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Cyclicity in Dirichlet Spaces

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Abstract. Let μ be a positive finite Borel measure on the unit circle and $\mathcal{D}(\mu)$ the associated harmonically weighted Dirichlet space. In this paper we show that for each closed subset E of the unit circle with zero c_μ-capacity, there exists a function $f \in \mathcal{D}(\mu)$ such that f is cyclic (*i.e.*, $\{pf : p \text{ is a polynomial}\}$ is dense in $\mathcal{D}(\mu)$), f vanishes on E, and f is uniformly continuous. Next, we provide a sufficient condition for a continuous function on the closed unit disk to be cyclic in $\mathcal{D}(\mu)$.

1 Introduction

A bounded operator T on a Hilbert space $\mathcal{H}$ is called *two-isometry* if $T^{*2}T^2 - 2T^*T + I = 0$, is called *cyclic* if there exists $x \in \mathcal{H}$ such that $\operatorname{span}\{T^n x : n \geq 0\}$ is dense in $\mathcal{H}$, and is said to be *analytic* if $\cap_{n\geq 0} T^n\mathcal{H} = \{0\}$. Richter proved in [18] that every cyclic, analytic, and two-isometry operator can be represented as multiplication by z on the Dirichlet-type space $\mathcal{D}(\mu)$ induced by a positive finite Borel measure μ on the unit circle. These spaces were later studied by several authors; see, for instance, [6, 9, 16, 19, 20].

In this paper we are interested in the study of the cyclicity in $\mathcal{D}(\mu)$. For the Hardy space H^2, by Beurling's theorem [1] the cyclic functions are exactly the outer functions. In the classical Dirichlet space $\mathcal{D}$, Brown and Shields proved that every cyclic function in $\mathcal{D}$ is an outer function whose zero set has zero logarithmic capacity. They conjectured that the converse is also true [3, Question 12]. Some partial results toward this conjecture were obtained by Hendenmalm and Shields in [16]. They proved that every outer function $f \in \mathcal{D} \cap A(\mathbb{D})$ with countable zero set is cyclic, where $A(\mathbb{D})$ is the disk algebra . In [11, 12], El-Fallah, Kellay, and Ransford gave the first example of an uncountable closed subset E of $\mathbb{T}$ such that every outer function $f \in \mathcal{D} \cap A(\mathbb{D})$ with zero set included in E is cyclic. Furthermore, they provided some sufficient conditions on E to ensure the cyclicity of every outer function $f \in \mathcal{D} \cap A(\mathbb{D})$ vanishing on E.

Carleson [4] proved that for every closed subset E of the unit circle that has zero logarithmic capacity, there exists a cyclic function in $\mathcal{D}$ that vanishes on E. Later, Brown and Cohn in [2] modified Carleson's construction and gave a cyclic function in $\mathcal{D} \cap A(\mathbb{D})$ vanishing on E. Moreover, the problem for cyclicity in $\mathcal{D}$ is still open [9]. For a brief history of the cyclicity problem in $\mathcal{D}(\mu)$, we refer the reader to [8].

Our first aim in this work is to extend the Brown–Cohn Theorem to the Dirichlet spaces $\mathcal{D}(\mu)$. Next, we give a capacitary sufficient condition for cyclicity in this space.

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Let $\mathbb{T}$ be the boundary of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. We denote by $\mathrm{Hol}(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let μ be a positive finite Borel measure on $\mathbb{T}$; the Dirichlet-type space $\mathcal{D}(\mu)$ is given by

$$\mathcal{D}(\mu) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{D}_\mu(f) = \int_{\mathbb{D}} |f'(z)|^2 P[\mu](z) dA(z) < \infty \right\},$$

where dA is the two-dimensional Lebesgue measure and $P[\mu]$ is the Poisson integral of μ

$$P[\mu](z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

The space $\mathcal{D}(\mu)$ is endowed with the norm

$$\|f\|_\mu^2 = |f(0)|^2 + \mathcal{D}_\mu(f), \quad f \in \mathrm{Hol}(\mathbb{D}). \tag{1.1}$$

