SOME MULTIPLICATIVE FUNCTIONALS

D. G. BOURGIN

This note concerns itself primarily with the representation of continuous multiplicative functionals on L_2 types of rings or Banach algebras to the real or complex fields where convolution is taken as the ring multiplication. In a recent publication [**1**] such functionals were studied for the continuous function ring $C(S)$ over a compact space S. It was shown that for each such multiplicative functional M there is an associated countable compactum, $D(M)$, termed a determining set in S, such that the values of $x(s)$ on $D(M)$ alone, fix $M(x)$ in the real case and $M|x|$ in the complex case. For the case considered in the present work, a similar result is valid except that a finite set enters in the role of $D(M)$.

For a Banach algebra the maximal ideals are associated with continuous functionals common to the family of linear functionals and to the family of multiplicative functionals. The first family bears on the underlying Banach space and has of course been extensively investigated. The results below and earlier results [**1**], study the second, hitherto neglected, family, which is associated with the underlying multiplicative semi-group. It seems promising also to consider our results from the viewpoint of a linear representation theory of this multiplicative semi-group. In this sense our work yields the representations of degree 1.

Suppose G is a compact Abelian group. We write G' for its discrete character group and use R and K for the real and complex fields respectively. We employ F to stand for either R or K. Let $L_2\ (G,\ F)$ be the ring of functions $x \sim x(g \mid G)$ with multiplication designated by a star and defined by convolution, i.e.,

$$(x*y)(h) = \int_G x(g)y(hg^{-1})\, dg. \tag{1}$$

Let $C^\circ(S, F)$, S discrete, be the Banach algebra of functions on S to F, vanishing except on a denumerable subset at most, and such that the function values converge to 0. The norm is that induced by $C(S, F)$ and the ring multiplication is pointwise multiplication of functions. $L_p(S, F)$ is the obvious ring with pointwise multiplication and elements designated by capitals, that is, $X \sim X(s \mid S)$. Plainly the elements of $L_p(S, F)$ are in $C^\circ(S, F)$. We use *countable* to cover either *finite* or *denumerable*. For convenience we quote two results of [**1**] that intervene in the sequel. We assume $M|x|$ is not identically 0 or 1.

THEOREM A. *If M is a norm continuous multiplicative functional on $C(Q, R)$ to R, where Q is Hausdorff compact, then $M(x)$ is determined by the values of $x(q)$ on a countable compactum D and $|M(x)|$ has the representation*

Received September 17, 1951.

$$|M(x)| = M|x| = \prod_{d \in D} |x(d)|^{\mu(d)},$$

where $\mu(d) > 0$.

THEOREM B. *If M is a norm continuous multiplicative functional on $C(Q, K)$ to K, where Q is Hausdorff compact, then $M|x|$ is determined by the values of $|x(q)|$ on a countable compactum D and $M|x|$ has the representation.*

$$M|x| = \exp \sum_{d \in D} \{\mu(d) + i\,\nu(d)\} \log |x(d)|,$$

where $\mu(d) > 0$ and $\nu(d)$ is unrestricted.

Since Theorem B is stated without proof in [**1**, p. 569] we sketch the demonstration. First $M|x| = |M|x|| \exp iF|x|$. Then $|M|x||$ is the transformation denoted by M_0 in [**1**] and has the determining set D. Evidently $F|x|$ is distributive. The association of a regular measure ν and the arguments concerning the measure adduced in [**1**, Theorem 3] apply here except of course that the inferences from $F|x_n| \to \infty$ do not differ from those from $F|x_n| \to -\infty$. The conclusion then is that ν may be of either sign or 0 and is concentrated on the countable compactum, D_i. The proof that $D \supset D_i$ is immediate, for otherwise $\{x^n\}$ exists with $x^n \to x$, where $x(s)$ vanishes at some point of D_i but is bounded away from 0 on D. Then $M|x_n|/|M|x_n||$ converges while $\exp iF|x_n|$ does not, a manifest absurdity.

We now give the two theorems fundamental for our conclusions.

THEOREM 1. *If M is a norm continuous single-valued multiplicative functional on E to K where E is either $C^{\circ}(S, K)$ or $L_p(S, K)$, $p \geqslant 1$, S discrete, then there is a finite set D in S and sets of complex numbers $\{\mu(d) + i\nu(d) | \mu(d) > 0, d \in D\}$ and integers $\{n(d)|D\}$ such that*

$$M(X) = \exp \Big(\sum_{d \in D} (\mu(d) + i\,\nu(d)) \log |X(d)| + i\,n(d) \arg X(d)\Big). \tag{2}$$

THEOREM 2. *If M is a norm continuous multiplicative functional on E to R where E is either $C^{\circ}(S, R)$ or $L_p(S, R)$ $p \geqslant 1$, S discrete, then there are a finite set D in S and real numbers*

$$\{\mu(d) \mid \mu(d) > 0, d \in D\}, \ \{n(d) \mid n(d) = 0, \text{ or } n(d) = 1\}$$

such that

$$M(X) = \prod_{d \in 0} (|X(d)|^{\mu(d)} (\operatorname{sgn} X(d))^{n(d)}). \tag{3}$$

We consider first the proof of Theorem 1.

