.

If B is the family described two paragraphs previously (with the p_i fixed and the u moving) then $B \cdot \Delta_T = 0$, $B \cdot \Delta_{T,u} = 10$, and $B \cdot D = 3280$. If B' is the family described in the preceding paragraph (with $p_1, \dots, p_9$ and u fixed), then $B' \cdot \Delta_T = 9$, $B' \cdot \Delta_{T,u} = 1$, and $B' \cdot D = \tau$. Hence $D \equiv \left(\frac{\tau - 328}{9}\right) \Delta_T + 328 \Delta_{T,u} \pmod{S}$.

4.8 Description of I as a Degeneracy Locus

Let $\pi\colon C \to S$ be a family of smooth genus 3 curves, and $\mathcal{L}$ an invertible sheaf on C of (relative) degree 4, with sections s_0, s_1, $s_2 \in h^0(C, \mathcal{L})$ giving a base-point free family of stable maps $C \to \mathbb{P}^2 \times S$. This induces a morphism $S \to \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^+$. Suppose this lifts to a morphism $\phi\colon S \to \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ (*e.g.* if S is normal), and suppose further that $\phi(S)$ is not contained in I. The subset of S where the curve maps to a line is a degeneracy locus (where the dimension of the vector space spanned by s_0, s_1, and s_2 in a fiber is at most 2 [F, Ch. 14]).

Lemma 4.9 *If m_{degen} is the multiplicity with which an irreducible Weil divisor D appears in the degeneracy locus, and m_I is the multiplicity with which D appears in $\phi^* I$, then $m_{\text{degen}} = m_I$.*

Proof If $S = \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ (with the universal family, and the sections given by s_0, s_1, s_2 given by pullbacks of the co-ordinates x, y, z on $\mathbb{P}^2$) and $D = I$, then $m_I = 1$, and m_{degen} is a positive integer k. By pulling back to an appropriate family, we see that $k = 1$—for example, fix a general genus 3 curve C and 3 general sections s_0, s_1, s_2' of $\mathcal{K}_C$, and consider the family $C \times \mathbb{A}^1 \to \mathbb{P}^2 \times \mathbb{A}^1$ (with co-ordinate t on $\mathbb{A}^1$) given by

$$C \xrightarrow{(s_0, s_1, t s_2')} \mathbb{P}^2$$

($t \in k$). This family has $m_{\text{degen}} = m_I = 1$.

Finally, if S is any other family of maps inducing a morphism $\phi\colon S \to \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$, then the degeneracy locus and $\phi^* I$ are both pullbacks of the analogous loci on $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$, so $m_{\text{degen}} = m_I$ on this family as well. ■

5 Proof of Theorem 4.2

This proof consists of rather involved case-by-case analysis, and the casual reader should probably skip it.

For simplicity, we divide the proof into a series of steps.

$g\backslash d$	1	2	3	4
0	2	5	8	11
1		6	9	12
2		8	10	13
3		10	12	14

Table 1: Maximum intersection dimension of families of maps of irreducible curves

5.1

If $\mathcal{C} \to \mathbb{P}^2$ is a family of stable maps over S, define the *intersection dimension* of the family (denoted $\operatorname{idim} S$) to be the largest integer n such that there is an integer a ($0 \leq a \leq n$) so there are maps in the family through a fixed general points and tangent to $n-a$ fixed general lines. (Recall that a line $\ell \subset \mathbb{P}^2$ is tangent to a map $\rho\colon C \to \mathbb{P}^2$ if $\rho^*\ell$ is not a union of reduced points.) Clearly $\operatorname{idim}(S) \leq \dim(S)$ (this is a consequence of [V2, Section 3]; $\operatorname{idim}(S)$ is also bounded by the image of the S in the moduli space of stable maps). Thus the theorem asserts that the only boundary divisors of $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ that have intersection dimension 13 are those listed. For the rest of the proof, suppose Ξ is an irreducible boundary divisor of intersection dimension 13.

5.2

If $\mathcal{C} \to \mathbb{P}^2$ is a family of degree d genus g maps over an irreducible scheme S ($1 \leq d \leq 4$, $0 \leq g \leq 3$) and the curve over a general k-point of S is irreducible, then it is easy to verify that the intersection dimension of the family is at most that given in Table 1, and that if equality holds, then the general source curve must be smooth. Note that if $d > g$ then the maximum is $3d + g - 1$ (which is the virtual dimension of the moduli space of degree d genus g stable maps to $\mathbb{P}^2$).

5.3

Suppose that the general (source) curve has a component of arithmetic genus 3 that maps with degree 4. Then this is the only component of the general curve. If the image of the general curve is reduced, then (as the general map in Ξ isn't an immersion of a smooth curve), the image of Ξ in $\mathbb{P}^{14}$ must be the discriminant locus. Then $\Xi = \Delta_0$.

If the image of the general curve is non-reduced, then it is either a double conic or a quadruple line. (As the general curve is irreducible, in the first case the conic must be smooth.) If the general map is a double cover of a smooth conic, then Ξ lies in H. As $\dim H = 13$ and H is irreducible, $\Xi = H$. If the general map is a quadruple cover of a line, then as $\Xi \subset \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$, the general map is a limit of canonical maps. As the general curve in Ξ is irreducible, the general map is given by the canonical sheaf (*i.e.*, the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ is isomorphic to the canonical sheaf), so $\Xi = I$ (as $\dim I = 13$ and I is irreducible).

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5.4

Suppose a component of the general curve over Ξ has arithmetic genus 3 and maps with degree 3. Then the general curve must have one other component, with genus 0 and degree 1 (and the two components meet at one node). As $\Xi \subset \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$, the map from a general curve is a limit of canonical maps. As the general curve is of compact type (*i.e.*, the dual graph is a tree), the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ to the general curve must be the line bundle described in the definition of T (see Section 4.1). Hence $\Xi \subset T$, so $\Xi = T$ (as $\dim T = 13$ and T is irreducible).

5.5

Suppose a component of the general curve has arithmetic genus 3 and maps with degree 2. Then the general curve must be one of the possibilities shown in Figure 4. In the first case, C_1 meets two components of genus 0, each mapping with degree 1. In the second case, C_1 meets (at one point q) a union of components of total arithmetic genus 0, mapping with total degree 2.

In case i), as the map is a limit of canonical maps, and the source curve is of compact type, then for some integers n_2, n_3 the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ to C_1 is $\mathcal{K}_{C_1}\big((1-n_2)y_2 + (1-n_3)y_3\big)$, and the pullback to C_i $(i = 2, 3)$ is $\mathcal{K}_{C_i}\big((1+n_i)y_i\big)$. From the degrees of the maps on the components, $n_2 = n_3 = 2$. As the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ to C_1 has at least 2 sections,

$$h^0\big(C_1, \mathcal{K}_{C_1}(-y_2 - y_3)\big) \geq 2,$$

so $\mathcal{K}_{C_1}(-y_2 - y_3)$ must be the hyperelliptic sheaf, and y_2 and y_3 must be hyperelliptically conjugate. Hence $\Xi \subset Y$, so (as Y is irreducible of dimension 13) $\Xi = Y$.

