

CO-UNIVERSAL C^* -ALGEBRAS ASSOCIATED TO APERIODIC k -GRAPHS

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Abstract. We construct a representation of each finitely aligned aperiodic k -graph Λ on the Hilbert space \mathcal{H}^{ap} with basis indexed by aperiodic boundary paths in Λ . We show that the canonical expectation on $\mathcal{B}(\mathcal{H}^{\text{ap}})$ restricts to an expectation of the image of this representation onto the subalgebra spanned by the final projections of the generating partial isometries. We then show that every quotient of the Toeplitz algebra of the k -graph admits an expectation compatible with this one. Using this, we prove that the image of our representation, which is canonically isomorphic to the Cuntz–Krieger algebra, is co-universal for Toeplitz–Cuntz–Krieger families consisting of non-zero partial isometries.

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1. Introduction. A directed graph is a quadruple (E^0, E^1, r, s) where E^0 and E^1 are countable sets, whose elements are thought of as vertices and edges respectively, and r and s are maps from E^1 to E^0 which indicate the directions of the edges: we think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$. Associated to each graph there are a number of C^* -algebras, the two most prominent being the Toeplitz algebra $\mathcal{TC}^*(E)$ and the Cuntz–Krieger algebra $C^*(E)$. These are far-reaching generalizations of the Toeplitz–Cuntz algebras \mathcal{TO}_n and the Cuntz algebras \mathcal{O}_n [3], and have been much studied in recent years. One way to think of $\mathcal{TC}^*(E)$ and $C^*(E)$ is as follows: $\mathcal{TC}^*(E)$ is the image of the natural analogue of a left-regular representation of a graph on the Hilbert space $\ell^2(E^*)$ with basis indexed by paths in the graph, and $C^*(E)$ is the quotient of $\mathcal{TC}^*(E)$ by its intersection with $\mathcal{K}(\ell^2(E^*))$ [6].

However, this spatial viewpoint is not the traditional one. In the seminal papers [11, 12] Kumjian, Pask, Raeburn and Renault considered graphs E in which $r^{-1}(v)$ is both finite and non-empty for every $v \in E^0$. They defined $C^*(E)$ as a groupoid C^* -algebra, and then showed how $C^*(E)$ could be described as the C^* -algebra which is universal for a system of generators and relations which have come to be known as a Cuntz–Krieger family: a collection $\{p_v : v \in E^0\}$ of mutually orthogonal projections, and a family $\{s_e : e \in E^1\}$ of partial isometries, such that¹ $s_e^*s_e = p_{s(e)}$ and $p_v = \sum_{r(e)=v} s_e s_e^*$ for each $v \in E^0$ and $e \in E^1$. The Toeplitz algebra is universal for elements

¹These relations look a little different from those of [11, 12]: the roles of r and s are reversed. See [14] for an explanation.

$\{q_v : v \in E^0\}$ and $\{t_e : e \in E^1\}$, such that $t_e^* t_e = q_{s(e)}$ and $q_v \geq \sum_{r(e)=v} t_e t_e^*$. It is standard that both algebras are spanned by elements of the form $t_\mu t_\nu^*$ where μ, ν are directed paths in E with the same source, and $t_\mu = t_{\mu_1} t_{\mu_2} \cdots t_{\mu_n}$ is the product of the generators associated to the edges occurring in the path μ . This universal groupoid C^* -algebra has now become standard [1, 7, 14].

Two key theorems about graph C^* -algebras are the uniqueness theorems. The first of these is an Huef and Raeburn's gauge-invariant uniqueness theorem. By virtue of its universal property, $C^*(E)$ carries an action γ of \mathbb{T} , called the gauge action, satisfying $\gamma_z(s_e) = z s_e$ for all $e \in E^1$. The gauge-invariant uniqueness theorem says that any two Cuntz–Krieger families consisting of non-zero partial isometries and carrying \mathbb{T} -actions compatible with the gauge action generate isomorphic C^* -algebras. The second uniqueness theorem, the one in which we are interested in the present paper, is called the Cuntz–Krieger uniqueness theorem. It says that if E satisfies an appropriate aperiodicity hypothesis, then any two Cuntz–Krieger families consisting of non-zero partial isometries generate isomorphic C^* -algebras.

In recent years, a number of generalizations of graph algebras have arisen, one of which is k -graphs and their C^* -algebras, first introduced by Kumjian and Pask [10]. A k -graph is a k -dimensional analogue of a directed graph, and the k -graph C^* -algebra is the universal C^* -algebra generated by a family of generators and relations analogous to the Cuntz–Krieger relations for directed graphs. However, the complexity of these relations for the finitely aligned k -graphs of [16] makes them cumbersome to verify in examples. In particular, while the relations which determine the Toeplitz algebra remain fairly natural—they are those which arise in the left-regular representation—the relation characterizing the quotient map from the Toeplitz algebra to the Cuntz–Krieger algebra is more complicated.

Katsura's recent work on C^* -algebras associated to Hilbert modules [8, 9] suggests a more elegant approach, which we call a co-universal property. The idea is that rather than specifying the Cuntz–Krieger algebra as the universal object for a complicated set of relations (which include the Toeplitz ones), we aim to specify it as the co-universal object for non-zero generators satisfying the Toeplitz relations. Specifically, given a k -graph satisfying the hypotheses of the Cuntz–Krieger uniqueness theorem, we aim to show that there exists a C^* -algebra generated by a Toeplitz–Cuntz–Krieger family consisting of non-zero partial isometries, which is the smallest such algebra in the sense that it occurs as a quotient of any other C^* -algebra generated by a Toeplitz–Cuntz–Krieger family consisting of non-zero partial isometries.

In this paper we construct such a co-universal algebra for every aperiodic finitely aligned k -graph; the Cuntz–Krieger uniqueness theorem is then a corollary. Our basic technique is fairly versatile and we believe that it is interesting in its own right. We construct a spatial representation π of $C^*(\Lambda)$ on a Hilbert space \mathcal{H}^{ap} for which the canonical expectation from $\mathcal{B}(\mathcal{H}^{\text{ap}})$ onto its diagonal subalgebra implements a map Φ_π on $\pi(C^*(\Lambda))$, which sends each $\pi(s_\mu s_\nu^*)$ to $\delta_{\mu,\nu} \pi(s_\mu s_\mu^*)$. It follows that Φ_π is a faithful conditional expectation. We then use algebraic arguments to show that given any Toeplitz–Cuntz–Krieger family $\{t_\lambda : \lambda \in \Lambda\}$ for Λ , there is an expectation Φ_t on $C^*(\{t_\lambda : \lambda \in \Lambda\})$ which satisfies $\Phi_t(t_\mu t_\nu^*) = \delta_{\mu,\nu} t_\mu t_\mu^*$. From this, we deduce that the image of the Toeplitz algebra under π is co-universal as described above.

2. Preliminaries. Let $\mathbb{N} = \{0, 1, \dots\}$ denote the monoid of natural numbers under addition, and regard \mathbb{N}^k an additive semi-group with identity 0. We write

e_1, \dots, e_k for the generators of \mathbb{N}^k , and n_i for the i th coordinate of $n \in \mathbb{N}^k$. For $m, n \in \mathbb{N}^k$, we say that $m \leq n$ if $m_i \leq n_i$ for each i . We write $m \vee n$ for the coordinatewise maximum of m and n .

A k -graph (Λ, d) consists of a countable small category Λ together with a *degree functor* $d : \Lambda \rightarrow \mathbb{N}^k$ satisfying the *factorization property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m, d(\nu) = n$. Elements $\lambda \in \Lambda$ are called *paths*. For $n \in \mathbb{N}^k$, we write $\Lambda^n := d^{-1}(n)$. The factorization property allows us to identify $\text{Obj}(\Lambda)$ with Λ^0 . So we may regard the codomain and domain maps in Λ as functions $r, s : \Lambda \rightarrow \Lambda^0$. For $v \in \Lambda^0, v\Lambda = \{\lambda \in \Lambda : r(\lambda) = v\}$ and $\Lambda v = \{\lambda \in \Lambda : s(\lambda) = v\}$.

