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AN ALMOST KRULL DOMAIN WITH DIVISORIAL HEIGHT ONE PRIMES

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ABSTRACT. E. Pirtle has conjectured that if D is an almost Krull domain in which the height one prime ideals are divisorial then D is a Krull domain. An example is given to show that this is not the case. Further, let $U = \{f \in D[x] \mid c(f)^{-1} = D\}$ and let $\mathcal{P}(D)$ denote the set of prime ideals of D which are minimal over some ideal $(a):(b)$, where $a, b \in D$. If Dp is a valuation ring for each $P \in \mathcal{P}(D)$ then Huckaba and Papick have asked whether $D[x]_U$ must be a Prufer domain. The given example shows that it need not be.

1. **Introduction**. Let D be an integral domain with quotient field L. D is an *almost Krull* domain provided D_P is a Krull domain for each prime ideal P of D. Clearly, if D is an almost Krull domain and $\{P_\alpha\}_{\alpha \in A}$ is the set of all height one prime ideals of D then

(i) each D_{P_α} is a rank one valuation ring and

(ii) $D = \bigcap_{\alpha \in A} D_{P_\alpha}$.

If in an integral domain D there exists a set $\{P_\alpha\}_{\alpha \in A}$ of height one prime ideals satisfying (i), (ii), and

(iii) each P_α is divisorial,

then D is called a K-domain ([14], p. 486). A K-domain need not be an almost Krull domain ([14], p. 491) and an almost Krull domain need not be a K-domain. Indeed, a one-dimensional almost Krull domain is an almost Dedekind domain and an almost Dedekind domain is Dedekind if and only if each maximal ideal is divisorial (cf. [15]). Pirtle has conjectured the following:

(1) CONJECTURE ([13], p. 433). *If D is an almost Krull domain and each height one prime ideal of D is divisorial (hence, D is a K-domain) then D is a Krull domain.*

As we have noted, the conjecture is true when D is one-dimensional. If for each polynomial $f \in D[x]$ we denote by $c(f)$ the content of f then

$$U = \{f \in D[x] \mid c(f)^{-1} = D\} = \{f \in D[x] \mid c(f)_v = D\}$$

is a multiplicative system of $D[x]$. Let $\mathcal{P}(D)$ denote the set of prime ideals of D which are minimal over some ideal $(a):(b)$, where $a, b \in D$. In ([8], p. 113) Huckaba and

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Papick have shown that if $D[x]_U$ is a Prüfer domain then D_P is a valuation ring for each $P \in \mathcal{P}(D)$. They ask the following:

(2) QUESTION ([8], p. 113). *If D_P is a valuation ring for each $P \in \mathcal{P}(D)$ is $R[x]_U$ a Prüfer domain?*

Two further related questions/conjectures that have appeared in the literature are:

(3) CONJECTURE ([6], p. 717). *There exists an essential ring which is not a Prüfer v-multiplication ring.*

(4) QUESTION ([9], note 14, p. 19). *Is every almost Krull domain a Prüfer v-multiplication ring?*

The main point of [7] is to provide an example illustrating that conjecture (3) is true. In a review of [9] Heinzer notes that the example given in [7] is an almost Krull domain and, thus, resolves question (4). In [11] Matsudu proposes an example to show that the answer to (2) is negative, but his proof relies on ([9], Example 2(d)) which is false. We provide here an example that resolves all four questions/conjectures. Indeed, one can show that the example presented in [7] suffices, but we shall present a somewhat altered version.

2. **The example**. Before giving the example we require three results.

LEMMA 1. *If $D[x]_U$ is a Prüfer domain then D is a Prüfer v-multiplication ring and $D[x]_U = D^\vee$, where $D^\vee$ is the Kronecker function ring with respect to the v-operation on D.*

PROOF. Assume that $D[x]_U$ is a Prüfer domain. In [10] Matsuda has shown that there is a family $\{V_\lambda\}_{\lambda \in \Lambda}$ of essential valuations overrings of D such that the set $\{V_\lambda^*\}_{\lambda \in \Lambda}$ of trivial extensions to $L(x)$ is the set of valuation overrings of $D[X]_U$. By Proposition 44.13 of [5] the v-operation on D is equivalent to the w-operation induced by the family $\{V_\lambda\}_{\lambda \in \Lambda}$ and hence, by Theorem 32.11 of [5], $D^\vee = \bigcap_\lambda V_\lambda^* = D[x]_U$. It now follows from Theorem 3 of [1] that D is a Prüfer v-multiplication ring.

LEMMA 2. *Let D be an almost Krull domain in which each height one prime ideal is divisorial. Then D is a Krull domain if and only if it is a Prüfer v-multiplication ring.*

PROOF. It is well known that a Krull domain is a Prüfer v-multiplication ring. Thus, assume that D is a Prüfer v-multiplication ring, let $\{P_\alpha\}_{\alpha \in A}$ be the set of height one prime ideals of D, set $V_\alpha = D_{P_\alpha}$ for each α, and let V_α^* denote the trivial extension of V_α to $L(x)$. Then $D^\vee = \bigcap_{\alpha \in A} V_\alpha^*$ ([5], Theorem 32.11 and Proposition 44.13) and since D is a K-domain with defining family $\{V_\alpha\}_{\alpha \in A}$, $D^\vee$ is a K-domain with defining family $\{V_\alpha^*\}_{\alpha \in A}$ ([14], Theorem 2.4 and Proposition 2.6). In particular, if $\beta \in A$ then $V_\beta^* \not\supset \bigcap_{\alpha \neq \beta} V_\alpha^*$ ([14], Proposition 1.7).