In case ii), a similar argument (using $h^0\big(C_1, \mathcal{K}_{C_1}(-2q)\big) \geq 2$) shows that q is a Weierstrass point of C_1. We claim next that the image of C_2 meets the image of C_1 at one point. Assume otherwise. Then the images intersect at two points: the image of q, and some other point $r \in \mathbb{P}^2$. (A dimension count shows that the image of C_2 cannot include the image of C_1—such maps form a family of dimension less than 13.) Then consider the germ of this map above a formal neighborhood of r. The branch of C_2 is immersed in $\mathbb{P}^2$ and is transverse to the image of C_1 (and the two branches are not connected), so we can construct the local intersection product of $C_2 \hookrightarrow \mathbb{P}^2$ and $C_1 \to \mathbb{P}^2$ (where C_1 locally consists of two immersed branches, or one branch which double-covers the image of C_1 with simple ramification). These branches (of C_1 and C_2) intersect with multiplicity 2. By continuity of intersection products, in any deformation of this germ of a map the two branches will continue to intersect. Thus in any deformation of this germ, the image will remain singular. Hence such a map cannot be the limit of smooth maps, so our assumption is false. (Remark: this possibility does not appear to be excluded by the theory of limit linear series.)

Therefore the image of C_2 is tangent to the image of C_1, so $\Xi \subset Q$, so (as Q is irreducible of dimension 13) $\Xi = Q$.

5.6

Suppose a component of the general curve has arithmetic genus 2 and maps with degree 2. Then a quick case check shows that the general curve must be one of the possibilities

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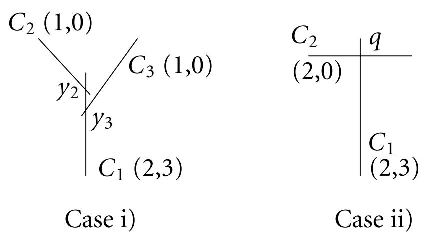

Figure 4: Possibilities for the general map in Ξ, Section 5.5 (components labeled (degree, genus))

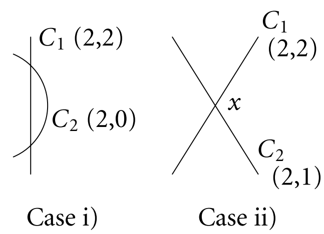

Figure 5: Possibilities for the general map in Ξ, Section 5.6 (components labeled (degree, genus))

shown in Figure 5, or the general curve has a contracted union of components of arithmetic genus 1. We save the latter case for the end of the proof, Section 5.10.

In case i), $\Xi \subset P$, so $\Xi = P$. In case ii), the map from C_2 to $\mathbb{P}^2$ is given by the line bundle $\mathcal{K}_{C_2}(2x) \cong \mathcal{O}_{C_2}(2x)$, so the double cover from C_2 ramifies at x. Hence $\Xi = X$.

We have now completed our list, so we now need to show that there are no more enumeratively relevant components.

5.7

Suppose that the general curve has no contracted components, and has no (arithmetic) genus 2 component mapping with degree 2, and no genus 3 components.

Replace Ξ by an open subscheme where the topological type of the source curve is constant. Then replace Ξ by an étale cover where the components are distinguishable (*i.e.*, the components of the universal curve correspond to components of a general k-fiber). Let c be the number of irreducible components, and let Ξ_i ($1 \leq i \leq c$) be the families of maps corresponding to the components of the universal curve over Ξ. It is straightforward to check that $\operatorname{idim} \Xi \leq \sum_i \operatorname{idim} \Xi_i$. Let n be the number of nodes connecting distinct components of the general fiber, and d_i and g_i the degree and arithmetic genus of the (map from the)

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i-th component (so $\sum_i d_i = 4$, $\sum_i (g_i - 1) + n = 2$). As $\operatorname{idim} \Xi_i \leq 3d_i + g_i - 1$,

$$\begin{aligned} 13 = \operatorname{idim} \Xi &\leq \sum_i \operatorname{idim} \Xi_i \\ &\leq 3 \sum_i d_i + \sum_i (g_i - 1) \\ &\leq 12 + (2 - n) \\ &= 14 - n. \end{aligned}$$

Hence $n = 0$ or 1. If $n = 0$, there is only one component, necessarily of arithmetic genus 3, contradicting the hypothesis of 5.7 that there are no genus 3 components. If $n = 1$, there are two components. But then one of the components must be genus 3, or genus 2 mapping with degree 2, violating the hypotheses of 5.7.

5.8

Finally, we show that the general curve of Ξ cannot have any components contracted by the map. If $C \to \mathbb{P}^2$ is a stable map, and D is a connected union of contracted components of C not meeting any other contracted components of C, we say D is a *contracted clump*. Note that if a stable map is "smoothable" (*i.e.*, can be deformed to a map from a smooth curve), then any contracted clump cannot just meet a single, immersed branch—it must meet either at least two non-contracted branches, or one contracted branch C at a point p such that the map $C \to \mathbb{P}^2$ ramifies at p. (More generally, it is also true—although not immediate—that if a stable map is smoothable, a contracted clump meets the rest of C at one point p, the image of the germ of C at p is reduced, and the map is unibranch over the image of p (no other branches of C "interfere" with the picture) then the arithmetic genus of the clump is at most the δ-invariant of the image of the germ of C at p.)

5.9

Suppose that the general curve of Ξ has at least one contracted component, no genus 2 component mapping with degree 2, and no genus 3 components.

As in 5.7, reduce to the case where the components of the universal curve over Ξ are distinguishable. Let c be the number of components mapping with positive degree to $\mathbb{P}^2$. Base change further if necessary so the nodes of the universal curve over Ξ are also distinguishable.

Construct the family Ξ' by (i) taking the closure (in the universal curve over Ξ) of the generic points of the non-contracted components (essentially discarding the contracted components), and (ii) for every contracted clump meeting more than two non-contracted branches, choose two of the branches and glue them together at a node. (To be precise, the schemes Ξ and Ξ' are the same, but the families above them are different.) Then as in 5.8, the maps in Ξ through a fixed point (resp. tangent to a fixed line) are the same as the maps in Ξ' through a fixed point (resp. tangent to a fixed line). (The gluing described above was to ensure that a line tangent to a map in Ξ because it passed through the image of a contracted clump is also tangent to the corresponding map in Ξ' because it passes through the image of a node.) Thus $\operatorname{idim}(\Xi) = \operatorname{idim}(\Xi')$: the contracted components "do not contribute to intersection dimension".

Next, let Ξ_i $(1 \leq i \leq c)$ be the families of maps corresponding to the components of the universal curve over Ξ', so c is the number of non-contracted components in Ξ. Let d_i and g_i $(1 \leq i \leq c)$ be the degree and genus of the maps in Ξ_i. Let b be the number of contracted clumps, and $h_1, \dots, h_b$ their arithmetic genera. Call the nodes on non-contracted components of the universal curve over Ξ *eligible nodes* (so each eligible node lies on at most one contracted clump). Let n be the number of eligible nodes, so

$$\sum_{i=1}^{c}(g_i - 1) + \sum_{j=1}^{b}(h_j - 1) + n = 2.$$

Then

$$\begin{aligned} 13 = \operatorname{idim}\Xi = \operatorname{idim}\Xi' &\leq \sum_{i=1}^{c} \operatorname{idim}\Xi_i \\ &\leq 3\sum_{i=1}^{c} d_i + \sum_{i=1}^{c}(g_i - 1) \\ &\leq 12 + (2 - n) + \sum_{j=1}^{b}(1 - h_j), \end{aligned}$$

so

$$n - 1 \leq \sum_{j=1}^{b}(1 - h_j). \tag{6}$$

For reasons of stability, a contracted clump with arithmetic genus 0 must have at least 3 eligible nodes. If r is the number of such "genus 0" contracted clumps, then the right side of (6) is at most r, while the left side is at least $3r - 1$, so $r = 0$.