NOTATION 1. For $\lambda \in \Lambda$ and $m \leq n \leq d(\lambda)$, by the factorization property we can express λ uniquely as $\lambda = \lambda_1\lambda_2\lambda_3$ with $d(\lambda_1) = m, d(\lambda_2) = n - m$ and $d(\lambda_3) = d(\lambda) - n$. We denote λ_2 by $\lambda(m, n)$, so $\lambda_1 = \lambda(0, m)$ and $\lambda_3 = \lambda(n, d(\lambda))$.

A graph morphism between two k -graphs (Λ, d_Λ) and (Γ, d_Γ) is a functor $x : \Lambda \rightarrow \Gamma$ such that $d_\Gamma(x(\lambda)) = d_\Lambda(\lambda)$ for all $\lambda \in \Lambda$.

Let $k \in \mathbb{N}$ and let $m \in (\mathbb{N} \cup \{\infty\})^k$. Then $(\Omega_{k,m}, d)$ is the k -graph with

$$\begin{aligned} \text{Obj}(\Omega_{k,m}) &:= \{p \in \mathbb{N}^k : p \leq m\}, \\ \text{Mor}(\Omega_{k,m}) &:= \{(p, q) \in \text{Obj}(\Omega_{k,m}) \times \text{Obj}(\Omega_{k,m}) : p \leq q\}, \\ r(p, q) &:= p, \quad s(p, q) := q \quad d(p, q) := q - p. \end{aligned}$$

If $x : \Omega_{k,m} \rightarrow \Lambda$ is a graph morphism and $n \in \mathbb{N}^k$ with $n \leq m$, then there is a graph morphism $\sigma^n(x) : \Omega_{k,m-n} \rightarrow \Lambda$ determined by $\sigma^n(x)(p, q) := x(n + p, n + q)$. Also, if $x : \Omega_{k,m} \rightarrow \Lambda$ is a graph morphism and $\lambda \in \Lambda r(x)$, then there is a unique graph morphism $\lambda x : \Omega_{k,m+d(\lambda)} \rightarrow \Lambda$ such that $(\lambda x)(0, d(\lambda)) = \lambda$ and $\sigma^{d(\lambda)}(\lambda x) = x$. If $x : \Omega_{k,m} \rightarrow \Lambda$ is a graph morphism, we define $d(x) = m$.

Let (Λ, d) be a k -graph. For $\mu, \nu \in \Lambda$, we say that λ is a *minimal common extension* of μ and ν if $d(\lambda) = d(\mu) \vee d(\nu)$ and $\lambda = \mu\mu' = \nu\nu'$ for some $\mu', \nu' \in \Lambda$. We write $\text{MCE}(\mu, \nu)$ for the set of all minimal common extensions of μ and ν . More generally, for a finite subset F of Λ , let

$$\text{MCE}(F) := \{\lambda \in \Lambda : d(\lambda) = \bigvee_{\alpha \in F} d(\alpha) \text{ and } \lambda(0, d(\alpha)) = \alpha \text{ for all } \alpha \in F\},$$

and let $\vee F := \bigcup_{G \subset F} \text{MCE}(G)$.

We say that Λ is *finitely aligned* if $\text{MCE}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$. Fix $v \in \Lambda^0$ and $E \subset v\Lambda$. We say that E is *exhaustive* if for each $\mu \in v\Lambda$ there exists $\lambda \in E$ such that $\text{MCE}(\mu, \lambda) \neq \emptyset$. If $|E| < \infty$, we say that E is *finite exhaustive*. Define $\text{FE}(\Lambda)$ to be the set of finite exhaustive sets of Λ and for each $v \in \Lambda^0$, define $v\text{FE}(\Lambda)$ to be the set of finite exhaustive sets whose elements all have range v .

DEFINITION 2.1. We say that a k -graph is *aperiodic* if, for all $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, there exists $\tau \in s(\mu)\Lambda$ such that $\text{MCE}(\mu\tau, \nu\tau) = \emptyset$.

REMARK 2.2. There have been many versions of ‘‘aperiodicity’’ proposed for k -graphs. For an account of the relationship between them, see [13, 17]. The version presented here first appeared in [13].

LEMMA 2.3 ([13], Lemma 4.4). Let (Λ, d) be an aperiodic finitely aligned k -graph, and fix $v \in \Lambda^0$ and a finite subset H of $v\Lambda$. Then there exists $\tau \in v\Lambda$ such that $\text{MCE}(\mu\tau, \nu\tau) = \emptyset$ for all distinct $\mu, \nu \in H$.

DEFINITION 2.4. Let (Λ, d) be a finitely aligned k -graph. A k -graph morphism $x : \Omega_{k,m} \rightarrow \Lambda$ is called a *boundary path* if for all $n \in \mathbb{N}^k$ with $n \leq d(x)$ and for all $E \in x(n)\text{FE}(\Lambda)$, there exists $m \leq d(x) - n$ such that $x(n, n + m) \in E$. We write $\partial\Lambda$ for the set of all boundary paths, and for $v \in \Lambda^0$, write $v(\partial\Lambda)$ for $\{x \in \partial\Lambda : r(x) = v\}$. Define the set of *aperiodic boundary paths* $\partial\Lambda^{\text{ap}}$ by

$$\partial\Lambda^{\text{ap}} := \{x \in \partial\Lambda : m \neq n \implies \sigma^m(x) \neq \sigma^n(x) \text{ for all } m, n \leq d(x)\}.$$

LEMMA 2.5. *Let (Λ, d) be a finitely aligned k -graph and let $x \in \partial\Lambda^{\text{ap}}$.*

- (1) *If $m \in \mathbb{N}^k$ and $m \leq d(x)$, then $\sigma^m(x) \in \partial\Lambda^{\text{ap}}$.*
- (2) *If $\lambda \in \Lambda r(x)$, then $\lambda x \in \partial\Lambda^{\text{ap}}$.*

Proof. Lemma 2.4 of [13] implies that each of $\sigma^m(x)$ and λx belong to $\partial\Lambda$. So we just have to check that they are aperiodic.

For (1), observe that $p, q \leq d(\sigma^m(x))$ and $\sigma^p(x) = \sigma^q(x)$ implies that $p + m, q + m \leq d(x)$ and $\sigma^{p+m}(x) = \sigma^{q+m}(x)$.

For (2), suppose that $p, q \leq d(\lambda x)$ and $\sigma^p(\lambda x) = \sigma^q(\lambda x)$. For $i \leq k$,

$$d(\lambda)_i + d(x)_i - p_i = d(\sigma^p(\lambda x))_i = d(\sigma^q(\lambda x))_i = d(\lambda)_i + d(x)_i - q_i.$$

Hence, whenever $p_i \neq q_i$, we have $d(x)_i = \infty$. Let $p' := p - p \wedge q$ and $q' := q - p \wedge q$. Since each $p'_i = p_i - \min\{p_i, q_i\}$, whenever p'_i is non-zero, we have $p_i \neq q_i$ and hence $d(x)_i = \infty$. Similarly, $q'_i \neq 0$ implies $d(x)_i = \infty$. Now let $m := p' + ((p \wedge q) \vee d(\lambda))$ and let $n := q' + ((p \wedge q) \vee d(\lambda))$. Then $p, q, d(\lambda) \leq m, n$. We claim that $m, n \leq d(\lambda x)$. Indeed, since $p, q, d(\lambda) \leq d(\lambda x)$, we certainly have $(p \wedge q) \vee d(\lambda) \leq d(\lambda x)$, and then since $p'_i \neq 0 \implies d(x)_i = \infty$, it follows that $m \leq d(\lambda x)$ also, and similarly for n . We now calculate

$$\sigma^{m-d(\lambda)}(x) = \sigma^m(\lambda x) = \sigma^{m-p}(\sigma^p(\lambda x)).$$