We tacitly assume below that the trivial cases $M(X) = 0$ or $M(X) = 1$ for all X are excluded. Let σ designate a finite subset of S. Write $1(\sigma)$ for the element of E which is 1 on σ and vanishes on the complement of σ. For arbitrary $X \in E$ let $Q(X) = \{s \mid X(s) \neq 0\}$. Suppose $Q(X)$ is denumerable. Order the finite subsets $\sigma(X)$ of $Q(X)$ by inclusion so $\{\sigma \mid \sigma = \sigma(X)\}$, for fixed X, is a directed set. Plainly $X\,1(\sigma)$ converges in the norm to X. Hence, since $M(X) \neq 0$ for say

https://doi.org/10.4153/CJM-1953-018-1 Published online by Cambridge University Press

$X = X_0$, there is a σ' in the collection $\{\sigma \mid \sigma = \sigma(X_0)\}$ for which $M(X_0\, 1(\sigma')) \neq 0$. If $Q(X)$ is finite, σ' may be taken as $Q(X)$ itself. Since $1(\sigma) = (1(\sigma))^2$ it follows that

$$M(X_0\, 1(\sigma')) = M(X_0\, 1(\sigma'))M(1(\sigma')).$$

Accordingly $M(1(\sigma')) = 1$. Therefore for arbitrary $X \in E$ we have

$$(4) \qquad M(X) = M(1(\sigma'))M(X) = M(X\, 1(\sigma')).$$

Thus we need consider the values of M on σ' alone; so we are reduced to consideration of multiplicative functionals over $C(\sigma', K)$.

Suppose σ' consists of N points. Then

$$(5) \qquad C(\sigma', K) = \prod_{j=1}^{N} K_j.$$

If $Z = (Z_1, \ldots, Z_N) \in C(\sigma', K)$ we refer to Z_j as the jth coordinate of Z. Write Z^j as the element of $C(\sigma', K)$ whose jth coordinate is Z_j and whose other coordinates are 1. We have

$$(6) \qquad M(Z) = \prod_{j=1}^{N} M(Z^j).$$

Evidently for fixed j, $M(Z^j)$ *may be considered as on* K *to* K. Accordingly, suppose $N = 1$ and let $W = \rho e^{i\theta}$. We have then

$$(7) \qquad M(W) = M(\rho)M(e^{i\theta}).$$

It is easy to see from the continuity condition that $M(\rho) = \exp\,(\mu + i\nu)\log\rho$, where $\mu > 0$ but ν is an unrestricted real number. For positive integers k and N and $\theta = 2\pi k/N$, $(M(e^{i\theta}))^N = M(1) = 1$. Hence $|M(e^{2k\pi i/N})| = 1$. Appeal to continuity establishes $|M(e^{i\theta})| = 1$ for arbitrary θ. The single-valuedness and continuity requirements on M imply now that M is a homomorphism on the topological group of the circle, P, into itself; that is to say that M is a character of P. It is well known then that $M(e^{i\theta}) = e^{in\theta}$ for n integral. In view of (6) the representation (2) for $N \geqslant 1$ is now fully verified.

The demonstration of Theorem 2 proceeds along similar lines. The significant part of the proof is the analogue of (5) with $C(\sigma', R)$ replacing $C(\sigma', K)$. Then a direct argument (or appeal to Theorem A, since σ' is compact) yields (3).

In the interests of completeness we note the effect of changing the continuity requirement on M.

THEOREM 3. *If M is a weakly continuous multiplicative functional on $C(Q, R)$ to R, where Q is Hausdorff compact, then D is finite and $M(x)$ has the representation* (3). *For R replaced by K the set D for $M|x|$ is finite and the representation for $M|x|$ falls under* (2).