Hence the right side of (6) is at most zero, so $n = 0$ or 1. As (by hypothesis of this step) there is a contracted component, $n > 0$, so $n = 1$, and the left side is 0. Hence $b = 1$ and $h_1 = 1$, so the map must be from a genus 2 curve C_1 mapping with degree 4, union a contracted genus 1 curve C_2, meeting at a single point p. By 5.8, the map from C_1 is not an immersion at p. The intersection dimension of the family is the same as that of the family of maps from C_1 (with the contracted component discarded), and if this is 13, then from Table 1 the general map from C_1 must be an immersion, giving a contradiction.

5.10

Finally, we take care of the remaining case from 5.6, if a component C_1 of the general curve has arithmetic genus 2 and maps with degree 2, and there is a contracted clump C_2 of arithmetic genus 1 (and at least one more component, for degree reasons). Then the genus $g_i = 2$ degree $d_i = 2$ map moves in a family of (intersection) dimension at most $(3d_i + g_i - 1) + 1$, so the same argument as in the previous step gives

$$n - 2 \leq \sum_{j=1}^{b} (1 - h_j).$$

If r is the number of "genus 0 contracted clumps", then the left side is at least $3r - 1$ (as there is at least one eligible node on the genus 1 contracted clump, and at least 3 on each genus 0), and the right side is at most r, so $r = 0$. Hence $b = 1$ and $n = 2$, and the other non-contracted component must be a rational curve C_3 mapping with degree 2. The map from C_1 moves in a family of intersection dimension at most 8, and the map from C_3 moves in a family of intersection dimension at most 5, so (as idim $\Xi = 13$) equality holds in both cases. For a general k-point in Ξ, the image of C_3 (a smooth conic) is transverse to the map of C_1 (a line) at two points; let b_1 and b_2 be these two (smooth, immersed) branches of C_3. Neither branch can be a smooth point of the total curve $C_1 \cup C_2 \cup C_3$, as then the map wouldn't be smoothable by the same argument as 5.5 Case ii). Hence one of the branches is a point attached to C_1, and the other is a point of attachment to the collapsed elliptic curve C_2 (and this accounts for both nodes of $C_1 \cup C_2 \cup C_3$). But then this contracted clump $(b_2 \cup C_2)$ isn't smoothable by Section 5.8, giving a contradiction.

This completes the proof of Theorem 4.2. ■

6 Determining Coefficients Using Test Families

We now determine as many of the unknown co-efficients in (4) and (5) as we can easily, using test families. (Although they will not be used here, other methods, such as pencils—as in Section 3—and torus actions give test families with which h, i, t, p, h', i', t', p' could be determined.)

Suppose $\pi \colon C \to S$ is a family of nodal genus 3 curves over a one-dimensional smooth base. For convenience, let $\omega := \omega_{C/S}$. Let L be an ample line bundle on S. Suppose Z is a union of components of fibers, and that the total space of C is smooth at all points of Z. Let $M = \omega(-Z) \otimes \pi^* L^n$. Suppose that for every $s \in S(k)$,

$$h^0\big(\mathcal{O}_{C_s}, \omega(-Z)|_{C_s}\big) = 3. \tag{7}$$

Then $\pi_* M$ is a rank 3 vector bundle on S (Grauert [Ha, Cor. III.12.9]). Suppose $n \gg 0$, so $\pi_* M$ is generated by global sections. Then for generally chosen sections, all degeneracy loci of $\pi_* M$ are reduced of the expected dimension [F, Example 14.3.2]). Thus three general sections of $H^0(C, M)$ determines a map of nodal curves $\rho \colon C \to \mathbb{P}^2 \times S$, and this linear system is base-point free.

6.1

Let I' be the (scheme-theoretic) degeneracy locus where the three sections are linearly dependent; I' is dimension 0, and (as S is smooth) we will denote the associated (Weil or Cartier) divisor I' as well. Note that I' and $\pi(Z)$ are disjoint. Away from the fibers above

$\pi(Z)$ and I', ρ is an immersion. For the rest of this section, assume $\rho\colon C \to \mathbb{P}^2 \times S$ is a family of stable maps whose general curve is smooth (so C has at worst A_n singularities), inducing a morphism $\phi\colon S \to \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$. (*A priori* the family only induces a morphism $S \to \overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^+$, but as S is normal, the morphism lifts to ϕ.)

Simple calculation using $\phi^*\alpha = M^2$ and $\phi^*\beta = M \cdot (M + \omega)$ [V2, 3.10] gives

$$\deg_S \phi^*\alpha = \deg_S(\omega^2 - 2\omega Z + Z^2) + 8n \deg_S L, \tag{8}$$

$$\deg_S \phi^*\beta = \deg_S(2\omega^2 - 3\omega Z + Z^2) + 12n \deg_S L. \tag{9}$$

Proposition 6.2 *If $\pi_*\mathcal{O}_C(Z) = \mathcal{O}_S$, and η is the locus of nodes of the family, then*

$$\deg_S \phi^* I = 3n \deg_S(L) + \frac{1}{12} \deg_C(\omega^2 - 6\omega Z + 6Z^2 + \eta).$$

Note that $\pi_*\mathcal{O}_C(Z) = \mathcal{O}_S$ if Z is a positive linear combination of components of fibers of π, and Z does not contain any fibers of π.

Proof By Proposition 4.9, $\phi^* I = I'$. As I' is a degeneracy locus, by [F, Ch. 14],

$$I' = c_1\big(\pi_*(M)\big) = c_1\Big(\pi_*\big(\omega(-Z)\big)\Big) + 3 \deg_S(L^n).$$

From (7) and Serre duality, $h^0\big(C_s, \mathcal{O}_C(Z)|_{C_s}\big) = 1$ for all $s \in S(k)$, so $R^1\pi_*\big(\omega(-Z)\big) = \big(\pi_*\mathcal{O}_C(Z)\big)^\vee = \mathcal{O}_S$ (by [HM, Ex. 3.12]).

By Grothendieck-Riemann-Roch,

$$\begin{aligned} ch\pi_*\omega(-Z) &= chR^1\pi_*\omega(-Z) + \pi_* \left(\left(1 - (\omega - Z) + \frac{(\omega - Z)^2}{2} \right) \cdot \left(1 - \frac{\omega}{2} + \frac{\omega^2 + \eta}{12} \right) \right) \\ &= 3 + \pi_* \left(\frac{\omega^2 + 6\omega Z + 6Z^2 + \eta}{12} \right), \end{aligned}$$

and the result follows after simple manipulation. ∎

6.3 Calculating i and i'

Fix a general genus 3 curve C_1, and let $C = C_1 \times \mathbb{P}^1$ and $S = \mathbb{P}^1$. Apply the set-up above with $Z = 0$ and $L = \mathcal{O}_S(1)$. Then of the divisors appearing in (4) and (5), only α, β, and I intersect the image of S in $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$. From (8), (9), and Prop. 6.2, $\deg_S \phi^*\alpha = 8n$, $\phi^*\beta = 12n$, $\phi^* I = 3n$. Substituting this into (4) and (5) (pulled back to S) yields $i = 12$ and $i' = 72$.

6.4 Calculating h and h'

Let $\psi\colon S \to \overline{\mathcal{M}}_3$ be any morphism from a smooth curve S, such that $\psi^* h$ and $\psi^*\delta_0$ are non-empty unions of reduced points and $\psi^*\delta_1$ is empty. (One such family is given in [HM, Ex. (3.166) part 3].) Let C be the pullback of the universal curve to S (so C is smooth).