Note that $\mathcal{D}(\mu)$ is a reproducing kernel Hilbert space. Denote by k^μ the reproducing kernel of $\mathcal{D}(\mu)$; we have $f(z) = \langle f, k_z^\mu \rangle_\mu$, $f \in \mathcal{D}(\mu)$, $z \in \mathbb{D}$, where $\langle \cdot, \cdot \rangle_\mu$ is the inner product in $\mathcal{D}(\mu)$ introduced by norm (1.1). The reproducing kernel k^μ satisfies the inequalities

$$2 \operatorname{Re} k^\mu(z, w) - 1 \geq 0, \tag{1.2}$$

$$|k^\mu(z, w)| \leq \frac{2}{|1 - z\overline{w}|}, \tag{1.3}$$

for each $z, w \in \mathbb{D}$ (see, for instance, [20, Theorem 2]).

To introduce the capacity associated with $\mathcal{D}(\mu)$, we recall the definition of the harmonic Dirichlet-type space $\mathcal{D}^h(\mu)$: the set of functions $f \in \mathrm{L}^2(\mathbb{T})$ such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} \left(\left| \frac{\partial f(z)}{\partial z} \right|^2 + \left| \frac{\partial f(z)}{\partial \overline{z}} \right|^2 \right) P[\mu](z) dA(z)$$

is finite, where $f(z) := P[f](z)$ is the harmonic extension for f to $\mathbb{D}$. The space $\mathcal{D}^h(\mu)$ equipped with the norm

$$\|f\|_\mu^2 := |f(0)|^2 + \mathcal{D}_\mu(f), \quad f \in \mathrm{L}^2(\mathbb{T}),$$

is a reproducing kernel Hilbert space containing $\mathcal{D}(\mu)$ as a closed subspace [6].

Note also that $\mathcal{D}^h(\mu)$ is a Dirichlet space in the sense of Beurling and Deny [13]. The c_μ-capacity of a subset E of $\mathbb{T}$ is defined by

$$c_\mu(E) := \inf \left\{ \|f\|_\mu^2 : f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \; m\text{-a.e. on a neighborhood of } E \right\}, \tag{1.4}$$

where m denotes the Lebesgue measure on $\mathbb{T}$. From (1.4) we have $m(O) \leq c_\mu(O)$, for every open subset O of $\mathbb{T}$. Hence any set of zero c_μ-capacity is m-negligible. Moreover, since $\mathcal{D}(\mu) \subset \mathrm{L}^2(\mu)$ (see [9, Theorem 8.1.2]), we have for every Borel subset E of $\mathbb{T}$

$$\mu(E) \leq \big(1 + \mu(\mathbb{T})\big) c_\mu(E).$$

The c_μ-capacity satisfies the strong-type inequality, see [5]. Namely,

$$\int_0^{+\infty} c_\mu\big(|f| > t\big) \, dt^2 \lesssim \|f\|_\mu^2, \quad f \in \mathcal{D}^h(\mu). \tag{1.5}$$

We say that a property holds c_μ-quasi-everywhere (c_μ-q.e.) if it holds everywhere outside a set of zero c_μ-capacity.

Recall that the polynomials are dense in $\mathcal{D}(\mu)$; see [18] and [9, Corollary 7.3.4]. A function $f \in \mathcal{D}(\mu)$ is called *cyclic* in $\mathcal{D}(\mu)$ if $[f]_{\mathcal{D}(\mu)} = \mathcal{D}(\mu)$, where

$$[f]_{\mathcal{D}(\mu)} := \overline{\{pf : p \text{ is a polynomial}\}}.$$

Recall also that the outer functions are given by

$$f(z) = \exp \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \psi(\zeta) dm(\zeta), \quad z \in \mathbb{D},$$

where ψ is a positive function such that $\log \psi \in L^1(\mathbb{T})$. If $\mu = m$, then c_m is comparable to the logarithmic capacity c, and the space $\mathcal{D}(m)$ coincides with the classical Dirichlet space $\mathcal{D}$. Brown and Shields proved in [3] that if f is cyclic in $\mathcal{D}$, then f is an outer function and $c(Z(f)) = 0$, where

$$Z(f) := \{\zeta \in \mathbb{T} : f(\zeta) = 0\}.$$

They conjectured that the converse is also true. The generalized Brown–Shields conjecture asserts that an outer function $f \in \mathcal{D}(\mu)$ is cyclic if and only if $c_\mu(Z(f)) = 0$. Guillot showed in [15] that this conjecture is true for finitely atomic measure. In [7], El-Fallah, Elmadani, and Kellay proved that this conjecture is also true for measures with countable support. The generalized Brown–Shields conjecture remains open.