A similar calculation gives $\sigma^{n-d(\lambda)}(x) = \sigma^{n-q}(\sigma^q(\lambda x))$. Since $\sigma^p(\lambda x) = \sigma^q(\lambda x)$ by assumption and since

$$m - p = m - p' - (p \wedge q) = ((p \wedge q) \vee d(\lambda)) - (p \wedge q) = n - q' - (p \wedge q) = n - q,$$

it follows that $\sigma^{m-d(\lambda)}(x) = \sigma^{n-d(\lambda)}(x)$. Since $x \in \partial\Lambda^{\text{ap}}$, it follows that $m - d(\lambda) = n - d(\lambda)$, and hence

$$0 = m - n = p' - q' = p - q.$$

That is, $p, q \leq d(\lambda x)$ and $\sigma^p(\lambda x) = \sigma^q(\lambda x)$ imply that $p = q$, so λx is aperiodic as required. □

DEFINITION 2.6. Let (Λ, d) be a finitely aligned k -graph. A *Toeplitz–Cuntz–Krieger Λ -family* is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^* -algebra satisfying

(TCK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;

(TCK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$; and

(TCK3) $t_\mu^* t_\nu = \sum_{\alpha \in \nu\beta \in \text{MCE}(\mu, \nu)} t_\alpha t_\beta^*$ for all $\mu, \nu \in \Lambda$.

A *Cuntz–Krieger Λ -family* is a Toeplitz–Cuntz–Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$, which satisfies

$$(CK) \prod_{\lambda \in E} (t_v - t_\lambda t_\lambda^*) = 0 \text{ for all } v \in \Lambda^0 \text{ and } E \subset v\text{FE}(\Lambda).$$

As in [4], given a finitely aligned k -graph Λ there is a C^* -algebra $\mathcal{T}C^*(\Lambda)$ called the *Toeplitz algebra* of Λ generated by a Toeplitz–Cuntz–Krieger Λ -family $\{s_\lambda : \lambda \in \Lambda\}$, which is universal in the sense that given any other Topelitz–Cuntz–Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$ in a C^* -algebra B , there exists a unique homomorphism $\pi_t : \mathcal{T}C^*(\Lambda) \rightarrow B$ such that $\pi_t(s_\lambda) = t_\lambda$ for every $\lambda \in \Lambda$.

PROPOSITION 2.7. *Let (Λ, d) be an aperiodic finitely aligned k -graph. Let $\{\xi_x : x \in \partial\Lambda^{\text{ap}}\}$ be the canonical basis for $\mathcal{H}^{\text{ap}} := \ell^2(\partial\Lambda^{\text{ap}})$. For $\lambda \in \Lambda$, define*

$$S_\lambda^{\text{ap}} \xi_x := \begin{cases} \xi_{\lambda x} & \text{if } s(\lambda) = r(x), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then, $\{S_\lambda^{\text{ap}} : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ -family on \mathcal{H}^{ap} . Furthermore, every S_ν^{ap} is non-zero.

Proof. First, we show that $S_\lambda^{\text{ap}} \neq 0$ for all $\lambda \in \Lambda$. Fix $\lambda \in \Lambda$. Since Λ is aperiodic, [13, Proposition 3.6] implies that there exists $x \in \partial\Lambda^{\text{ap}}$ with $r(x) = s(\lambda)$. Then $\lambda x \in \partial\Lambda^{\text{ap}}$ by Lemma 2.5. So $S_\lambda^{\text{ap}} \xi_x = \xi_{\lambda x} \neq 0$, which implies $S_\lambda^{\text{ap}} \neq 0$.

Lemma 4.6 of [18] applied with $\mathcal{E} = \text{FE}(\Lambda)$ implies that there is a Cuntz–Krieger Λ -family on $\ell^2(\partial\Lambda)$ satisfying (2.1). Lemma 2.5 shows that $\ell^2(\partial\Lambda^{\text{ap}})$ is an invariant subspace for the corresponding representation and the result follows. \square

DEFINITION 2.8. Let (Λ, d) be an aperiodic finitely aligned k -graph. We define $C_{\min}^*(\Lambda)$ to be the C^* -subalgebra of $\mathcal{B}(\mathcal{H}^{\text{ap}})$ generated by $\{S_\lambda^{\text{ap}} : \lambda \in \Lambda\}$.

The notation $C_{\min}^*(\Lambda)$ will be justified in Theorem 3.11.

3. The Co-universal algebras.

DEFINITION 3.1. Let (Λ, d) be a finitely aligned k -graph. A *boolean representation* of Λ in a C^* -algebra B is a map $q : \lambda \mapsto q_\lambda$ from Λ to B such that each q_λ is a projection, and

$$q_\mu q_\nu = \sum_{\gamma \in \text{MCE}(\mu, \nu)} q_\gamma. \tag{3.1}$$

REMARK 3.2.

- (i) When $\text{MCE}(\mu, \nu) = \emptyset$, (3.1) is intended to mean that $q_\mu q_\nu = 0$.
- (ii) Since $\text{MCE}(\mu, \nu) = \text{MCE}(\nu, \mu)$ for all μ, ν , (3.1) implies that the projections in a boolean representation of Λ pairwise commute.

LEMMA 3.3. [15, Proposition 8.6] *Let (Λ, d) be a finitely aligned k -graph, let q be a boolean representation of Λ , and F be a finite subset of Λ such that $\lambda \in F$ implies $s(\lambda) \in F$. For $\lambda \in \vee F$, define*

$$Q_\lambda^{\vee F} := q_\lambda \prod_{\lambda\alpha \in \vee F \setminus \{\lambda\}} (q_\lambda - q_{\lambda\alpha}). \tag{3.2}$$

Then, the $Q_\lambda^{\vee F}$ are mutually orthogonal projections and for each $\mu \in \vee F$,

$$q_\mu = \sum_{\mu\mu' \in \vee F} Q_{\mu\mu'}^{\vee F}. \tag{3.3}$$

REMARK 3.4. Observe that $\mu' = s(\mu)$ indexes a term in the sum (3.3), so the sum includes the term $Q_\mu^{\vee F}$.

Proof of Lemma 3.3 The argument which establishes the displayed equation on [15, p.421] immediately below equation (8.5) uses only the relation $t_\mu t_\mu^* t_\nu t_\nu^* = \sum_{\lambda \in \text{MCE}(\mu, \nu)} t_\lambda t_\lambda^*$ for representations of product systems of graphs. So, after identifying finitely aligned product systems of graphs over \mathbb{N}^k with finitely aligned k -graphs as in [5, Example 1.4], an identical arguments works here. \square

REMARK 3.5. If $F \subset \Lambda$ is finite and closed under minimal common extensions, then $\vee F = F$. Hence, $\{Q_\lambda^F : \lambda \in F\}$ are mutually orthogonal projections and $q_\mu = \sum_{\mu\mu'} Q_{\mu\mu'}^F$ for $\mu \in F$.

