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ON THE ASYMPTOTIC BEHAVIOR OF THE LINEARITY DEFECT

HOP D. NGUYEN AND THANH VU

Abstract. This work concerns the linearity defect of a module M over a Noetherian local ring R, introduced by Herzog and Iyengar in 2005, and denoted $\mathrm{ld}_R M$. Roughly speaking, $\mathrm{ld}_R M$ is the homological degree beyond which the minimal free resolution of M is linear. It is proved that for any ideal I in a regular local ring R and for any finitely generated R-module M, each of the sequences $\big(\mathrm{ld}_R(I^n M)\big)_n$ and $\big(\mathrm{ld}_R(M/I^n M)\big)_n$ is eventually constant. The first statement follows from a more general result about the eventual constancy of the sequence $(\mathrm{ld}_R C_n)_n$ where C is a finitely generated graded module over a standard graded algebra over R.

§1. Introduction

This paper concerns the asymptotic behavior of the powers of an ideal in a Noetherian local ring R. A paradigm for our results is the statement, due to Hilbert, and to Samuel, that for any ideal I that is primary to the maximal ideal of R, the sequence $\big(\mathrm{length}_R(R/I^n)\big)_n$ is eventually given by a polynomial in n. Another example, due to Brodmann [3], is that for any ideal I, the sequence $\big(\mathrm{depth}_R(R/I^n)\big)_n$ is eventually constant. We are also interested in the value of n beyond which the asymptotic behavior sets in, and the asymptotic value of the relevant invariants. The papers [5, 13] and their references discuss various aspects of similar asymptotic results.

In this paper, we study the linearity defect introduced by Herzog and Iyengar [15] (see Section 2). One of the motivations for studying the linearity defect is the research on the linear part of minimal free resolutions over the exterior algebras in [9]. The finiteness of the linearity defect has strong consequences on the structure of a module: if $\mathrm{ld}_R M$ is finite, then the Poincaré series of M is rational with denominator depending only on R

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(see [15, Proposition 1.8]). However, there remain many open questions on the finiteness of the linearity defect (see [1, 6, 23] for details).

The linearity defect was studied by many authors (see e.g., [1, 6, 15, 19, 21, 23, 24]). Nevertheless, it is still an elusive invariant. The problem is highly nontrivial as to bound efficiently the linearity defect even for familiar classes of ideals like monomial ideals. Beyond componentwise linear ideals [12] (which have linearity defect zero), there are few interesting and large enough classes of ideals whose linearity defect is known. In this paper, the above remarks notwithstanding, we show that the linearity defect behaves in a pleasant way asymptotically. Let $\operatorname{gl\,ld} R$ be the supremum of the numbers $\operatorname{ld}_R M$, where M runs through all the finitely generated R-modules (see Section 2). The main result of this paper is

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\operatorname{gl\,ld} R$ is finite. For any ideal $I \subseteq \mathfrak{m}$ and finitely generated R-module M, the sequences $\left(\operatorname{ld}_R(I^n M)\right)_n$, $\left(\operatorname{ld}_R(I^n M/I^{n+1}M)\right)_n$, and $\left(\operatorname{ld}_R(M/I^n M)\right)_n$ are eventually constant.*

We obtain the assertion concerning the linearity defect of the sequences $(I^n M)_n$ and $(I^n M/I^{n+1}M)_n$ as a corollary of Theorem 1.2 below, whose proof makes crucial use of work of Şega [23], and the theory of Rees algebras. The assertion involving $(M/I^n M)_n$ uses, in addition, a result of Avramov [2] concerning small homomorphisms of modules. Below, recall that S is called a standard graded algebra over R if it is an $\mathbb{N}$-graded ring with $S_0 = R$ and S is generated over R by finitely many elements of S_1.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\operatorname{gl\,ld} R < \infty$. Let S be a standard graded algebra over R, and let C be a finitely generated graded S-module. Then the sequence $(\operatorname{ld}_R C_n)_n$ is eventually constant.*

This result is motivated by previous work of Herzog and Hibi [13, Theorem 1.1] on depth. In the last part of Section 4, particularly Remark 4.3, we discuss variations of Theorem 1.2.