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Apply the set-up above with $Z = 0$ and L any degree 1 (ample) divisor on S. Then the image of S in $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ misses all divisors in (4) and (5) except α, β, I, H, and Δ_0. From (8) and (9) and Prop. 6.2, $\deg_S \phi^*\alpha = \deg_C \omega^2 + 8n$, $\deg_S \phi^*\beta = 2\deg_C \omega^2 + 12n$, and $12 \deg_S \phi^* I = \deg_C \omega^2 + 36n + \deg_S \phi^*\Delta_0$. By [HM, pp. 158, 188],

$$4h = 3\pi_* \omega^2_{\tilde{C}/\overline{\mathcal{M}}_3} - \delta_0 - 9\delta_1$$

as divisors on the stack $\overline{\mathcal{M}}_3$ (where $\pi\colon \tilde{C} \to \overline{\mathcal{M}}_3$ is the universal curve). Now $\psi^* h$ is a union of reduced points, and by Section 4.5 $\psi^* h = a\phi^* H$, so $a = 1$ and $\psi^* h = \phi^* h$. Thus

$$4 \deg_S \phi^* H = 3 \deg_C \omega^2 - \deg_S \phi^* \Delta_0.$$

Substituting into (4) and (5) yields $h = 4$ and $h' = 28$.

6.5 Calculating t and t'

Fix a general genus 3 curve C_1, and let C be the blow-up of $C_1 \times \mathbb{P}^1$ at a general point with exceptional divisor E, and let $S = \mathbb{P}^1$. Apply the usual set-up with $Z = 2E$ and $L = \mathcal{O}_S(1)$. All divisors in (4) and (5) are 0 except α, β, I, and T. Then $\deg_S \phi^*\alpha = -1 + 8n$ and $\deg_S \phi^*\beta = 12n$. Also, $\deg_S \phi^* T = 1$ by Criterion 4.4. By Proposition 6.2, $12 \deg_S \phi^* I = 36n - 12$. Substituting into (4) and (5) yields $t = 6$ and $t' = 45$.

6.6 Calculating p and p'

Let $\psi\colon S \to \overline{\mathcal{M}}_3$ be any morphism from a smooth curve S such that $\psi^*\delta_1$ is empty and $\psi^*\delta_0$ is a union of reduced points plus one point p with multiplicity 2. (For example, double-cover the base of the family in Section 6.4 ramified at one of the points mapping to δ_0, and at other generally chosen points.) Let $C' \to S$ be the pullback of the universal curve over $\overline{\mathcal{M}}_3$, so C' is smooth except for an A_1-singularity above p. Let $b\colon C \to C'$ be the blow-up of C' at the singularity, with exceptional divisor E, so C is smooth and $b^*\omega_{C'/S} = \omega$.

Apply the usual construction, with $Z = E$, and L a degree 1 (ample) line bundle on S. The divisors appearing in (4) and (5) intersecting this family are α, β, P, H, I, and Δ_0. One may check that on C, $\omega \cdot Z = 0$ and $Z^2 = -2$, so $\deg_S \phi^*\alpha = \deg_C \omega^2 - 2 + 8n$, $\deg_S \phi^*\beta = 2\deg_C \omega^2 - 2 + 12n$, $\deg_S \phi^* P = 1$ (by Criterion 4.4), and $12 \deg_S \phi^* I = 36n + \deg_C \omega^2 - 12 + \deg_S \phi^*\Delta_0 + 2$ (by Proposition 6.2).

From the family $C' \to S$ (as $\deg_S \psi^*\delta_0 = \deg_S \phi^*\Delta_0 + 2$) we have (as in Section 6.4)

$$\begin{aligned} 4 \deg_S \phi^* H &= 4\psi^* h \\ &= 3 \deg_{C'} \omega^2_{C'/S} - (\deg_S \psi^*\delta_0 + 2) \\ &= 3 \deg_C \omega^2 - \deg_S \phi^*\Delta_0 - 2. \end{aligned}$$

Pulling back (4) and (5) to S and solving for p and p' yields $p = 2$ and $p' = 20$.

6.7 Calculating x and x'

Let $\psi\colon S \to \overline{\mathcal{M}}_3$ be any morphism from a smooth curve S such that $\psi^*\delta_0$ and $\psi^*\delta_1$ are unions of reduced points and $\psi^*\delta_1$ is non-empty. Let C be the pullback of the universal curve to S (so C is smooth). Let $m = \deg_S \psi^*\delta_1$, and let Z be the union of the (m) genus 1 components of fibers. Apply the usual construction with L a degree 1 (ample) line bundle on S. All divisors in (4) and (5) are 0 except α, β, I, Δ_0, H, and X. Simple calculations yield $\deg_C Z^2 = -m$ and $\deg_C \omega Z = m$, so $\deg_S \phi^*\alpha = \deg_C \omega^2 - 3m + 8n$, $\deg_S \phi^*\beta = 2\deg_C \omega^2 - 4m + 12n$. By Criterion 4.4, $\deg_C \phi^* X = m$ (so $c = 1$). By Proposition 6.2, $12\deg_S \phi^* I = \deg_C \omega^2 + \deg_S \Delta_0 - 11m + 36n$. As in Section 6.4, $4h = 3\pi_*\omega^2_{\tilde{C}/\overline{\mathcal{M}}_3} - \delta_0 - 9\delta_1$ on $\overline{\mathcal{M}}_3$, so

$$4\deg_S \phi^* H = 3\deg_C \omega^2 - \deg_S \phi^*\Delta_0 - 9m.$$

Substituting these values into (4) and (5) gives $x = 6$ and $x' = 48$.

6.8 Aside: Multiplicities of Discriminants

As a consequence, we can compute the multiplicity of the discriminant hypersurface Δ (in the parameter space $\mathbb{P}^{14}$ of quartics) at various points. Let p be a general point of the locus in $\mathbb{P}^{14}$ corresponding to the divisor H (respectively I, T, P). Then construct a family of maps by taking a general pencil through p. If m is the multiplicity of the discriminant at p, and a is the order of the automorphism group of the limit map, so $a = 2$ (resp. 4, 3, 2) then the pencil meets α with degree 1, Δ_0 with degree $(\deg\Delta - m) = 27 - m$, and H (resp. I, T, P) with multiplicity $1/a$. Then from (4), using $h' = 28$ (resp. $i' = 72$, $t' = 45$, $p' = 20$), the multiplicity of Δ at p is $m = 14$ (resp. 18, 15, 10), recovering examples of Aluffi and F. Cukierman [AC, Example 3.1].

7 Characteristic Numbers of Boundary Divisors

We next calculate the characteristic numbers of the boundary divisors; the final answers are given in Table 2. Maple code computing many of the characteristic numbers described here is available upon request. As Zeuthen had essentially calculated these before [Z, p. 391], see Section 9, we have a quick check on our numbers.

7.1 Everything But Δ_0

The characteristic numbers of the components of the families over each boundary divisor (involving maps of lower genus and/or degree) are already known. Then using [V2, Section 3], we can calculate the characteristic numbers of the boundary divisors. For the sake of brevity, we will explicitly calculate one characteristic number for each boundary divisor, and hope that the general method is clear.