LEMMA 3.6. *Let (Λ, d) be a finitely aligned k -graph and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Topfplitz–Cuntz–Krieger Λ -family. Let $q_\alpha = t_\alpha t_\alpha^*$ for $\alpha \in \Lambda$. Then $q : \alpha \mapsto q_\alpha$ is a boolean representation of Λ . If $\{t_\lambda : \lambda \in \Lambda\}$ satisfies (CK), then for a finite exhaustive set $F \subseteq s(\alpha)\Lambda$,*

$$\prod_{\alpha\alpha' \in F \setminus \{\alpha\}} (q_\alpha - q_{\alpha'}) = 0.$$

Proof. Multiplying (TCK3) on the left by t_μ and on the right by t_ν^* shows that q_λ is a boolean representation of Λ . Fix $\lambda \in \Lambda$ and let F be a finite exhaustive set such that $F \subseteq s(\lambda)\Lambda$. Then $\prod_{\lambda\mu \in F \setminus \{\lambda\}} (q_\lambda - q_{\lambda\mu}) = \prod_{\lambda\mu \in F \setminus \{\lambda\}} (t_\lambda t_\lambda^* - t_{\lambda\mu} t_{\lambda\mu}^*) = t_\lambda (\prod_{\lambda\mu \in F \setminus \{\lambda\}} (t_{s(\lambda)} - t_\mu t_\mu^*)) t_\lambda^* = 0$ by (CK). \square

LEMMA 3.7. *Let (Λ, d) be a finitely aligned k -graph and let $q : \lambda \mapsto q_\lambda$ a boolean representation of Λ with each $q_\lambda \neq 0$. Let F be a finite subset of Λ which is closed under minimal common extensions. If $\alpha \in F$ and $\{\alpha' \in \Lambda \setminus \Lambda^0 : \alpha\alpha' \in F\}$ is not exhaustive, then the projection Q_α^F of (3.2) is non-zero.*

Proof. Suppose that $\{\alpha' \in \Lambda \setminus \Lambda^0 : \alpha\alpha' \in F\}$ is not exhaustive. Then there exist $\tau \in s(\alpha)\Lambda$ such that $\text{MCE}(\alpha', \tau) = \emptyset$ whenever $\alpha\alpha' \in F \setminus \{\alpha\}$. Since $\text{MCE}(\alpha', \tau) = \emptyset$ implies that $\text{MCE}(\alpha\alpha', \alpha\tau) = \emptyset$, it follows that $q_{\alpha\alpha'} q_{\alpha\tau} = \sum_{\gamma \in \text{MCE}(\alpha\alpha', \alpha\tau)} q_\gamma = 0$ for all $\alpha\alpha' \in F \setminus \{\alpha\}$. So, $Q_\alpha^F q_{\alpha\tau} = q_\alpha \prod_{\alpha\alpha' \in F \setminus \{\alpha\}} (q_\alpha - q_{\alpha\alpha'}) q_{\alpha\tau} = q_{\alpha\tau} \neq 0$. \square

For the following proposition, we need some notation. Let Λ be a finitely aligned k -graph. For each $\lambda \in \Lambda$, define $P_\lambda^{\text{ap}} := S_\lambda^{\text{ap}} (S_\lambda^{\text{ap}})^*$ where $\{S_\lambda^{\text{ap}} : \lambda \in \Lambda\}$ is the Cuntz–Krieger Λ -family of Proposition 2.7.

PROPOSITION 3.8. *Let (Λ, d) be a finitely aligned k -graph and let q be a boolean representation of Λ such that $q_\lambda \neq 0$ for each $\lambda \in \Lambda$. Then there is a homomorphism $\psi_q : \overline{\text{span}}\{q_\lambda : \lambda \in \Lambda\} \rightarrow \overline{\text{span}}\{P_\lambda^{\text{ap}} : \lambda \in \Lambda\}$ satisfying $\psi_q(q_\lambda) = P_\lambda^{\text{ap}}$ for all $\lambda \in \Lambda$. Moreover, ψ_q is injective if and only if $\prod_{\mu \in F} (q_\lambda - q_{\lambda\mu}) = 0$ for each $\lambda \in \Lambda$ and finite exhaustive set $F \subseteq s(\lambda)\Lambda$.*

Proof. For the first assertion, fix a finite set $F \subseteq \Lambda$ and scalars $\{a_\lambda : \lambda \in F\}$. Since $q_\lambda \mapsto P_\lambda^{\text{ap}}$ preserves products and adjoints, it suffices to show that

$$\left\| \sum_{\lambda \in F} a_\lambda P_\lambda^{\text{ap}} \right\| \leq \left\| \sum_{\lambda \in F} a_\lambda q_\lambda \right\|.$$

For $\alpha \in \Lambda$, define

$$T_\alpha^{\vee F} := P_\alpha^{\text{ap}} \prod_{\alpha\alpha' \in \vee F \setminus \{\alpha\}} (P_\alpha^{\text{ap}} - P_{\alpha\alpha'}^{\text{ap}}),$$

and let $Q_\alpha^{\vee F}$ be as in (3.2). For each $\lambda \in \vee F \setminus F$, define $a_\lambda := 0$. Then Lemma 3.3 implies that

$$\sum_{\lambda \in F} a_\lambda P_\lambda^{\text{ap}} = \sum_{\lambda \in \vee F} a_\lambda P_\lambda^{\text{ap}} = \sum_{\alpha \in \vee F} \left(\sum_{\substack{\lambda \in \vee F \\ \alpha = \lambda\lambda'}} a_\lambda \right) T_\alpha^{\vee F} \text{ and}$$

$$\sum_{\lambda \in F} a_\lambda q_\lambda = \sum_{\lambda \in \vee F} a_\lambda q_\lambda = \sum_{\alpha \in \vee F} \left(\sum_{\substack{\lambda \in \vee F \\ \alpha = \lambda\lambda'}} a_\lambda \right) Q_\alpha^{\vee F}.$$

Also, Lemmas 3.6 and 3.7 imply that $\{\alpha \in \vee F : T_\alpha^{\vee F} \neq 0\} \subseteq \{\alpha \in \vee F : Q_\alpha^{\vee F} \neq 0\}$. Thus,

$$\left\| \sum_{\lambda \in F} a_\lambda P_\lambda^{\text{ap}} \right\| = \max_{T_\alpha^{\vee F} \neq 0} \left| \sum_{\lambda \in \vee F, \alpha = \lambda\lambda'} a_\lambda \right| \leq \max_{Q_\alpha^{\vee F} \neq 0} \left| \sum_{\lambda \in \vee F, \alpha = \lambda\lambda'} a_\lambda \right| = \left\| \sum_{\lambda \in F} a_\lambda q_\lambda \right\|. \tag{3.4}$$

Now suppose that $\prod_{\mu \in F} (P_\lambda^{\text{ap}} - P_{\lambda\mu}^{\text{ap}}) = 0$ for each $\lambda \in \Lambda$ and a finite exhaustive set $F \in s(\lambda)\Lambda$. Then for each finite exhaustive set F , we have $\{\alpha \in F : T_\alpha^{\vee F} \neq 0\} = \{\alpha \in F : Q_\alpha^{\vee F} \neq 0\}$. Thus we have equality at the second step in (3.4), which implies that ψ_p is isometric. To show the other direction, suppose that ψ_p is injective. Fix $\lambda \in \Lambda$ and a finite exhaustive set $F \subseteq s(\lambda)\Lambda$. Lemma 3.6 implies that $\prod_{\mu \in F} (P_\lambda^{\text{ap}} - P_{\lambda\mu}^{\text{ap}}) = 0$. Hence, $\psi_q(\prod_{\mu \in F} (q_\lambda - q_{\lambda\mu})) = \prod_{\mu \in F} (\psi_q(q_\lambda) - \psi_q(q_{\lambda\mu})) = 0$. Since ψ_q is injective, this forces $\prod_{\mu \in F} (q_\lambda - q_{\lambda\mu}) = 0$. \square

PROPOSITION 3.9. *Let (Λ, d) be an aperiodic finitely aligned k -graph. Let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz–Cuntz–Krieger Λ -family. For $\lambda \in \Lambda$, let $q_\lambda := t_\lambda t_\lambda^*$. Then for a finite subset F of Λ and collection $\{a_{\mu, \nu} : \mu, \nu \in F\}$ of scalars,*

$$\left\| \sum_{\mu \in F} a_{\mu, \mu} q_\mu \right\| \leq \left\| \sum_{\mu, \nu \in F} a_{\mu, \nu} t_\mu t_\nu^* \right\|. \tag{3.5}$$