We do not know how to bound effectively the asymptotic values of the sequences in Theorem 1.1. A rare result in this direction is [14, Theorem 2.4]. There the authors establish a necessary and sufficient condition for all the powers of a polynomial ideal to have linearity defect zero, using the theory of d-sequences [17]. It would be interesting to study possible generalizations and analogues of this result.

We use [4, 8] as our reference for standard concepts and facts in commutative algebra.

§2. Linearity defect

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring with residue field k. Let I be a proper ideal of R. Let M be a finitely generated R-module. We call

$$\mathrm{gr}_I M = \bigoplus_{j\geqslant 0} \frac{I^j M}{I^{j+1}M}$$

the *associated graded module* of M with respect to I.

Let F be the minimal free resolution of M:

$$F : \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0.$$

Since $\partial(F) \subseteq \mathfrak{m}F$ the graded submodule

$$\mathcal{F}^i F : \cdots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow \mathfrak{m}F_{i-1} \longrightarrow \cdots \longrightarrow \mathfrak{m}^{i-j}F_j \longrightarrow \cdots$$

of F is stable under the differential; said otherwise, $\mathcal{F}^i F$ is a subcomplex of F. The *linear part* of F is the associated graded complex

$$\mathrm{lin}^R F := \bigoplus_{i=0}^{\infty} \frac{\mathcal{F}^i F}{\mathcal{F}^{i+1} F}.$$

Note that $\mathrm{lin}^R F$ is a complex of graded modules over $\mathrm{gr}_{\mathfrak{m}} R$, and $(\mathrm{lin}^R F)_i = (\mathrm{gr}_{\mathfrak{m}} F_i)(-i)$ for every $i \geqslant 0$. Following [15], the *linearity defect* of M is defined by

$$\mathrm{ld}_R M := \sup\{i : H_i(\mathrm{lin}^R F) \neq 0\}.$$

If $M \cong 0$, we set $\mathrm{ld}_R M = 0$. This convention guarantees that the maximal ideal (0) of the field k has linearity defect zero.

In order to establish our main results, we use the following characterization of the linearity defect due to Şega [23, Theorem 2.2].

THEOREM 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, M be a finitely generated R-module, and $d \geqslant 0$ be an integer. The following statements are equivalent:*

(i) $\mathrm{ld}_R M \leqslant d$;
(ii) *the natural morphism* $\mathrm{Tor}_i^R(R/\mathfrak{m}^{q+1}, M) \longrightarrow \mathrm{Tor}_i^R(R/\mathfrak{m}^q, M)$ *is the zero map for every $i > d$ and every $q \geqslant 0$.*

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The following result is useful to study the sequence $(\mathrm{ld}_R(M/I^nM))_n$ in Theorem 1.1.

LEMMA 2.2. *Let $0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$ be an exact sequence of nonzero, finitely generated R-modules. Let F and G be the minimal free resolutions of M and P, respectively. Assume that there is a lifting $\varphi : F \longrightarrow G$ of $M \to P$ such that $\varphi(F) \subseteq \mathfrak{m}^2G$. Then there is an equality*

$$\mathrm{ld}_R N = \max\{\mathrm{ld}_R P, \mathrm{ld}_R M + 1\}.$$

Proof. Since $\varphi(F) \subseteq \mathfrak{m}^2G$, the mapping cone, say W, of φ, is a minimal free resolution of N. By simple computations, we get a direct sum decomposition

$$\mathrm{lin}^R W \cong \mathrm{lin}^R G \oplus (\mathrm{lin}^R F)[-1].$$

Hence by accounting, $\mathrm{ld}_R N = \max\{\mathrm{ld}_R P, \mathrm{ld}_R M + 1\}$, as desired. □

Our results require the finiteness of the *global linearity defect* of R, which is defined by

$$\mathrm{gl\,ld}\, R = \sup\{\mathrm{ld}_R M : M \text{ is a finitely generated } R\text{-module}\}.$$

There is an ample supply of rings with finite global linearity defect: regular rings, or more generally local rings which are both Koszul and Golod (see [15, Corollary 6.2]).

§3. Asymptotic behavior of the linearity defect

Let R be a Noetherian local ring with $\mathrm{gl\,ld}\, R < \infty$. Assuming that S is a standard graded polynomial ring over R, we can make Theorem 1.2 more precise by giving an upper bound for the smallest integer from which the sequence $(\mathrm{ld}_R C_n)_n$ becomes constant. This requires certain information about the minimal graded free resolution of C as an S-module.