On [Z, pp. 390–391], Zeuthen computes the characteristic numbers of the boundary divisors when $a = 2$, $b = 11$ as sums, without further explanation. Although his method of computing the summands is different, his summands agree with the summands computed by this method. (See Section 9 for a comparison, a glossary of notation, and further discussion.) The interested reader can use this method and use Zeuthen's sums as a check.

a	$\gamma_a \Delta_0$	$\gamma_a H$	$\gamma_a I$	$\gamma_a T$
13	27	0	0	0
12	162	0	0	0
11	972	0	0	0
10	5832	0	0	0
9	34992	0	0	0
8	209952	0	0	0
7	1256352	0	0	0
6	7453872	0	0	0
5	43393596	4096	0	0
4	242612208	110592	0	54τ
3	1268876232	1635840	0	$103320 + 1170\tau$
2	5919651072	14805120	16ι	$1523720 + 10120\tau$
1	23328812592	90549360	288ι	$8651280 + 40920\tau$
0	74651593680	403572312	2535ι	0

a	$\gamma_a P$	$\gamma_a Q$	$\gamma_a X$	$\gamma_a Y$
13	0	0	0	0
12	0	0	0	0
11	0	0	0	0
10	0	0	0	0
9	0	0	0	0
8	0	0	0	0
7	168	0	0	0
6	4536	72	0	0
5	69860	1972	0	150
4	716688	24210	4032	2700
3	5332320	177300	105840	19170
2	29220576	842160	1164240	59400
1	115886232	2561724	7609140	0
0	308287980	4487769	33648615	0

Table 2: Characteristic numbers of boundary divisors ($\gamma_a = \alpha^a \beta^{13-a}$ for brevity)

7.2 The Divisor H

We count the double covers of conics ramified at 8 points, passing through a fixed general point, and tangent to 12 fixed general lines. The double cover is tangent to a line if

(i) a branch point lies on the line, or
(ii) the image curve is tangent to the line (which will give a multiplicity of 2, for the choice of the two branches to be tangent to the line).

Thus the characteristic number is a sum over non-negative integers a, b with $a + b = 12$ (where a of the 12 lines are tangent in the sense of (i) and b are tangent in the sense of (ii)). Of the b lines, there are $4-a$ pairs such that the conic passes through the intersection of that pair (thus fixing the conic, up to a finite number of choices). The double cover branches at those $4 - a$ points, plus at one point of the conic's intersection with each of the remaining $b - 2(4 - a)$ lines; this accounts for all $8 = (4 - a) + \big(b - 2(4 - a)\big)$ ramifications. The number of such maps is a product of several terms:

- $\binom{12}{a}$ from the choice of the a lines,
- $\frac{b!}{(4-a)!\,2^{4-a}(b-2(4-a))!}$ from the choice of the $(4 - a)$ pairs of lines,
- $2^{b-2(4-a)}$ from the choice of intersection of the $b - 2(4 - a)$ lines with the conic, and
- the number of conics tangent to a general lines and through $5 - a$ general points (*i.e.*, a characteristic number of plane conics).

The multiplicity with which each such map appears is also a product of terms:

- $\frac{1}{2}$ from the automorphism of the stable map,
- 2 from the choice of pre-image of the fixed point, and
- 2^a from the choice of tangent point to the a lines.

Adding these products for $0 \le a \le 4$ gives $\alpha\beta^{12}[H] = 90549360$.

7.3 The Divisor X

Note that the general map in X has an automorphism group of order 2 (from the genus 1 component). The divisor on X corresponding to maps tangent to a line ℓ has three components. The first (resp. second) is where the genus 1 (resp. genus 2) double cover branches over ℓ, but not at a node of the source curve; this divisor appears with multiplicity 1. The third divisor is the locus where the node of the source curve maps to the line, and this divisor appears with multiplicity 3 (by [V2, Theorem 3.1]): two from the node, plus one because the genus one component ramifies simply over a general line through the node.

We now count the maps in X passing through a fixed point and tangent to 12 fixed general lines (with appropriate multiplicities). There are seven cases to consider. For convenience, let ℓ_1 be the image of the genus 1 component, and ℓ_2 the image of the genus 2 component (so ℓ_1 and ℓ_2 are lines).

The first case is if none of the 12 lines pass through the image of the node $\ell_1 \cap \ell_2$, ℓ_1 passes through the fixed point, and ℓ_1 also passes through the intersection p of a pair of the 12 lines (thus fixing the choice of ℓ_1). The line ℓ_2 passes through the pairwise intersection of two pairs of lines (fixing ℓ_2). The genus 2 cover branches at those two points, plus where

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ℓ_2 intersects 4 other of the 12 lines. The genus 1 curve branches at $\ell_1 \cap \ell_2$, the point p, and where ℓ_1 intersects 2 other of the 12 lines. Note that we have partitioned the 12 lines into 2 (whose intersection is on ℓ_1), 2 (where the genus 1 cover also branches), 2×2 (in 2 pairs, whose intersections are on ℓ_2, and 4 (where the genus 2 cover also branches).

The degree of this locus is a product of several terms:

- $\frac{1}{2}$ from the automorphism group of the map
- 2 from the choice of pre-image of the fixed point on the double cover
- $\frac{1}{2}\binom{12}{2,2,4,2,2}$ from the choice of partition of the 12 lines.

Hence this case contributes 623700.

The remaining cases are similar, and are listed in Table 3. The $\frac{1}{2}$ from the automorphisms of the map and the 2 from the choice of pre-image of the fixed point always cancel, and are omitted in the table. The total of the contributions is $\alpha\beta^{12}[X] = 7609140$.

7.4 The Divisor Y

The image of a curve in Y has a point that looks like an "asterisk". The divisor of maps in Y corresponding to maps tangent to a line ℓ includes the locus where the genus 3 curve branches over ℓ (with multiplicity 1), and the locus where the asterisk lies on ℓ (with multiplicity 4: two from each of the nodes of the source curve mapping to ℓ, by [V2, Theorem 3.1]).

We count the maps in Y through 2 general points and tangent to 11 general lines. The two genus 0 components must each pass through one of the fixed points. If m is the image of the genus 3 component, then m must pass through two intersections of pairs of the 11 lines (and there are $\frac{1}{2}\binom{11}{2,2,7} = 1980$ ways of choosing these two pairs). Of these two points plus the 7 intersections of m with the remaining lines, the genus 3 double cover must branch at 8 of them, and the asterisk must be at the ninth (contributing a multiplicity of 4 or 16, depending on the number of lines through the asterisk). Hence the characteristic number is $\alpha^2\beta^{11}[Y] = 1980(2 \times 16 + 7 \times 4) = 59400$. (A similar calculation appeared in Section 3.4.)

7.5 The Divisor P

We count the maps in P tangent to 13 fixed general lines. For convenience, let c denote the image of the rational component (a conic), and ℓ the image of the genus 2 component (a line). Note that there are two stable maps with the same c and ℓ and given branch points of the double cover of ℓ (coming from the choice of which branch of the cover the conic is glued to). This will contribute a factor of 2 to each of our calculations below.

We consider the cases where x of the lines pass through one of the nodes of the image, and y lines pass through the other ($0 \le x \le y \le 2$). Our results are summarized in Tables 4 and 5.