Moreover, there exists a conditional expectation $\Phi_t : C^*(\{t_\lambda : \lambda \in \Lambda\}) \rightarrow \overline{\text{span}}\{q_\lambda : \lambda \in \Lambda\}$ satisfying $\Phi_t(t_\mu t_\nu^*) = \delta_{\mu, \nu} q_\mu$. In particular, if $\pi_t : \mathcal{TC}^*(\Lambda) \rightarrow C^*(\{t_\lambda : \lambda \in \Lambda\})$ and $\pi_{\mathcal{S}^{\text{ap}}} : \mathcal{TC}^*(\Lambda) \rightarrow C^*(\{\mathcal{S}_\lambda^{\text{ap}} : \lambda \in \Lambda\})$ are the homomorphisms induced by the universal property of $\mathcal{TC}^*(\Lambda)$, then

$$\psi_q \circ \Phi_t \circ \pi_t = \Phi_{\mathcal{S}^{\text{ap}}} \circ \pi_{\mathcal{S}^{\text{ap}}}. \tag{3.6}$$

Proof. To prove (3.5), fix a finite subset $F \subseteq \Lambda$ and scalars $\{a_{\mu, \nu} : \mu, \nu \in F\}$. Assume without loss of generality that F is closed under minimal common extensions. Define

$$F' = \bigcup_{\mu, \nu \in F} \{\lambda\beta', \lambda\delta' : \lambda = \mu\beta = \nu\delta \in \text{MCE}(\mu, \nu) \text{ and } \beta\beta' = \delta\delta' \in \text{MCE}(\beta, \delta)\}.$$

Then F' is finite since F is finite and each $\text{MCE}(\beta, \delta)$ is finite. Let $\bar{F} = F \cup F'$.

For each $\lambda \in F$, let $B_\lambda^{\sqrt{F}} = \{\lambda' \in s(\lambda)\Lambda \setminus \{s(\lambda)\} : \lambda\lambda' \in \sqrt{F}\}$. For each $\lambda \in F$, if $B_\lambda^{\sqrt{F}}$ is not exhaustive, then there exists $\alpha_1 \in s(\lambda)\Lambda$ such that $\text{MCE}(\mu, \alpha_1) = \emptyset$ for all $\mu \in B_\lambda^{\sqrt{F}}$. Fix $\alpha \in s(\alpha_1)\Lambda$ such that for any $i \leq k$ satisfying $d(\alpha)_i < \max_{\mu \in \sqrt{F}} d(\mu)_i$, we have $s(\alpha)\Lambda^{e_i} = \emptyset$. Define $\alpha^\lambda := \alpha_1\alpha$. Then $\text{MCE}(\mu, \alpha^\lambda) = \emptyset$ for each $\mu \in F$ by choice of α_1 ; and our choice of α ensures that

$$(\mu \in \sqrt{F} \text{ and } \text{MCE}(\lambda\alpha^\lambda, \mu) \neq \emptyset) \implies \lambda\alpha^\lambda = \mu\mu' \text{ for some } \mu' \in \Lambda. \tag{3.7}$$

Let $G = \{\epsilon \in \Lambda : \lambda\alpha^\lambda = \mu\epsilon \text{ for some } \lambda, \mu \in \sqrt{F}\}$. By Lemma 2.3, for each $v \in s(G) := \{s(\epsilon) : \epsilon \in G\}$, there exists $\tau_v \in v\Lambda$ such that $\text{MCE}(\epsilon\tau_v, \delta\tau_v) = \emptyset$ for all distinct $\epsilon, \delta \in Gv$. For $\lambda \in \sqrt{F}$, define $\tau^\lambda := \tau_{s(\alpha^\lambda)}$. For $\lambda \in F$, we define

$$\phi_\lambda^{\sqrt{F}} := \begin{cases} q_{\lambda\alpha^\lambda\tau^\lambda} & \text{if } B_\lambda^{\sqrt{F}} \text{ is not exhaustive} \\ Q_\lambda^{\sqrt{F}} & \text{otherwise.} \end{cases} \tag{3.8}$$

By definition, each $\phi_\lambda^{\sqrt{F}} \leq Q_\lambda^{\sqrt{F}}$. So since the $Q_\lambda^{\sqrt{F}}$ are mutually orthogonal, the $\phi_\lambda^{\sqrt{F}}$ are mutually orthogonal. Hence, for $\lambda \in F$

$$\left\| \sum_{\mu, v \in \sqrt{F}} a_{\mu, v} t_\mu t_v^* \right\| \geq \left\| \sum_{\lambda \in F} \phi_\lambda^{\sqrt{F}} \left(\sum_{\mu, v \in \sqrt{F}} a_{\mu, v} t_\mu t_v^* \right) \phi_\lambda^{\sqrt{F}} \right\|. \tag{3.9}$$

For $\lambda, \mu, v \in F$, we claim that

$$\phi_\lambda^{\sqrt{F}} t_\mu t_v^* \phi_\lambda^{\sqrt{F}} = \begin{cases} \phi_\lambda^{\sqrt{F}} & \text{if } \mu = v \text{ and } \lambda = \mu\lambda' \text{ for some } \lambda' \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

To prove (3.10), we consider a number of cases separately.

Case 1: $\mu = v$. We first show that (3.10) holds if either $\lambda = \mu\lambda'$ or $\text{MCE}(\lambda, \mu) = \emptyset$. If $\lambda = \mu\lambda'$, then $\phi_\lambda^{\sqrt{F}} \leq Q_\lambda^{\sqrt{F}} \leq q_\lambda \leq q_\mu$, and hence $\phi_\lambda^{\sqrt{F}} t_\mu t_\mu^* \phi_\lambda^{\sqrt{F}} = \phi_\lambda^{\sqrt{F}} q_\mu \phi_\lambda^{\sqrt{F}} = \phi_\lambda^{\sqrt{F}}$. And if $\text{MCE}(\mu, \lambda) = \emptyset$, then $q_\mu q_\lambda = 0$; and hence the identities $q_{\lambda\alpha^\lambda\tau^\lambda} q_\mu q_{\lambda\alpha^\lambda\tau^\lambda} = q_{\lambda\alpha^\lambda\tau^\lambda} q_\lambda q_\mu q_{\lambda\alpha^\lambda\tau^\lambda} = 0$ and $Q_\lambda^{\sqrt{F}} q_\mu Q_\lambda^{\sqrt{F}} = Q_\lambda^{\sqrt{F}} q_\lambda q_\mu Q_\lambda^{\sqrt{F}} = 0$ give $\phi_\lambda^{\sqrt{F}} q_\mu \phi_\lambda^{\sqrt{F}} = 0$. So to complete Case 1, we suppose that $\lambda \neq \mu\lambda'$ and that $\text{MCE}(\mu, \lambda) \neq \emptyset$; we must show that $\phi_\lambda^{\sqrt{F}} q_\mu \phi_\lambda^{\sqrt{F}} = 0$. We consider two subcases.

Case 1a: $\mu = v$ and $B_\lambda^{\sqrt{F}}$ is not exhaustive. Then $\phi_\lambda^{\sqrt{F}} = q_{\lambda\alpha^\lambda\tau^\lambda}$, so since $q_\lambda \geq q_{\lambda\alpha^\lambda\tau^\lambda}$,

$$\phi_\lambda^{\sqrt{F}} q_\mu \phi_\lambda^{\sqrt{F}} = q_{\lambda\alpha^\lambda\tau^\lambda} q_\mu q_{\lambda\alpha^\lambda\tau^\lambda} = \sum_{\gamma \in \text{MCE}(\mu, \lambda)} q_{\lambda\alpha^\lambda\tau^\lambda} q_\gamma q_{\lambda\alpha^\lambda\tau^\lambda}.$$

By choice of α^λ , $\text{MCE}(\lambda', \alpha^\lambda) = \emptyset$ for all $\lambda' \in B_\lambda^{\sqrt{F}}$. Hence, $\text{MCE}(\gamma, \lambda\alpha^\lambda) = \text{MCE}(\lambda\lambda', \lambda\alpha^\lambda) = \emptyset$. Thus, $q_\gamma q_{\lambda\alpha^\lambda} = 0$, giving $\phi_\lambda^{\sqrt{F}} q_\mu \phi_\lambda^{\sqrt{F}} = 0$. This completes Case 1a.