DEFINITION 3.1. Let $S = R[y_1, \ldots, y_m]$ be a polynomial extension of R, where $m \geqslant 0$, each variable y_i has degree 1. For each finitely generated graded S-module C, let $\mathrm{pdeg}_S(C)$ be the minimal number such that $C_i = 0$ for all $i \geqslant \mathrm{pdeg}_S(C)$ or $C_i \neq 0$ for all $i \geqslant \mathrm{pdeg}_S(C)$. If $C = 0$, we set $\mathrm{pdeg}_S(0) = -\infty$. Note that $\mathrm{pdeg}_S(C)$ is well defined since S is standard graded.

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We can compute the number $\operatorname{pdeg}_S(C)$ effectively, using two simple facts:

(i) $\operatorname{pdeg}_S(C) = \operatorname{pdeg}_{S/\mathfrak{m}S}(C/\mathfrak{m}C)$. This is by Nakayama's lemma.

(ii) $\operatorname{pdeg}_{S/\mathfrak{m}S}(C/\mathfrak{m}C)$ is bounded above by one plus the postulation number of $C/\mathfrak{m}C$, viewed as a module over $S/\mathfrak{m}S = k[y_1, \ldots, y_m]$. The latter number is given, for example, in [4, Proposition 4.12].

DEFINITION 3.2. Given a finitely generated graded S-module C, define the constant $N(C)$ as follows. For $i = 0$, denote $n(0) = \operatorname{pdeg}_S(C)$. For $1 \leqslant i \leqslant \min\{\operatorname{gl\,ld} R, \operatorname{pd}_S C\}$, let $c(i, q) := \operatorname{pdeg}_S\left(\operatorname{Im} \mu^{i,q}\right)$, where $\mu^{i,q}$ denotes the map

$$\operatorname{Tor}_i^S(S/\mathfrak{m}^{q+1}S, C) \to \operatorname{Tor}_i^S(S/\mathfrak{m}^q S, C).$$

Let

$$\cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

be the minimal graded free resolution of C over S (see [4, Section 1.5]).

Set $M_i = \operatorname{Im}(F_i \to F_{i-1})$ and let

$$T(i) = \min\left\{h \geqslant 1 : \mathfrak{m}^q F_{i-1} \cap M_i = \mathfrak{m}^{q-h}(\mathfrak{m}^h F_{i-1} \cap M_i) \text{ for all } q \geqslant h\right\}.$$

Note that $T(i)$ is finite, by the Artin–Rees lemma. Denote

$$n(i) = \max\left\{c(i, 1), \ldots, c(i, T(i))\right\}.$$

Finally, define $N(C) := \max\left\{n(0), n(1), \ldots, n(\min\{\operatorname{gl\,ld} R, \operatorname{pd}_S C\})\right\}$.

Since $\operatorname{gl\,ld} R$ is a finite number, $N(C)$ is also finite.

REMARK 3.3. In principal, the numbers $T(i)$ in the definition of $N(C)$ should not be difficult to compute. Indeed, consider the graded ring $\operatorname{gr}_{\mathfrak{m}S} S$ and the ideal $\mathfrak{n} = \bigoplus_{j \geqslant 1}\left(\mathfrak{m}^j S/\mathfrak{m}^{j+1} S\right)$. Denote by K_i the kernel of the natural surjective map $\operatorname{gr}_{\mathfrak{m}S}(F_{i-1}) \to \operatorname{gr}_{\mathfrak{m}S}(F_{i-1}/M_i)$. Then there is an equality

$$T(i) = \sup\left\{q : (K_i/\mathfrak{n}K_i)_q \neq 0\right\}.$$

The proof is straightforward (see [16, Proposition 2.1] for an analogous statement).

The main technical result of this paper is as follows.

THEOREM 3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with* $\operatorname{gl\,ld} R < \infty$. *Let S be a standard graded polynomial ring over R, and let C be a finitely generated graded S-module. Then for all $n \geqslant N(C)$, there is an equality* $\operatorname{ld}_R C_n = \operatorname{ld}_R C_{N(C)}$.