From the proof of Theorem 2.5 of [4] we know that $D^\vee = D[x]_U$ and, since $D[x]$ is an almost Krull domain ([12], Theorem 2.11), it follows that $D^\vee$ is an almost Krull domain. But $D^\vee$ is a Prüfer domain so $D^\vee$ is one-dimensional; that is, $D^\vee$ is an almost

Dedekind domain. Thus, $D^{\vee}$ is a Dedekind domain ([3], Theorem 3) and, hence, the family $\{V_{\alpha}^{*}\}_{\alpha\in A}$ has finite character. But then so does the family $\{V_{\alpha}\}_{\alpha\in A}$ so D is a Krull domain.

LEMMA 3. *If D is an almost Krull domain then* $\mathcal{P}(D) = \{P \in \mathrm{Spec}(D) \mid \text{height } P \leqslant 1\}$.

PROOF. Certainly each height one prime ideal of D is in $\mathcal{P}(D)$. Therefore, assume that $P \in \mathcal{P}(D)$ with P minimal over $(a):(b)$ and $P \neq (0)$. Then $a \neq 0$ and $b/a \notin D_P$ so PD_P is minimal over $aD_P : bD_P$. But D_P is a Krull domain and $aD_P : bD_P$ is a v-ideal, so P has height one ([5], Corollary 44.8. Also see the proof of Theorem 3.1c in [8]).

In view of the preceding results, a counterexample to the first conjecture resolves all four questions/conjectures.

EXAMPLE. (cf. [2], Example 1.6, and [9], Example 166). Let $R = Z[\{x/p_i, y/p_i\}_{i=1}^{\infty}]$, where Z is the ring of integers, $\{p_i\}_{i=1}^{\infty}$ is the set of positive primes, and x, y are indeterminates over Z.

(a) *R is an almost Krull domain but is not a Krull domain.*

PROOF. If p is a prime integer then $R_{Z\setminus(p)} = Z_{(p)}[x/p, y/p]$ so, in the terminology of [2], R is locally polynomial over Z. If M is a maximal ideal of R such that $M \cap Z = (0)$ then R_M is a localization of $Q[x, y]$. Otherwise, $M \cap Z = (p)$ for some prime integer p and R_M is a localization of the polynomial ring $Z_{(p)}[x/p, y/p]$. Thus, R is an almost Krull domain. But p_iR is a height one prime ideal of R for each i ([2], (1.9) and (1.11)) and $x \in \bigcap_{i=1}^{\infty} \mathcal{P}_iR$, so R is not a Krull domain.

(b) *Each height one prime ideal of R is divisorial.*

PROOF. Let $\{f_j\}_{j=1}^{\infty} \subset Q[x, y]$ be a set of irreducible polynomials such that $\{f_jQ[x, y]\}_{j=1}^{\infty}$ is the set of height one prime ideals of $Q[x, y]$. It follows from ([2], (1.9) and (1.11)) that $\{p_iR\}_{i=1}^{\infty} \cup \{f_jQ[x, y] \cap R\}_{j=1}^{\infty}$ is the set of height one prime ideals of R. Further, $R_{p_iR} = Z_{(p_i)}[x/p_i, y/p_i]_{p_iZ[x/p_i, y/p_i]}$ and $R_{f_jQ[x,y]\cap R} = Q[x, y]_{f_jQ[x,y]}$. For each prime integer p_i, let v_i be the p_i-adic valuation on Q. Then R_{p_iR} is the valuation ring associated with the trivial extension v_i^{*} of v_i to $Q(x, y)$ determined by $v_i^{*}(x) = v_i^{*}(y) = v_i^{*}(p) = 1$. It is straightforward to see that for each $\xi \in Q(x, y)$ there exists a positive integer m such that $v_i^{*}(\xi) \geqslant -m$ for all i. Thus, $x^m\xi$ and $y^m\xi$ are in $\bigcap_{i=1}^{\infty} R_{p_iR}$.

To complete the proof it suffices to show that if $\{P_{\alpha}\}_{\alpha\in A}$ is the set of height one prime ideals of R and $\beta \in A$ then there exists $\zeta \in (\bigcap_{\alpha\neq\beta} R_{P_{\alpha}})\setminus R_{P_{\beta}}$ ([14], Proposition 1.7). If $P_{\beta} = p_iR$ we may take $\zeta = 1/p_i$. If $P_{\beta} = xQ[x, y] \cap R$ we take $\zeta = y/x$ and, similarly, if $P_{\beta} = yQ[x, y] \cap R$ we take $\zeta = x/y$. If $P_{\beta} = fQ[x, y] \cap R$ and $x, y \notin fQ[x, y]$ then choose a positive integer m such that $x^m/f \in \bigcap_{i=1}^{\infty} R_{p_iR}$ and take $\zeta = x^m/f$.

(c) *R is not a Prüfer v-multiplication ring.*

PROOF. Apply Lemma 2.

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