Case 1b: $\mu = v$ and $B_\lambda^{\sqrt{F}}$ is exhaustive. Then $\phi_\lambda^{\sqrt{F}} = Q_\lambda^{\sqrt{F}}$. Whenever $\beta, \beta\beta' \in \sqrt{F}$, we have $Q_\beta^{\sqrt{F}} \leq q_\beta - q_{\beta\beta'}$ and $(q_\beta - q_{\beta\beta'}) \perp q_{\beta\beta'}$. Since $\lambda, \mu \in F$ and F is closed under minimal common extensions, $\text{MCE}(\lambda, \mu) \subset F \subset \sqrt{F}$. So,

$$\begin{aligned} Q_\lambda^{\sqrt{F}} q_\mu Q_\lambda^{\sqrt{F}} &= Q_\lambda^{\sqrt{F}} q_\mu q_\lambda \prod_{\lambda\alpha \in \sqrt{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\alpha}) \\ &= Q_\lambda^{\sqrt{F}} \sum_{\lambda\lambda' \in \text{MCE}(\lambda, \mu)} q_{\lambda\lambda'} \prod_{\lambda\alpha \in \sqrt{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\alpha}) = 0. \end{aligned}$$

This completes Case 1b.

Case 2: $\mu \neq \nu$. Again we consider two subcases.

Case 2a: $\mu \neq \nu$ and $B_\lambda^{\vee\bar{F}}$ is exhaustive. Then $\phi_\lambda^{\vee\bar{F}} = Q_\lambda^{\vee\bar{F}}$. We calculate:

$$t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* = q_\lambda q_\mu t_\mu t_\nu^* q_\nu q_\lambda = \sum_{\substack{\lambda\alpha=\mu\beta \in \text{MCE}(\lambda, \mu) \\ \lambda\tau=\nu\delta \in \text{MCE}(\lambda, \nu)}} q_{\mu\beta} t_\mu t_\nu^* q_{\nu\delta}.$$

Applying (TCK3) and the definition of the q_λ to each term in the sum gives

$$t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* = \sum_{\substack{\lambda\alpha=\mu\beta \in \text{MCE}(\lambda, \mu) \\ \lambda\tau=\nu\delta \in \text{MCE}(\lambda, \nu) \\ \beta\beta'=\delta\delta' \in \text{MCE}(\beta, \delta)}} t_{\lambda\alpha\beta'} t_{\lambda\tau\delta'}^*.$$

Thus,

$$\begin{aligned} Q_\lambda^{\vee\bar{F}} t_\mu t_\nu^* Q_\lambda^{\vee\bar{F}} &= q_\lambda \prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) t_\mu t_\nu^* q_\lambda \prod_{\lambda\lambda' \in \vee\bar{F}} (q_\lambda - q_{\lambda\lambda'}) \\ &= \prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) \left(\sum_{\substack{\lambda\alpha=\mu\beta \in \text{MCE}(\lambda, \mu) \\ \lambda\tau=\nu\delta \in \text{MCE}(\lambda, \nu) \\ \beta\beta'=\delta\delta' \in \text{MCE}(\beta, \delta)}} t_{\lambda\alpha\beta'} t_{\lambda\tau\delta'}^* \right) \prod_{\lambda\lambda' \in \vee\bar{F}} (q_\lambda - q_{\lambda\lambda'}) \\ &= \prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) \left(\sum_{\substack{\lambda\alpha=\mu\beta \in \text{MCE}(\lambda, \mu) \\ \lambda\tau=\nu\delta \in \text{MCE}(\lambda, \nu) \\ \beta\beta'=\delta\delta' \in \text{MCE}(\beta, \delta)}} q_{\lambda\alpha\beta'} t_{\lambda\alpha\beta'} t_{\lambda\tau\delta'}^* q_{\lambda\tau\delta'} \right) \prod_{\lambda\lambda' \in \vee\bar{F}} (q_\lambda - q_{\lambda\lambda'}). \end{aligned}$$

Fix $\lambda\alpha = \mu\beta \in \text{MCE}(\lambda, \mu)$, $\lambda\tau = \nu\delta \in \text{MCE}(\lambda, \nu)$ and $\beta\beta' = \delta\delta' \in \text{MCE}(\beta, \delta)$. If either $d(\alpha) \neq 0$ or $d(\tau) \neq 0$, then

$$\prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) q_{\lambda\alpha\beta'} t_{\lambda\alpha\beta'} t_{\lambda\tau\delta'}^* q_{\lambda\tau\delta'} \prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) = 0,$$

because $\prod_{\lambda\lambda' \in \vee\bar{F}} (q_\lambda - q_{\lambda\lambda'}) q_{\lambda\alpha\beta'} \leq (q_\lambda - q_{\lambda\alpha}) q_{\lambda\alpha\beta'} = 0$ and similarly on the right. So suppose that $d(\alpha) = d(\tau) = 0$. Then $\mu = \lambda(0, d(\mu))$ and $\nu = \lambda(0, d(\nu))$. So $\mu \neq \nu$ implies $d(\beta') - d(\delta') = d(\nu) - d(\mu) \neq 0$, whence one of $d(\beta')$ and $d(\delta')$ is non-zero. Since $\lambda\beta', \lambda\delta' \in \vee\bar{F}$ by construction, this forces

$$\begin{aligned} \prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) q_{\lambda\alpha\beta'} t_{\lambda\alpha\beta'} t_{\lambda\tau\delta'}^* q_{\lambda\tau\delta'} \prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) \\ = \prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) q_{\lambda\beta'} t_{\lambda\beta'} t_{\lambda\delta'}^* q_{\lambda\delta'} \prod_{\lambda\lambda' \in \vee\bar{F} \setminus \{\lambda\}} (q_\lambda - q_{\lambda\lambda'}) = 0, \end{aligned}$$

as above. This completes Case 2a.

Case 2b: $\mu \neq \nu$ and $B_\lambda^{\vee\bar{F}}$ is not exhaustive. Then,

$$\begin{aligned} \phi_\lambda^{\vee\bar{F}} t_\mu t_\nu^* \phi_\lambda^{\vee\bar{F}} &= q_{\lambda\alpha^\lambda\tau^\lambda} q_\mu t_\mu t_\nu^* q_\nu q_{\lambda\alpha^\lambda\tau^\lambda} \\ &= \sum_{\substack{\lambda\alpha^\lambda\tau^\lambda\eta=\mu\gamma\in\text{MCE}(\lambda\alpha^\lambda\tau^\lambda,\mu) \\ \lambda\alpha^\lambda\tau^\lambda\zeta=\nu\delta\in\text{MCE}(\lambda\alpha^\lambda\tau^\lambda,\nu)}} q_{\lambda\alpha^\lambda\tau^\lambda\eta} t_\mu t_\nu^* q_{\lambda\alpha^\lambda\tau^\lambda\zeta} \\ &= \sum_{\substack{\lambda\alpha^\lambda\tau^\lambda\eta=\mu\gamma\in\text{MCE}(\lambda\alpha^\lambda\tau^\lambda,\mu) \\ \lambda\alpha^\lambda\tau^\lambda\zeta=\nu\delta\in\text{MCE}(\lambda\alpha^\lambda\tau^\lambda,\nu)}} t_{\lambda\alpha^\lambda\tau^\lambda\eta} t_{\lambda\alpha^\lambda\tau^\lambda\eta}^* t_\mu t_\nu^* t_{\lambda\alpha^\lambda\tau^\lambda\zeta} t_{\lambda\alpha^\lambda\tau^\lambda\zeta}^*. \end{aligned}$$