Proof. Since S is a flat R-algebra, there is an isomorphism of R-modules

$$\operatorname{Tor}_i^R(R/\mathfrak{m}^q, C_n) \cong \operatorname{Tor}_i^S(S/\mathfrak{m}^q S, C)_n \tag{3.1}$$

for all $i, q \geqslant 0, n \in \mathbb{Z}$.

Let $N = N(C)$ and $e = \sup\{\operatorname{ld}_R C_n : n \geqslant N\} \leqslant \operatorname{gl\,ld} R < \infty$. We prove that $\operatorname{ld}_R C_n = e$ for any $n \geqslant N$. There is nothing to do if $e = 0$, so we assume that $e \geqslant 1$. Note that, since $\operatorname{Tor}_i^S(S/\mathfrak{m}^q S, C) = 0$ for $i > \operatorname{pd}_S C$, Isomorphism (3.1) yields $e \leqslant \min\{\operatorname{gl\,ld} R, \operatorname{pd}_S C\}$.

Denote by $\mu_n^{e,q}$ the following map

$$\operatorname{Tor}_e^S(S/\mathfrak{m}^{q+1} S, C)_n \longrightarrow \operatorname{Tor}_e^S(S/\mathfrak{m}^q S, C)_n.$$

Choose $m \geqslant N$ such that $\operatorname{ld}_R C_m = e$. Since $\operatorname{ld}_R C_m = e > e - 1$, Theorem 2.1 implies that $\mu_m^{e,\overline{q}} \neq 0$ for some $\overline{q} \geqslant 0$. To prove the inequality $\operatorname{ld}_R C_n \geqslant e$, also by Theorem 2.1, it suffices to show that $\mu_n^{e,q} \neq 0$ for some $q \geqslant 0$.

Firstly, consider the case $\overline{q} < T(e)$. Since $n, m \geqslant N \geqslant c(e, \overline{q})$, the definition of $c(e, \overline{q})$ implies that $\mu_n^{e,\overline{q}}$ and $\mu_m^{e,\overline{q}}$ are both zero or both nonzero. This implies that $\mu_n^{e,\overline{q}} \neq 0$, as desired.

Secondly, consider the case $\overline{q} \geqslant T(e)$. Denote $T = T(e)$, we claim that $\mu_n^{e,T} \neq 0$. As $m, n \geqslant c(e, T)$, $\mu_m^{e,T}$ and $\mu_n^{e,T}$ are both zero or both nonzero, so it suffices to prove that $\mu_m^{e,T} \neq 0$. Assume the contrary, so $\mu_m^{e,T} = 0$. Let F be the minimal graded free resolution of C over S. Denote $M_i = \operatorname{Im}(F_i \to F_{i-1})$, the ith syzygy of C as an S-module if $i \geqslant 1$ and $M_0 = C$. Denote $M = M_e$ and $P = F_{e-1}$. Clearly

$$\operatorname{Tor}_e^S(S/\mathfrak{m}^q S, C) \cong \operatorname{Tor}_1^S(S/\mathfrak{m}^q S, M_{e-1}) \cong \frac{\mathfrak{m}^q P \cap M}{\mathfrak{m}^q M},$$

$$\operatorname{Im} \mu^{e,q} \cong \frac{\mathfrak{m}^{q+1} P \cap M + \mathfrak{m}^q M}{\mathfrak{m}^q M}.$$

The equality $\mu_m^{e,T} = 0$ then yields

$$(\mathfrak{m}^{T+1} P \cap M + \mathfrak{m}^T M)_m = (\mathfrak{m}^T M)_m. \tag{3.2}$$

We show that $\mu_m^{e,\overline{q}} = 0$. Indeed,

$$\begin{aligned}\left(\mathfrak{m}^{\overline{q}+1}P\cap M+\mathfrak{m}^{\overline{q}}M\right)_m &= \left(\mathfrak{m}^{\overline{q}-T}(\mathfrak{m}^{T+1}P\cap M+\mathfrak{m}^{T}M)\right)_m\\ &= \mathfrak{m}^{\overline{q}-T}\left(\mathfrak{m}^{T+1}P\cap M+\mathfrak{m}^{T}M\right)_m\\ &= \mathfrak{m}^{\overline{q}-T}(\mathfrak{m}^{T}M)_m=(\mathfrak{m}^{\overline{q}}M)_m.\end{aligned}$$

In the above string, the first equality holds because of the inequality $\overline{q}\geqslant T=T(e)$ and the definition of $T(e)$, the second and fourth because $\mathfrak{m}\subseteq S_0$, the third because of (3.2).