If $x = y = 0$, then the conic c must be tangent to 5 of the lines (fixing c), and the line ℓ must pass through 2 intersections of pairs of lines (fixing ℓ); the double cover branches at these 2 points, and also where ℓ intersects the 4 remaining lines. There are $\frac{1}{2}\binom{13}{5,4,2,2}$ ways of partitioning the lines in this way, giving (along with the factor of 2 described above) a total of 540540. If $(x, y) = (0, 2)$ or $(2, 2)$, the argument is similar. These three cases are the first three columns of Table 4.

number of lines through node	0	0	2	2
point condition on cover of genus	1	2	1	2
number of pairwise intersections of lines lines ℓ_1 passes through	1	2	0	1
number of other lines where genus 1 cover branches	2	1	3	2
number of pairwise intersections of lines ℓ_2 passes through	2	1	1	0
number of other lines where genus 2 cover branches	4	5	5	6
multiplicity from lines through node	1	1	9	9
number of partitions of 12 lines	$\frac{1}{2}\binom{12}{2,2,4,2,2}$	$\frac{1}{2}\binom{12}{1,2,2,5,2}$	$\binom{12}{2,5,3,2}$	$\binom{12}{2,2,2,6}$
total contribution (product of previous two rows)	623700	249480	1496880	748440

number of lines through node	1	1	1	1
point condition on cover of genus	1	1	2	2
number of pairwise intersections of lines lines ℓ_1 passes through	1	0	2	1
number of other lines where genus 1 cover branches	2	3	1	2
number of pairwise intersections of lines ℓ_2 passes through	1	2	0	1
number of other lines where genus 2 cover branches	5	4	6	5
multiplicity from lines through node	3	3	3	3
number of partitions of 12 lines	$\binom{12}{1,2,2,5,2}$	$\frac{1}{2}\binom{12}{1,3,4,2,2}$	$\frac{1}{2}\binom{12}{1,1,2,2,6}$	$\binom{12}{1,2,2,5,2}$
total contribution (product of previous two rows)	1496880	1247400	249480	1496880

Table 3: Calculating the characteristic number $\alpha\beta^{12}[X]$ of X

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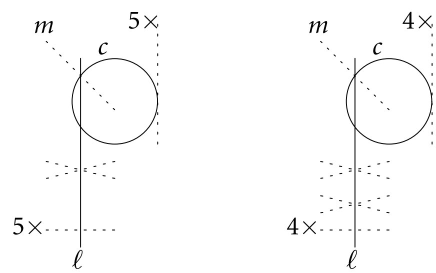

Conic c fixed first   Line ℓ fixed first

Figure 6: Calculating characteristic numbers of P: the case $(x, y) = (0, 1)$

If $(x, y) = (0, 1)$ (so an intersection of c and ℓ is required to lie on some line m), there are two possibilities, described pictorially in Figure 6. First, c could be tangent to five of the lines; ℓ would pass through one of the two intersections of c with m, and the pairwise intersection of another pair of the lines; the cover of ℓ branches at the latter point, and the intersection of ℓ with the remaining 5 lines. There are $\binom{13}{1,5,5,2} = 216216$ ways of partitioning the lines in such a way, and the other factors involved are 2 (from the 2 stable maps with the same ℓ, c, and branch points), 2 (from the choice of intersection of c with m), and 2 (the multiplicity from the line m through the node) for a total of 1729728. In this case, we say that the conic c was *fixed first* by the conditions (and then the choice of ℓ was determined using c).

Second, if the line ℓ is fixed first, the argument is similar (see the fifth column of Table 4 and the second half of Figure 6).

The case $(x, y) = (1, 2)$ breaks into two analogous subcases as well (first two columns of Table 5).

If $(x, y) = (1, 1)$, then the line can be fixed first before choosing the conic (third column of Table 5), or the conic can be fixed first (fourth column), but there is one additional case (the last column). Let m_1 and m_2 be the two lines such that c and ℓ are to intersect once on each line. The conic c is required to be tangent to 4 of the other lines, and the line ℓ is required to pass through the intersection of 2 others. (The double cover of ℓ is required to branch there, and at the intersection of ℓ with the remaining 5 lines.) The number of ways of partitioning the lines in this way is $\binom{13}{2,4,2,5} = 540540$. The four tangent lines restrict c to move in a one-parameter family, and the requirement on ℓ restricts ℓ to a one-parameter family. How often in this (combined) two-parameter family do c and ℓ intersect at two distinct points, one on m_1 and one on m_2? This straightforward enumerative question was addressed (in much more generality) in [V1]; we sketch a solution here. Let n_1 and n_2 be two $\mathbb{P}^1$'s with fixed isomorphisms $n_i \cong m_i$. Consider the surface $n_1 \times n_2$. Let P be the point on the surface corresponding to the point $m_1 \cap m_2$ in each factor. As c moves in its one-parameter family, it sweeps out a path in $n_1 \times n_2$ corresponding to pairs of points on m_1 and m_2; this path is in class $(4, 4)$—for a fixed general point on $m_2 \in \mathbb{P}^2$, there are 2 conics c tangent to the 4 lines and passing through the point, and each of those conics intersects m_1 in 2 points, and similarly with the roles of m_1 and m_2 reversed. The curve c passes through P with multiplicity 2 (by a similar argument). As l moves in a one-parameter

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(x,y)	(0,0)	(0,2)	(2,2)	(0,1)	(0,1)
component fixed first				c	ℓ
number of lines tangent to c	5	4	3	5	4
number of pairwise intersections of lines lines ℓ passes through	2	1	0	1	2
number of other lines where genus 2 cover branches	4	5	6	5	4
number of choices for conic c	1	2	4	1	2
number of choices for line ℓ	1	1	1	2	1
multiplicity from lines through nodes	1	4	16	2	2
number of partitions of 13 lines	$\frac{1}{2}\binom{13}{5,4,2,2}$	$\binom{13}{2,4,2,5}$	$\frac{1}{2}\binom{13}{2,2,3,6}$	$\binom{13}{1,5,5,2}$	$\frac{1}{2}\binom{13}{1,4,4,2,2}$
total contribution ($2\times$ product of previous 4 rows)	540540	8648640	23063040	1729728	10810800

Table 4: Calculating the characteristic number $\beta^{13}[P]$ of P, part 1

family, it sweeps out a path of pairs of points as well, and this path is in class $(1,1)$, passing through P with multiplicity 1. These paths intersect with multiplicity 8, and it can be checked that the paths intersect at P with multiplicity 2 (corresponding to when both c and l pass through $m_1 \cap m_2$). Away from P, the two paths intersect at 6 points. Hence there are 6 configurations where c and l are in the one-parameter families described above, and intersect at two distinct points, one on m_1 and one on m_2. Thus the factors contributing in this case are thus 6, 540540 (from the choice of lines), 2 (the factor described at the beginning of this note), and 4 (the multiplicity from the two lines passing through the two nodes), giving a product of 25945920.

The sum of these ten numbers is the characteristic number $\beta^{13}[P] = 308287980$.

7.6 The Divisor Q

We count the maps in Q tangent to 13 fixed general lines. This case is very similar to the case P above. For convenience, let c denote the image of the rational component (a conic), and ℓ the image of the genus 3 component (a line). The divisor on Q corresponding to maps tangent to a line m has 3 components. The first is the locus where the conic c is tangent to m. The second is the locus where the double cover of ℓ branches over m, but not at the node of the source curve. (Both of these components appear with multiplicity 1.) The third is the locus where the node of the source curve maps to m. This component appears with multiplicity 3 for the same reason as in the case X above: two from the node, plus one because the double cover of ℓ ramifies simply over a general line through the image of the node.

We consider the cases where 0, 1, and 2 lines pass through the image of the node. In

(x,y)	(1,2)	(1,2)	(1,1)	(1,1)	(1,1)
component fixed first	c	ℓ	c	ℓ	neither
number of lines tangent to c	4	3	5	3	4
number of pairwise intersections of lines lines ℓ passes through	0	1	0	2	1
number of other lines where genus 2 cover branches	6	5	6	4	5
number of choices for conic c	2	4	1	4	*
number of choices for line ℓ	2	1	4	1	*
multiplicity from lines through nodes	8	8	4	4	4
number of partitions of 13 lines	$\binom{13}{1,2,4,6}$	$\binom{13}{1,2,3,2,5}$	$\binom{13}{2,5,6}$	$\frac{1}{2}\binom{13}{2,2,2,4,3}$	$\binom{13}{2,4,2,5}$
total contribution ($2\times$ product of previous 4 rows)	11531520	138378240	1153152	86486400	25945920

Table 5: Calculating the characteristic number $\beta^{13}[P]$ of P, part 2

each of these cases, the conditions can immediately fix (up to a finite number of choices) one of the two components c or ℓ (and then the choice of that component along with the remaining conditions fix the other component, up to a finite number of choices). These possibilities are summarized in the first six columns of Table 6. (The multiplicity of 1/2 in the last row comes from the fact that the general map in Q has automorphism group of order 2.)