By (3.7), if $\lambda\alpha^\lambda \neq \mu\epsilon'$ or $\lambda\alpha^\lambda \neq \nu\delta'$, then $\phi_\lambda^{\vee\bar{F}} t_\mu t_\nu^* \phi_\lambda^{\vee\bar{F}} = 0$. So suppose that $\lambda\alpha^\lambda = \mu\epsilon'$ and $\lambda\alpha^\lambda = \nu\delta'$. Then,

$$t_{\lambda\alpha^\lambda\tau^\lambda\eta}^* t_\mu t_\nu^* t_{\lambda\alpha^\lambda\tau^\lambda\zeta} = t_{\epsilon'\tau^\lambda\eta}^* t_\mu t_\nu^* t_{\nu\delta'\tau^\lambda\zeta} = t_{\epsilon'\tau^\lambda\eta}^* t_{\delta'\tau^\lambda\zeta}.$$

By choice of τ^λ , for distinct ϵ', δ' such that $\lambda\alpha^\lambda = \mu\epsilon' = \nu\delta'$, we have $\text{MCE}(\epsilon'\tau^\lambda, \delta'\tau^\lambda) = \emptyset$, which implies that $\text{MCE}(\epsilon'\tau^\lambda\eta, \delta'\tau^\lambda\zeta) = \emptyset$ for any $\eta, \zeta \in \Lambda$. Thus, $t_{\lambda\alpha^\lambda\tau^\lambda\eta}^* t_\mu t_\nu^* t_{\lambda\alpha^\lambda\tau^\lambda\zeta} = t_{\epsilon'\tau^\lambda\eta}^* t_{\nu\delta'\tau^\lambda\zeta} = 0$, forcing $\phi_\lambda^{\vee\bar{F}} t_\mu t_\nu^* \phi_\lambda^{\vee\bar{F}} = 0$. This completes Case 2b, establishing (3.10).

Now, to establish (3.5), we first show that

$$\{\lambda \in \vee\bar{F} : \phi_\lambda^{\vee\bar{F}} \neq 0\} = \{\lambda \in \vee\bar{F} : Q_\lambda^{\vee\bar{F}} \neq 0\}. \tag{3.11}$$

If $B_\lambda^{\vee\bar{F}}$ is exhaustive, then $\phi_\lambda^{\vee\bar{F}} = Q_\lambda^{\vee\bar{F}}$, which implies that $\phi_\lambda^{\vee\bar{F}} \neq 0 \Leftrightarrow Q_\lambda^{\vee\bar{F}} \neq 0$; and if $B_\lambda^{\vee\bar{F}}$ is not exhaustive, then $\phi_\lambda^{\vee\bar{F}} = q_{\lambda\alpha^\lambda\tau^\lambda} = q_{\lambda\alpha^\lambda\tau^\lambda} Q_\lambda^{\vee\bar{F}}$, and since $q_{\lambda\alpha^\lambda\tau^\lambda} \neq 0$ by hypothesis, it follows that both $\phi_\lambda^{\vee\bar{F}}$ and $Q_\lambda^{\vee\bar{F}}$ are non-zero. This gives (3.11).

Define $a_{\mu,\nu} := 0$ for $\mu, \nu \in \vee\bar{F} \setminus F$. Then

$$\begin{aligned} \left\| \sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^* \right\| &= \left\| \sum_{\mu,\nu \in \vee\bar{F}} a_{\mu,\nu} t_\mu t_\nu^* \right\| \\ &\geq \left\| \sum_{\lambda \in F} \phi_\lambda^{\vee\bar{F}} \left(\sum_{\mu,\nu \in \vee\bar{F}} a_{\mu,\nu} t_\mu t_\nu^* \right) \phi_\lambda^{\vee\bar{F}} \right\| \text{ by (3.9)} \\ &= \left\| \sum_{\lambda \in F} \phi_\lambda^{\vee\bar{F}} \left(\sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^* \right) \phi_\lambda^{\vee\bar{F}} \right\| \\ &= \left\| \sum_{\lambda \in F} \left(\sum_{\mu \in F} a_{\mu,\mu} t_\mu t_\mu^* \right) \phi_\lambda^{\vee\bar{F}} \right\| \text{ by (3.10)} \\ &= \max_{\phi_\lambda^{\vee\bar{F}} \neq 0} \left| \sum_{\substack{\mu \in F \\ \lambda = \mu\mu' \in F}} a_{\mu,\mu} \right|. \end{aligned}$$

Moreover,

$$\begin{aligned} \left\| \sum_{\mu \in F} a_{\mu, \mu} q_{\mu} \right\| &= \left\| \sum_{\mu \in \sqrt{F}} a_{\mu, \mu} q_{\mu} \right\| \\ &= \left\| \sum_{\lambda \in \sqrt{F}} \left(\sum_{\mu \in \sqrt{F}} a_{\mu, \mu} t_{\mu} t_{\mu}^* \right) Q_{\lambda}^{\sqrt{F}} \right\| \text{ by (3.3)} \\ &= \max_{Q_{\lambda}^{\sqrt{F}} \neq 0} \left| \sum_{\substack{\mu \in \sqrt{F} \\ \lambda = \mu \mu' \in \sqrt{F}}} a_{\mu, \mu} \right| \\ &= \max_{Q_{\lambda}^{\sqrt{F}} \neq 0} \left| \sum_{\substack{\mu \in F \\ \lambda = \mu \mu' \in F}} a_{\mu, \mu} \right| \text{ since } a_{\mu, \mu} = 0 \text{ for } \mu \in \sqrt{F} \setminus F. \end{aligned}$$

Hence, (3.11) establishes (3.5).

It follows that the map $t_{\mu} t_{\nu}^* \mapsto \delta_{\mu, \nu} t_{\mu} t_{\nu}^*$ extends to a well-defined linear map Φ_t from $C^*(\{t_{\lambda} : \lambda \in \Lambda\})$ to $\overline{\text{span}}\{q_{\lambda} : \lambda \in \Lambda\}$. Since Φ_t is a linear idempotent of norm one, it is a conditional expectation by [2, Theorem II.6.10.2]. The final statement is straightforward to check since the two maps in question agree on spanning elements. \square

LEMMA 3.10. *Let (Λ, d) be an aperiodic finitely aligned k -graph, and let $C_{\min}^*(\Lambda)$ be as in Definition 2.8. Then the expectation $\Phi_{S^{\text{ap}}} : C_{\min}^*(\Lambda) \rightarrow \overline{\text{span}}\{P_{\lambda}^{\text{ap}} : \lambda \in \Lambda\}$ obtained from Proposition 3.9 is faithful on positive elements.*

Proof. Let $\{\xi_x : x \in \partial \Lambda^{\text{ap}}\}$ be the canonical orthonormal basis for $\mathcal{H}^{\text{ap}} = \ell^2(\partial \Lambda^{\text{ap}})$, and for each $x \in \partial \Lambda^{\text{ap}}$, let θ_{ξ_x, ξ_x} be the rank-one projection onto $\mathbb{C}\xi_x$. Since the canonical expectation $\Phi(T)(h) = \sum_x \theta_{\xi_x, \xi_x} T \theta_{\xi_x, \xi_x}(h)$ on $B(\mathcal{H}^{\text{ap}})$ is faithful, it suffices to show that for $a \in C_{\min}^*(\Lambda)$, we have $\Phi_{S^{\text{ap}}}(a) = \sum_{x \in \partial \Lambda^{\text{ap}}} \langle a \xi_x | \xi_x \rangle \theta_{\xi_x, \xi_x}$.