Therefore, $\mu_m^{e,\overline{q}}=0$. But this is a contradiction, so the proof of the theorem is finished. □

From Theorem 3.4, it is easy to deduce the

Proof of Theorem 1.2. Let Q be a standard graded polynomial ring over R which surjects onto S, then by scalar restriction, C is a finitely generated graded Q-module. The conclusion follows by applying Theorem 3.4. □

Now we present the proof of Theorem 1.1. Recall that Rees $(I)=R\oplus I\oplus I^2\oplus\cdots$ denotes the Rees algebra of I, whose grading is given by deg $I^n=n$. Since R is Noetherian, Rees (I) is a standard graded R-algebra.

Proof of Theorem 1.1. Clearly $\bigoplus_{n\geqslant 0} I^nM$ and $\bigoplus_{n\geqslant 0} I^nM/I^{n+1}M$ are finitely generated graded modules over Rees (I). By Theorem 1.2, we see that each of the sequences $(\mathrm{ld}_R I^nM)_n$ and $(\mathrm{ld}_R I^nM/I^{n+1}M)_n$ is eventually constant.

Next, we prove the eventual constancy of the sequence $(\mathrm{ld}_R(M/I^nM))_n$. If $I^nM=0$ then so is $I^{n+1}M$, hence below, we assume that $I^nM\neq 0$ for all $n\geqslant 0$.

Applying [2, Corollary A.4] for M, there exists $d\geqslant 1$ such that for any $P\subseteq\mathfrak{m}^dM$, the map

$$\mathrm{Tor}_i^R(k,P)\to\mathrm{Tor}_i^R(k,M)$$

is zero for all $i\geqslant 0$. Applying the same result for $\mathfrak{m}^dM$, there exists $e\geqslant 1$ such that for any $P\subseteq\mathfrak{m}^{d+e}M$, the map

$$\mathrm{Tor}_i^R(k,P)\to\mathrm{Tor}_i^R(k,\mathfrak{m}^dM)$$

is zero for all $i\geqslant 0$.

Take $n\geqslant\max\{N,d+e\}$. Then the maps $\mathrm{Tor}_i^R(k,I^nM)\to\mathrm{Tor}_i^R(k,I^dM)$ and $\mathrm{Tor}_i^R(k,I^dM)\to\mathrm{Tor}_i^R(k,M)$ are zero for all $i\geqslant 0$. Let F,G,H be the minimal free resolution of I^nM, I^dM, M, respectively. Take any lifting $\lambda: F\to G$ of the map $I^nM\to I^dM$, then $\lambda(F)\subseteq\mathfrak{m}G$. Similarly, for any lifting

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$\psi : G \to H$ of the map $I^d M \to M$, we have $\psi(G) \subseteq \mathfrak{m}H$. Therefore, we obtain a lifting $\phi = \psi \circ \lambda : F \to H$ on the level of minimal free resolutions of the map $I^n M \to M$ which satisfies $\phi(F) \subseteq \mathfrak{m}^2 G$.

By Lemma 2.2, we have for any $n \geqslant \max\{N, d+e\}$ the equality

$$\mathrm{ld}_R(M/I^n M) = \max\bigl\{\mathrm{ld}_R M, \mathrm{ld}_R(I^n M) + 1\bigr\}.$$

As explained above, for n large enough, $\mathrm{ld}_R(I^n M)$ is a constant independent of n. Hence the same is true for $\mathrm{ld}_R(M/I^n M)$. This concludes the proof. □

Theorem 3.4 has the following consequence on the linearity defect of the integral closure of powers (see [18] for the definition of the integral closure $\overline{I}$ of an ideal I).

COROLLARY 3.5. *Let $(R, \mathfrak{m})$ be a regular local ring, and let $I \subseteq \mathfrak{m}$ be an ideal. Then the sequence $(\mathrm{ld}_R \overline{I^n})_n$ is eventually constant.*

Proof. Denote $C = R \oplus \overline{I} \oplus \overline{I^2} \oplus \cdots$, then C is a graded module over Rees (I) with deg $\overline{I^n} = n$. By [18, Proposition 5.3.4], C is a finitely generated Rees (I)-module. An application of Theorem 1.2 yields the desired conclusion. □

§4. Examples and remarks

The following example illustrates how $N(C)$ can be computed using Macaulay2 [10].