The one remaining case is the final column of the table. In this case, the conic c is tangent to 4 of the lines, restricting c to a one-dimensional family. The line ℓ passes through the intersection of a pair of the lines (and the double cover is required to branch there, as well as where ℓ meets 6 more lines), restricting ℓ to a pencil. The line ℓ and the conic c are required to intersect on the remaining line (call it m), and be tangent there. We determine how often this happens.

Consider the surface $S = \mathbb{P}(T_{\mathbb{P}^2}|_m)$, *i.e.*, the $\mathbb{P}^1$-bundle over m corresponding to the ordered pair (point p on m, line through p). This surface is the rational ruled surface $\mathbb{F}_1 = \mathbb{P}\big(\mathcal{O}_m \oplus \mathcal{O}_m(1)\big)$ with Picard group freely generated by the class E corresponding to ordered pairs (any point p, m), and F corresponding to ordered pairs (a fixed point p_0, line through p_0), with $E^2 = -1$, $E \cdot F = 1$, $F^2 = 0$. Define the class $C = E + F$, so $C^2 = 1$; if p_1 is a fixed point of $\mathbb{P}^2 \setminus m$, C is the class of the set (any point p, line $\overline{pp_1}$).

As ℓ moves in a pencil, it describes a curve B_1 in S corresponding to (point $\ell \cap m$, ℓ); this is in class C. As c moves in a one-parameter family, it describes a curve B_2 in S corresponding to (point p on $c \cap m$, tangent line to c at p). As there are two conics tangent to 4 fixed general lines through a fixed point $p_0 \in m$, $B_2 \cdot F = 2$. As there is one curve tangent to 4 fixed general lines and tangent to m, $B_2 \cdot E = 1$. Hence B_2 is in class $C + 2F$, so $B_1 \cdot B_2 = C \cdot (C + 2F) = 3$. (Of course, one must check that, for general choice of the lines,

number of lines through node	0	0	2	2	1	1	1
component fixed first	c	ℓ	c	ℓ	c	ℓ	neither
number of lines tangent to c	5	4	4	3	5	3	4
number of pairwise intersections of lines lines ℓ passes through	1	2	0	1	0	2	1
number of other lines where genus 3 cover branches	6	5	7	6	7	5	6
number of choices for conic c	1	1	2	1	1	1	*
number of choices for line ℓ	2	1	1	1	2	1	*
multiplicity from lines through node	1	1	9	9	3	3	3
number of partitions of 13 lines	$\binom{13}{5,2,6}$	$\frac{1}{2}\binom{13}{4,2,2,5}$	$\binom{13}{2,4,7}$	$\binom{13}{2,3,2,6}$	$\binom{13}{1,5,7}$	$\frac{1}{2}\binom{13}{1,3,2,2,5}$	$\binom{13}{1,4,2,6}$
total contribution ($\frac{1}{2}\times$ product of previous 4 rows)	36036	135135	231660	1621620	30888	1621620	810810

Table 6: Calculating the characteristic number $\beta^{13}[Q]$ of Q

all of these intersections are transverse.)

In conclusion, there are 3 ordered pairs (c, l) tangent at a point of m. Multiplying this by 3 (the multiplicity arising from the line m passing through the image of the node), $\binom{13}{1,4,2,6}$ (from the ways of partitioning the 13 lines), and $\frac{1}{2}$ (from the automorphism group of the general map in X), we see that this case contributes 810810.

Adding up the seven subtotals gives the characteristic number $\beta^{13}[Q] = 4487769$.

7.7 The Divisor I

We count the maps in I tangent to 13 fixed general lines. Let ℓ be the image of one such map in I. Such maps are in one of two forms.

The line ℓ could pass through the intersection of two pairs of the 13 lines. The quadruple cover must ramify at those 2 points, as well as the points of intersection of ℓ with the remaining 9 lines. (This specifies the canonical quadruple cover, up to a finite number of choices.) This number is ι by definition (see Section 4.6). There are $\frac{1}{2}\binom{13}{2,2,9} = 2145$ ways of partitioning the 13 lines in this case.

On the other hand, the line ℓ could pass through the intersection of a pair of the 13 lines (restricting ℓ to a pencil), and intersect the remaining 11 lines in distinct points; the cover is required to branch at these 12 points, and be a canonical map. This describes a one-parameter family C in $\overline{M}_{0,12}$ intersecting Δ_I transversely at $\binom{11}{2} = 55$ points, and missing the divisors in S (see Section 4.6 for notation). Hence the number of points in I in this family is $C \cdot \left(\frac{\iota}{11}\Delta_I\right) = 5\iota$. The number of ways of partitioning the 13 lines is $\binom{13}{2}$.

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Thus the characteristic number $\beta^{13}[I]$ is

$$\left(2145 + 5\binom{13}{2}\right)\iota = 2535\iota.$$

7.8 The Divisor T

This case is similar to the previous one. We count the maps in T tangent to 9 fixed general lines and passing through 4 fixed points. Let ℓ be the image of the genus 3 triple cover, and let m be the image of the genus 0 component (ℓ and m are both lines). Then ℓ must pass through 2 of the 4 points, and m must pass through the other 2 (so there are 6 ways of partitioning the points between the components). The 2 points on ℓ contribute a multiplicity of 3 each (from the 3 possible choices of pre-image of the point in the triple cover). The triple cover must branch where ℓ meets the 9 lines, and the node of the source curve must map to $\ell \cap m$, so by Section 4.7 there are τ points of T satisfying these conditions. Thus the characteristic number $\alpha^4\beta^9[T]$ is $6 \times 3^2 \times \tau = 54\tau$.

7.9 Characteristic Numbers of Δ_0

To calculate the characteristic numbers of Δ_0, we need to calculate

- N_a the number of degree 4 maps of smooth genus 2 curves through a fixed general points, and tangent to $13 - a$ fixed general lines (the characteristic numbers of genus 2 quartics),
- N_a^L the number of degree 4 maps of smooth genus 2 curves through a fixed general points, and tangent to $12 - a$ fixed general lines, and with the node of the image lying on another fixed general line, and
- N_a^p the number of degree 4 maps of smooth genus 2 curves through a fixed general points, and tangent to $11 - a$ fixed general lines, and with the node of the image at a fixed general point.

Then, by [V2, Theorem 3.15],

$$\deg \alpha^a \beta^{13-a}[\Delta_0] = N_a + 2\binom{13-a}{1}N_a^L + 4\binom{13-a}{2}N_a^p.$$

(The 2 and 4 come from the multiplicity from the node, and the binomial coefficients come from the choice of the $13 - a$ lines passing through the node.) The values of N_a, N_a^L, and N_a^p appear in [S, p. 187 (Section IV)].