Recall the definition of $\{S_{\lambda}^{\text{ap}} : \lambda \in \Lambda\}$ from Proposition 2.7. It is not hard to calculate

$$(S_{\lambda}^{\text{ap}})^* \xi_x = \begin{cases} \xi_y & \text{if } x = \lambda y \text{ for some } y, \\ 0 & \text{otherwise.} \end{cases}$$

Fix $\mu, \nu \in \Lambda$ and $x \in \partial \Lambda^{\text{ap}}$. Then,

$$\langle S_{\mu}^{\text{ap}} (S_{\nu}^{\text{ap}})^* \xi_x | \xi_x \rangle = \langle (S_{\nu}^{\text{ap}})^* \xi_x | (S_{\mu}^{\text{ap}})^* \xi_x \rangle = \begin{cases} 1 & \text{if } x = \mu y = \nu y \text{ for some } y, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $x = \mu y = \nu y$. Let $m = d(\mu)$ and $n = d(\nu)$. Then Lemma 2.5 implies that $\sigma^m(x) = \sigma^m(\mu y) = y \in \partial \Lambda^{\text{ap}}$ and $\sigma^n(x) = \sigma^n(\nu y) = y$. Since $x \in \partial \Lambda^{\text{ap}}$, this forces $m = n$, i.e. $d(\mu) = d(\nu)$. Since $x = \mu y = \nu y$, the factorization property implies $\mu = \nu$. Hence,

$$\sum_{x \in \partial \Lambda^{\text{ap}}} \langle S_{\mu}^{\text{ap}} (S_{\nu}^{\text{ap}})^* \xi_x | \xi_x \rangle \theta_{\xi_x, \xi_x} = \delta_{\mu, \nu} \text{proj}_{\overline{\text{span}}\{\xi_x : x \in \partial \Lambda^{\text{ap}}\}} = \delta_{\mu, \nu} P_{\mu}^{\text{ap}} = \Phi_{S^{\text{ap}}}(S_{\mu}^{\text{ap}} (S_{\nu}^{\text{ap}})^*).$$

Since $\overline{\text{span}}\{S_\mu^{\text{ap}}(S_\nu^{\text{ap}})^* : \mu, \nu \in \Lambda\}$ is dense in $C_{\min}^*(\Lambda)$, we now have

$$\Phi_{S^{\text{ap}}}(a) = \sum_{x \in \partial \Lambda^{\text{ap}}} \langle a \xi_x | \xi_x \rangle \theta_{\xi_x, \xi_x} \text{ for all } a \in C_{\min}^*(\Lambda).$$

□

THEOREM 3.11. *Let (Λ, d) be an aperiodic finitely aligned k -graph. Then $C_{\min}^*(\Lambda) := C^*(S^{\text{ap}})$ is co-universal in the sense that given any Toeplitz–Cuntz–Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$ in which t_ν is non-zero for each $\nu \in \Lambda^0$, there is a homomorphism $\psi_t : C^*(t) \rightarrow C_{\min}^*(\Lambda)$ such that $\psi_t(t_\lambda) = S_\lambda^{\text{ap}}$ for all $\lambda \in \Lambda$.*

The pair $(C_{\min}^(\Lambda), S^{\text{ap}})$ is unique up to canonical isomorphism: if B is a C^* -algebra generated by Toeplitz–Cuntz–Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$ with the same co-universal property, then there is an isomorphism $C_{\min}^*(\Lambda) \cong B$ which carries each S_λ^{ap} to t_λ .*

Proof. Let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz–Cuntz–Krieger Λ -family with each t_ν non-zero. We will show that $\ker(\pi_t) \subset \ker(\pi_{S^{\text{ap}}})$ in $\mathcal{T}C^*(\Lambda)$.

Since each $t_\lambda^* t_\lambda \neq 0$, Propositions 3.8 and 3.9 imply that there is a homomorphism ψ_q from $\overline{\text{span}}\{t_\lambda t_\lambda^* : \lambda \in \Lambda\}$ to $\overline{\text{span}}\{P_\lambda^{\text{ap}} : \lambda \in \Lambda\}$ taking each $t_\lambda t_\lambda^*$ to P_λ^{ap} . So, we have

$$\begin{aligned} \pi_t(a) = 0 &\iff \pi_t(a^*a) = 0 \\ &\implies \psi_q \circ \Phi_t \circ \pi_t(a^*a) = 0 \\ &\iff (\Phi_{S^{\text{ap}}} \circ \pi_{S^{\text{ap}}})(a^*a) = 0 \text{ by (3.6)} \\ &\iff \pi_{S^{\text{ap}}}(a^*a) = 0 \text{ by Lemma 3.10} \\ &\iff \pi_{S^{\text{ap}}}(a) = 0. \end{aligned}$$

Thus, $\ker(\pi_t) \subset \ker(\pi_{S^{\text{ap}}})$. Therefore, there exists a well-defined homomorphism $\psi_t : C^*(t) \rightarrow C^*(S^{\text{ap}}) := C_{\min}^*(\Lambda)$.

For the final statement of the theorem, we use the same argument as in the proof of [19, Theorem 3.1]. □

REMARK 3.12. We have denoted both the homomorphism of commutative algebras arising in Proposition 3.8 and the homomorphism arising in Theorem 3.11 by ψ . This notation is compatible: Lemma 3.6 shows that each Toeplitz–Cuntz–Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$ determines a boolean representation $\{q_\lambda : \lambda \in \Lambda\}$ of Λ , and then the homomorphism ψ_t agrees upon restriction to $\overline{\text{span}}\{s_\lambda s_\lambda^* : \lambda \in \Lambda\}$ with ψ_q .

THEOREM 3.13. *Let (Λ, d) be an aperiodic finitely aligned k -graph with no sources.*

- (1) *If $\{t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger Λ -family with each t_ν non-zero, then the homomorphism ψ_t of Theorem 3.11 is injective if and only if*
 - (a) $\prod_{\lambda \in F} (t_\nu - t_\lambda t_\lambda^*) = 0$ whenever $\nu \in \Lambda^0$ and $F \subset \nu \Lambda^0$ is finite exhaustive; and
 - (b) the expectation Φ_t is faithful.
- (2) *If ϕ is a homomorphism from $C_{\min}^*(\Lambda)$ to a C^* -algebra C such that each $\phi(P_\nu^{\text{ap}})$ is non-zero, then ϕ is injective.*

Proof. The proof is essentially the same as that of [19, Theorem 3.2]. □

COROLLARY 3.14 (The Cuntz–Krieger uniqueness theorem). *Let (Λ, d) be an aperiodic finitely aligned k -graph, and let $C^*(\Lambda)$ be the Cuntz–Krieger algebra of [16]. There is an isomorphism $C^*(\Lambda) \cong C_{\min}^*(\Lambda)$ which carries each s_λ to S_λ^{ap} . Moreover, every Toeplitz–Cuntz–Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$ in which each t_ν is non-zero, and which*

satisfies condition (1a) of Theorem 3.13 generates an isomorphic copy of $C^*(\Lambda)$. In particular, for Toeplitz–Cuntz–Krieger families, Condition (1a) of Theorem 3.13 implies Condition (1b).

Proof. By [16, Proposition 2.12], $C^*(\Lambda)$ is generated by a Toeplitz–Cuntz–Krieger family $\{s_\lambda : \lambda \in \Lambda\}$ in which each s_ν is non-zero. Hence, Theorem 3.11 implies that there is a homomorphism ψ_s from $C^*(\Lambda)$ to $C_{\min}^*(\Lambda)$ carrying each s_λ to S_λ^{ap} . The map Φ_s agrees with that obtained by first averaging over the gauge-action on $C^*(\Lambda)$ onto its AF core, and then applying the canonical expectation from the AF algebra onto its diagonal subalgebra. Since each of these expectations is faithful, so is Φ_s . We have $\prod_{\lambda \in F} (s_\nu - s_{\lambda, s_\lambda^*}) = 0$ whenever $\nu \in \Lambda^0$ and $F \subset \nu \Lambda^0$ is finite exhaustive by (CK). Hence, Theorem 3.13 implies that π_s is an isomorphism.

Now fix a Cuntz–Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$. The universal property of $C^*(\Lambda)$ gives a homomorphism $\rho_t : C^*(\Lambda) \rightarrow C^*(t)$ satisfying $\rho_t(s_\lambda) = t_\lambda$. The co-universal property of $C_{\min}^*(\Lambda)$ gives a homomorphism $\psi_t : C^*(t) \rightarrow C_{\min}^*(\Lambda)$ satisfying $\psi_t(t_\lambda) = S_\lambda^{\text{ap}}$. Since $\psi_s^{-1} \circ \psi_t$ is an inverse for ρ_t , the result follows. \square

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