EXAMPLE 4.1. Let $R = \mathbb{Q}[x, y, z]$ be a polynomial ring of dimension 3, and let $I = (x^2, xy, z^2)$. Denote $S = R[w_0, w_1, w_2]$ a standard graded polynomial extension of R which surjects onto the Rees algebra $E = \text{Rees}\,(I)$ by mapping $w_0 \mapsto x^2, w_1 \mapsto xy, w_2 \mapsto z^2$. The ring E has the following presentation

$$E \cong \frac{S}{(p_1, p_2, p_3)},$$

where $p_1 = w_0 y - w_1 x$, $p_2 = w_0 z^2 - w_2 x^2$, $p_3 = w_1 z^2 - w_2 xy$.

The minimal graded free resolution of E over S is as follows

$$F : 0 \longrightarrow \begin{matrix} S(-2) \\ \bigoplus \\ S(-1) \end{matrix} \xrightarrow{\begin{pmatrix} w_2 x & -z^2 \\ -w_1 & y \\ w_0 & -x \end{pmatrix}} S(-1)^3 \xrightarrow{\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}} S \longrightarrow 0.$$

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Using the notation of the proof of Theorem 3.4, we show that $N = 1$, namely all the powers of I have the same linearity defect, which turns out to be 1. Since $\operatorname{pd}_S E = 2 < \operatorname{gl\,ld} R = 3$, $N = \max\{n(0), n(1), n(2)\}$. The graded structure of E tells us that $n(0) = \operatorname{pdeg}_S(E) = 0$.

Let $J \subseteq S$, $M_2 \subseteq G$ be the first and second syzygies of E, where G denotes the module $F_1 = S(-1)^3$. We claim that $T(1) = 2$ and $T(2) = 1$, namely,

$$\mathfrak{m}^q S \cap J = \mathfrak{m}^{q-2}(\mathfrak{m}^2 S \cap J) \quad \text{for all } q \geqslant 2, \tag{4.1}$$

$$\mathfrak{m}^q G \cap M_2 = \mathfrak{m}^{q-1}(\mathfrak{m} G \cap M_2) \quad \text{for all } q \geqslant 1. \tag{4.2}$$

For (4.1): one sees immediately that both sides are equal to $\mathfrak{m}^{q-1}(w_0 y - w_1 x)S + \mathfrak{m}^{q-2}(w_0 z^2 - w_2 x^2, w_1 z^2 - w_2 xy)S$.

For (4.2): we have $M_2 = (w_2 x e_1 - w_1 e_2 + w_0 e_3, -z^2 e_1 + y e_2 - x e_3)$, where e_1, e_2, e_3 is the standard basis of G sitting in degree 1. It is not hard to check that both sides of (4.2) are equal to

$$\mathfrak{m}^q(w_2 x e_1 - w_1 e_2 + w_0 e_3) + \mathfrak{m}^{q-1}(-z^2 e_1 + y e_2 - x e_3).$$

The above arguments yield $n(1) = \max\{c(1,1), c(1,2)\}$ and $n(2) = c(2,1)$. We prove that $n(1) = 1$ and $n(2) = -\infty$.

For each $q \geqslant 1$, $\operatorname{Tor}_1^S(S/\mathfrak{m}^q S, E) = \operatorname{Tor}_1^S(S/\mathfrak{m}^q S, S/J) = (J \cap \mathfrak{m}^q S)/(J\mathfrak{m}^q S)$. Therefore, the image of $\operatorname{Tor}_1^S(S/\mathfrak{m}^{q+1} S, E) \to \operatorname{Tor}_1^S(S/\mathfrak{m}^q S, E)$ is $\operatorname{Im} \mu^{1,q} = (J \cap \mathfrak{m}^{q+1} S + J\mathfrak{m}^q S)/(J\mathfrak{m}^q S)$. Computations show that

$$\operatorname{Im} \mu^{1,1} = \frac{S^2}{\mathfrak{m} S^2 + (w_0 e_1^1 - w_1 e_2^1)},$$