Every functional continuous in the weak topology is surely continuous in the norm topology. Accordingly, the M's consistent with our hypotheses form a subset of those described in Theorem A and in Theorem B. Suppose that the determining set D could be infinite. Let $e(s) = 1$ for all s in S. Plainly $M(e) = 1$. Any weak neighbourhood of e is of the form

$$N(e, \sigma, \epsilon) = \{x \mid |x(s) - 1| < \epsilon, s \in \sigma\}$$

where σ is a finite subset of S. Evidently D contains a point, d_σ, not in σ. Let x_σ be a continuous function vanishing at d_σ, taking on the value 1 on σ and otherwise subject to $0 \leqslant x_\sigma(s) \leqslant 1$. Then $x_\sigma \in N(e, \sigma, \epsilon)$. Yet $M(x_\sigma) = 0$. Since σ and ϵ are arbitrary this shows M cannot be continuous in the weak topology if D is nonfinite. If D consists of a single point M satisfies the conditions of the theorem whence by combination D can be taken as a finite point set.

Let $I = \{t \mid 0 \leqslant t \leqslant 1\}$ in the sequel but interpret the elements of $L_2(I)$ as even periodic functions over $2I$ $(-1 \leqslant t \leqslant 1)$ *with convolutions over* $2I$. The functions $\{\psi_n(t) \mid \psi_0(t) = 2^{-1},\ \psi_n(t) = \cos n\pi t,\ n > 0\}$ constitute a complete orthogonal set for $L_2(I)$ and the expansion of x *in terms of* $\{\psi_n(t)\}$ we call the Fourier cosine series expansion.

THEOREM 4. *If M is a norm continuous multiplicative functional on the real Banach algebra $L_2(I)$ to R where ring multiplication in $L_2(I)$ is interpreted as convolution, then, for some finite set of integers, D,*

$$M(x) = \prod_{d \in D} |X(d)|^{\mu(d)} (\operatorname{sgn} X(d))^{n(d)}, \tag{8}$$

where $\mu(d) > 0$, $n(d) = 0$ or 1 and $\{X(n) \mid n = 0, 1, 2, \ldots\}$ are the coefficients in the Fourier cosine series expansion of x.

Let

$$x(t) \sim \sum_{j=0}^{\infty} X(j)\psi_j(t) \tag{9}$$

Then of course $\{X(j)\} \in l_2$. In view of the Parseval identity,

$$\begin{aligned}(x*y)(t) &= \int_{-1}^{1} x(\tau)y(t-\tau)\,d\tau \\ &\sim \sum_{j=0}^{\infty} (X(j)Y(j))\psi_j(t).\end{aligned} \tag{10}$$

The correspondence $x \leftrightarrow \{X(j)\}$ is a linear homeomorphism of $L_2(I)$ onto l_2. Indeed, it is compounded of $x \leftrightarrow 2^{\frac{1}{2}} x \leftrightarrow \{2^{-\frac{1}{2}} X(0),\ X(j \mid j \geqslant 1)\} \leftrightarrow \{X(j \mid j \geqslant 0)\}$, where the first map merely recognizes that the norm is taken over I and not $2I$, while the next map is a linear isometry etc. Accordingly, Theorem 2 can be applied in combination with (10) to establish (8).

It is well known [2] that a Fourier transform T can be defined on $L_2(G, K)$ to $L_2(G', K)$. Indeed $X(g') = (Tx)(g')$ is simply the coefficient in the development of x in terms of the character g' and so corresponds exactly to $X(j)$ in (9). The inverse Fourier transform T' from $L_2(G', K)$ to $L_2(G, K)$ satisfies $T'Tx = x$. Furthermore it is known that T and T' are unitary and

$$T(x*y) = Tx\,.\,Ty = X\ Y. \tag{11}$$

THEOREM 5. *If M is a norm continuous single-valued multiplicative functional on the ring $L_2(G, K)$ to K, where ring multiplication is taken as convolution, then there is a finite set D in G', complex numbers $\{\mu(d) + i\nu(d) \mid \mu(d) > 0, d \in D\}$ and integers $\{n(d) \mid D\}$ such that*

$$M(x) = \prod_{d \in D} |(T(x))(d)|^{\mu(d)+i\nu(d)} \exp i \sum_{d \in D} n(d) \arg ((T(x))(d)). \tag{12}$$

Let

$$N(X) = M(T'X) = M(x).$$

We remark

$$N(XY) = M(T'(XY) = M(x*y) = M(x)M(y) = N(X)N(Y).$$

Thus N is multiplicative and single-valued on $L_2(G', K)$ to K. Moreover since x and X are related by a unitary transformation the norm continuity of M implies norm continuity of N and conversely. Accordingly, Theorem 1 may be invoked to yield the representation (12).

REFERENCES

1. D. G. Bourgin, *Multiplicative transformations*, Proc. Nat. Acad. Sci., *36* (1950), 564–570.
2. A. Weil, *L'integration dans les groupes topologiques et ses applications*, Actualités sci. et industr., *869* (Publications de l'Inst. de Math. de Clermont-Ferrand, IV), 1–158.

University of Illinois