In [GP], T. Graber and R. Pandharipande give recursions for the characteristic numbers of genus 2 plane curves in $\mathbb{P}^2$, and compute N_a, verifying Zeuthen's degree 4 numbers N_a. Their method also works for the numbers N_a^L and N_a^p ([G], although they have not explicitly verified Zeuthen's degree 4 numbers for N_a^L and N_a^p).

a	$\deg \alpha^a \beta^{14-a}[\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*]$
14	1
13	6
12	36
11	216
10	1296
9	7776
8	46656
7	279600
6	1668096
5	9840040
4	56481396
3	308389896
2	1530345504
1	6533946576
0	23011191144

Table 7: Characteristic numbers of smooth plane quartics

8 Linear Algebra

By Section 6, equations (4) and (5) can be rewritten

$$6\alpha = \beta + 4H + 12I + 6T + 2P + qQ + 6X + yY \tag{10}$$

$$27\alpha = \Delta_0 + 28H + 72I + 45T + 20P + q'Q + 48X + y'Y \tag{11}$$

(modulo enumeratively irrelevant divisors). Intersecting these relations with $\alpha^a\beta^{13-a}$ ($0 \leq a \leq 13$) and using Table 2 yields 28 equations linear in the unknowns q, q', y, y', ι, τ, and the characteristic numbers $\deg \alpha^a \beta^{13-a}[\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*]$. (Clearly $\deg \alpha^{14}[\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*] = 1$: there is one quartic through 14 general points.) Solving these equations (with the aid of Maple) yields $q = 6$, $q' = 64$, $y = 4$, $y' = 46$, $\iota = 451440$, $\tau = 1552$, and the characteristic numbers of smooth quartics:

Theorem 8.1 *The characteristic number numbers of smooth plane quartics are as given in Table 7.*

These numbers confirm Zeuthen's predictions [S, p. 187], [Z, p. 391], and the first ten confirm the calculations of Aluffi [A2] and van Gastel [vG]. For unusual consequences of $\iota = 451440$, see [V3].

Theorem 8.2 *Modulo enumeratively irrelevant divisors,*

$$6\alpha = \beta + 4H + 12I + 6T + 2P + 6Q + 6X + 4Y$$

$$27\alpha = \Delta_0 + 28H + 72I + 45T + 20P + 64Q + 48X + 46Y.$$

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Notation here	α	β	Δ_0	H	I	T	P	Q	X	Y
Zeuthen's notation	μ	μ'	α	ϑ	ν	λ	ξ	η	κ	ζ

Table 8: Notation for analogous divisors on compactifications of $\mathcal{M}$

9 Comparison with Zeuthen's Method

Zeuthen's long article [Z] is devoted to the goal of calculating the characteristic numbers of smooth plane quartics. His approach has many similarities to this one. Here is a summary based on the author's understanding of [S, pp. 184–7] and the french summary to [Z], and suggestions by P. Aluffi.

Zeuthen's aim appears to be to give a general blueprint for all degrees, and then illustrate it with cubics and quartics.

Note that the dual of a smooth plane quartic has degree 12. The parameter space of smooth plane quartics is naturally a dimension 14 locally closed subvariety $\mathcal{M}$ of $\mathbb{P}^{14} \times \mathbb{P}^{90}$ (where the k-points correspond to (smooth quartic C, dual to C); for Zeuthen $k = \mathbb{C}$ of course). Let $\overline{\mathcal{M}}$ be the closure of $\mathcal{M}$ in $\mathbb{P}^{14} \times \mathbb{P}^{90}$.

Not surprisingly, $\overline{\mathcal{M}}$ has boundary divisors corresponding to the enumeratively relevant divisors given in Theorem 4.2. A dictionary between our notation and Zeuthen's is given in Table 8.

Zeuthen's description of points on the boundary of $\overline{\mathcal{M}}$ can be interpreted in modern language. For example, a general point on ϑ (our H) corresponding to a double cover of a conic branched at eight points is described as twice the class of a conic with eight "sommets" (in [S] in German, "Rangpunkte") on the conic. The projection of this point in $\mathbb{P}^{14}$ is the square of the equation of the conic, and the projection of this point in $\mathbb{P}^{90}$ is the square of the equation of the dual conic, times the equations of the eight lines in the dual plane corresponding to the lines through the 8 sommets. In our language, this corresponds to the fact that lines through the branch points should count for single tangencies.

Similarly, a general point of ξ (our P) is described as having double sommets at the nodes, corresponding to the fact that lines through nodes count for two simple tangencies. A general point of η (our Q) has a triple sommet at the singular point (analogous to the multiplicity of 3 in Section 7.6), and the same is true of κ (our X, analogous to the multiplicity of 3 in Section 7.3). A general point of ζ (our Y) has a quadruple sommet at the singular point (analogous to the multiplicity of 4 in Section 7.4).

Using these multiplicities, Zeuthen appears to calculate the characteristic numbers of the boundary divisors in the same way as described here. However, rather than having two unknowns ι and τ (from the divisors I and T), he has five unknowns $H = \tau$, $I = 3280$, $K = 5\tau + 1640$, $L = \iota$, and $M = 6\iota$ corresponding to various enumerative problems (from the analogous divisors ν and λ).

Zeuthen gives equations analogous to Theorem 8.2 [Z, p. 389]:

$$\mu' = 6\mu - 2\xi - 3\eta - 4\zeta - 3\kappa - 6\lambda - 12\nu - 2\vartheta \tag{12}$$

$$\alpha = 27\mu - 20\xi - 32\eta - 46\zeta - 24\kappa - 45\lambda - 72\nu - 14\vartheta. \tag{13}$$

The coefficients of η, κ, and ϑ (corresponding to Q, X, and H) are half the analogous

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coefficients in Theorem 8.2, because the isotropy group of the generic point of those divisors on $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$ is $\mathbb{Z}/2$ (the general such map has an automorphism group of order 2). Zeuthen's characteristic numbers differ from those in Table 2 for the same reason. (Thus equations (12) and (13) can be interpreted as equality on the coarse moduli scheme of $\overline{\mathcal{M}}_3(\mathbb{P}^2, 4)^*$, modulo enumeratively irrelevant divisors.)

It is not clear to the author how Zeuthen obtained the co-efficients in (12) and (13), which is the crux of the calculation. He certainly does not provide details of what he considered routine calculations. P. Aluffi has pointed out the following intriguing passage [Z, p. XI, Section 26 of the summary]:

> Ayant trouvé . . . les ordres des distances et des angles infiniment petits qui séparerent les points et les tangents des courbes singulières de ceux de leurs courbes voisines, nous pouvons faire usage de cette règle pour déterminer directement les coefficients des formules . . . .

Aluffi suggests that he may have determined co-efficients by computing angles (or orders of vanishing of angles), and the detailed figures at the end of the article seem to corroborate this.

9.1

There is one small (but interesting) point where Zeuthen is not correct (without throwing off his calculation). One of his unknowns corresponds to the number of solutions to the following problem [Z, Section 70]: given a line in the plane and 11 general sommets on the line, how many choices of a twelfth sommet are there so the resulting configuration lies in $\nu \subset \mathbb{P}^{14} \times \mathbb{P}^{90}$ (corresponding to our I)? In more modern language, given 11 points on a line, how many choices are there for a twelfth so there is a canonical cover (of genus 3, degree 4) branched at those 12 points? This is a (slightly) different question from that asked in Section 4.6: given 11 general points on a line, how many canonical covers are there branched at those 11 points?

In fact, for each general genus 3, degree 4 canonical cover, there are 119 other canonical maps branched at the same points! In other words, the natural rational map from the (11-dimensional) space of smooth genus 3 canonical covers of a line L, to its image in $\mathrm{Sym}^{12} L \cong \mathbb{P}^{12}$ is not birational, as one would naively expect, but of degree 120! (This is because the corresponding divisor in $\mathrm{Sym}^{12} L$ is very unusual. This divisor will be discussed further in [V3].) Zeuthen gives the answer to his question as 451440 points. The actual answer is 3762 points, each with multiplicity 120.

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Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Ave.
Cambridge, MA 02139
USA
email: vakil@math.mit.edu