where e_1^1, e_2^1 is a basis for S^2, both of degree 1, and

$$\operatorname{Im} \mu^{1,2} = \frac{S^5}{\mathfrak{m} S(e_3^2, e_4^2, e_5^2) + \mathfrak{m}^2 S(e_1^2, e_2^2) + (-x e_1^2 + y e_2^2, w_0 e_1^2 - w_1 e_2^2 + w_2 e_5^2)},$$

where $e_1^2, \ldots, e_5^2$ are a basis for S^5, all of them of degree 1. Thanks to routine Gröbner basis arguments, the residue classes $\overline{w_0^i e_2^1} \in \operatorname{Im} \mu^{1,1}$ and $\overline{w_0^i e_2^2} \in \operatorname{Im} \mu^{1,2}$ are always nonzero for every $i \geqslant 0$. Hence $c(1,1) = c(1,2) = 1$, and thus $n(1) = 1$.

Denote by f_1, f_2 the standard basis of F_2 where $\deg f_1 = 2, \deg f_2 = 1$. Since $\operatorname{Tor}_2^S(S/\mathfrak{m}^2 S, E) = H_2(F \otimes_S S/\mathfrak{m}^2 S)$, computations show that

(i) $\operatorname{Tor}_2^S(S/\mathfrak{m}^2 S, E)$ is generated by $\overline{x f_2}, \overline{y f_2}, \overline{z f_2} \in F_2 \otimes (S/\mathfrak{m}^2 S)$;

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(ii) $\mathrm{Tor}_2^S(S/\mathfrak{m}S, E)$ is generated by $\overline{f_2} \in F_2 \otimes (S/\mathfrak{m}S)$.

As $\mathrm{Tor}_2^S(S/\mathfrak{m}S, E)$ is killed by $\mathfrak{m}S$, the map $\mathrm{Tor}_2^S(S/\mathfrak{m}^2S, E) \to \mathrm{Tor}_2^S(S/\mathfrak{m}S, E)$ is zero; this yields $n(2) = c(2, 1) = -\infty$.

Putting everything together, $N = \max\{n(0), n(1), n(2)\} = \max\{0, 1, -\infty\} = 1$.

Recall that the saturation of I is $\widetilde{I} = \{x \in R : x\mathfrak{m}^d \subseteq I \text{ for some } d \geqslant 1\}$. The next example shows that the (graded) analog of Corollary 3.5 for saturation of powers does not hold.

EXAMPLE 4.2. Consider the ideal $I = \big(x(y^3 - z^3), y(x^3 - z^3), z(x^3 - y^3)\big) \subseteq R = \mathbb{C}[x, y, z]$. The ideal I defines a reduced set of 12 points in $\mathbb{P}^2$, the so-called *Fermat configuration* (see the proof of [11, Proposition 2.1]). We show that the saturation ideals $\widetilde{I^s}$ do not have eventually constant linearity defect.

For $s \geqslant 1$, denote by $I^{(s)} = R \cap \bigcap_{P \in \mathrm{Ass}\,(I)} I^s R_P$ the sth symbolic power of I. Since I is the defining ideal of a reduced set of points, we get that $\widetilde{I^s} = I^{(s)}$. From [11, Proposition 1.1], we deduce that $\widetilde{I^{3s}} = (\widetilde{I^3})^s$. By [14, Theorem 2.4], $\mathrm{ld}_R \widetilde{I^{3s}} = 0$ for all $s \geqslant 1$. Indeed, computations with Macaulay2 [10] show that $x, y + z, z$ is a d-sequence with respect to Rees $(\widetilde{I^3})$.

Now we show that $\mathrm{ld}_R \widetilde{I^{3s+1}} = 1$ for all $s \geqslant 1$. First, since depth $R/\widetilde{I^{3s+1}} \geqslant 1$, by [6, Proposition 6.3], $\mathrm{ld}_R R/\widetilde{I^{3s+1}} \leqslant \dim R - 1 = 2$. Hence $\mathrm{ld}_R \widetilde{I^{3s+1}} \leqslant 1$.

Let $H = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3)$. We show that the minimal nonzero component of $\widetilde{I^{3s+1}} = I^{(3s+1)}$ is of degree $9s + 4$ and

$$I^{(3s+1)}_{\langle 9s+4\rangle} = (H^s) I_{\langle 4\rangle} \cong I(-9s).$$

If this is the case, then $\widetilde{I^{3s+1}}_{\langle 9s+4\rangle}$ has linearity defect at least 1, as I does. (For the inequality $\mathrm{ld}_R I \geqslant 1$, use Römer's theorem [22, Theorem 3.2.8] and the fact that I is generated in degree 4 but does not have 4-linear resolution.) Hence $\mathrm{ld}_R \widetilde{I^{3s+1}} \geqslant 1$ for every $s \geqslant 1$. All in all, we obtain $\mathrm{ld}_R \widetilde{I^{3s+1}} = 1$ for every $s \geqslant 1$.

Now for our purpose, it suffices to prove the following claim:

$$I^{(3s+1)}_{\langle d\rangle} = (H^s) I_{\langle d-9s\rangle} \tag{4.3}$$

holds for all $d \leqslant 9s + 4$. We are grateful to Alexandra Seceleanu for providing us the following nice argument.

We proceed by induction on s; the starting case $s = 0$ is trivial. Assume that $s > 0$.

Let G be a homogeneous element of $I^{(3s+1)}$ of degree d. Here the geometry of the Fermat configuration comes into play. We have a decomposition $H = \prod_{i=1}^{9} h_i$, where each h_i is a linear form and no two of them are proportional. According to [11, Section 1.1], for each i, h_i passes through exactly 4 points (among the 12 points of the configuration). Moreover, each point of the configuration lies on 3 of the 9 lines defined by the h_is.

Now as G lies in $I^{(3s+1)}$, G passes through each point of the configuration with multiplicity at least $3s+1$. Thus the curves (G) and (h_i) intersect with multiplicity at least $4(3s+1)$, which is strictly larger than $d = (\deg G)\cdot(\deg h_i)$. From that, Bezout's theorem forces G to be divisible by h_i for all $1 \leqslant i \leqslant 9$. In particular, G is divisible by H. Writing $G = HG'$, then as H vanishes exactly 3 times at each of the points, we must have $G' \in I^{(3s+1-3)}_{\langle d-9\rangle} = I^{(3(s-1)+1)}_{\langle d-9\rangle}$. Finally, the induction hypothesis gives us the claim.

So we conclude that the sequence $\operatorname{ld}_R \widetilde{I^s}$ is not eventually constant when s goes to infinity.

REMARK 4.3. More generally than Theorem 1.2, one can prove the following: if S is a Noetherian R-algebra which is generated by elements of positive degrees, and C is a finitely generated graded S-module, then the sequence $(\operatorname{ld}_R C_n)_n$ is quasiperiodic, namely there exist a number $p \geqslant 1$ and integral constants $\ell_0, \ldots, \ell_{p-1}$ such that for all $n \gg 0$, we have $\operatorname{ld}_R \widetilde{C_n} = \ell_i$ if n is congruent to $i \in \{0, \ldots, p-1\}$ modulo p.

The proof uses the fact that any high enough Veronese subring of S is standard graded (after normalizing the grading), and Theorem 1.2. We leave the details to the interested reader (see an analogous statement in [7, Theorem 4.3]).

By [20, Theorem 4.3], for the ideal I in Example 4.2, the graded R-algebra $R \oplus \widetilde{I} \oplus \widetilde{I^2} \oplus \cdots$ is finitely generated. This fact and the above general version of Theorem 1.2 guarantee the quasiperiodic behavior of the sequence $(\operatorname{ld}_R \widetilde{I^n})_n$ in the example.

We do not know any example where the sequence $(\operatorname{ld}_R \widetilde{I^n})_n$ is not quasiperiodic. In view of [7, Example 4.4] on bad behavior of regularity for saturations of powers, it is desirable to seek for one.

REMARK 4.4. The theory in Section 2 (the linear part, linearity defect) works also for standard graded algebras and finitely generated graded modules over them. Furthermore, there are obvious analogues of our results in that setting.

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Hop D. Nguyen
Fachbereich Mathematik/Informatik
Institut für Mathematik
Universität Osnabrück
Albrechtstr. 28a
49069 Osnabrück
Germany

and

Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146 Genoa
Italy

ngdhop@gmail.com

Thanh Vu
Department of Mathematics
University of Nebraska-Lincoln
Lincoln
NE 68588
USA

tvu@unl.edu