The disk algebra $A(\mathbb{D})$ is the set of continuous functions on the closed unit disk $\overline{\mathbb{D}}$ that are holomorphic in $\mathbb{D}$. Our first result is the following theorem.

Theorem 1.1 *Let μ be a positive finite Borel measure on $\mathbb{T}$ and let E be a closed subset of $\mathbb{T}$. If $c_\mu(E) = 0$, then there exists a function $f \in \mathcal{D}(\mu) \cap A(\mathbb{D})$ that is cyclic in $\mathcal{D}(\mu)$ and $Z(f) = E$.*

It is clear that if E is a closed subset of $\mathbb{T}$, then $c(E) = \lim_{t \to 0^+} c(E_t)$, where $E_t := \{\zeta \in \mathbb{T} : d(\zeta, E) \le t\}$ and d denotes the distance with respect to arc-length. El-Fallah, Kellay, and Ransford proved in [10] that an outer function $f \in \mathcal{D} \cap A(\mathbb{D})$ is cyclic in $\mathcal{D}$ if $c(E_t)$ goes to zero "sufficiently rapidly" as $t \to 0$, where $E = Z(f)$.

In the following theorem we extend this result to Dirichlet spaces $\mathcal{D}(\mu)$.

Theorem 1.2 *Let $f \in \mathcal{D}(\mu) \cap A(\mathbb{D})$ be an outer function and let $E = Z(f)$ and $E_t = \{\zeta \in \mathbb{T} : d(\zeta, E) \le t\}$. If*

$$\int_0 c_\mu(E_t) \frac{\log(1/t)}{t} dt < \infty, \tag{1.6}$$

then f is cyclic in $\mathcal{D}(\mu)$.

Let K be a closed subset of $\mathbb{T}$. Consider the measure $d\mu(\zeta) = d(\zeta, K)^\alpha dm(\zeta)$ for some $\alpha \in (0,1)$. The measure μ provides some examples where condition (1.6) is satisfied. Indeed, by the same calculation as [7, Theorem 5.4], we obtain

$$c_\mu(E_t) \lesssim \Big(\int_t \frac{ds}{s^\alpha m(E_s)} \Big)^{-1}$$

for every subset E of K. If E is a Cantor-type set, we obtain $c_\mu(E_t) = O(t^{\alpha - \sigma})$, where σ is the Hausdorff dimension of E; hence, (1.6) holds, for all $\alpha > \sigma$.

In the next section, we will give some properties of the c_μ-capacity. The proof of Theorem 1.1 is given in Section 3. Section 4 is devoted to the proof of Theorem 1.2.

Throughout the paper, we use the following notation:

- $A \lesssim B$ means that there is an absolute constant C such that $A \le CB$.
- $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$ hold.
- $C(\mathbb{T})$ is the space of all continuous functions on $\mathbb{T}$.
- $\mathcal{M}^+(\mathbb{T})$ denotes the set of all positive Borel measures on $\mathbb{T}$.

2 Capacity

In this section we state some properties that will be needed in the proof of our results. First we recall some definitions. A function $u \in L^2(\mathbb{T})$ is called *quasi-continuous* if for every $\epsilon > 0$, there exists a subset A of $\mathbb{T}$ with $c_\mu(A) < \epsilon$ and such that the restriction of u to $\mathbb{T} \smallsetminus A$ is continuous. A function v is said to be a *quasi-continuous modification of* u if v is quasi-continuous and $v = u$ a.e. on $\mathbb{T}$. We denote by $\widehat{u}$ a quasi-continuous modification of u.

Theorem 2.1 *Each $u \in \mathcal{D}^h(\mu)$ admits a quasi-continuous modification $\widehat{u}$.*

Proof See [14, Theorem 23]. ■

It is well known that for a given closed subset E of $\mathbb{T}$, there exists a unique measure $\nu_E \in \mathcal{M}^+(\mathbb{T})$ supported on E such that $c(E) = \nu_E(\mathbb{T})$, where the energy of ν_E defined by

$$I(\nu_E) := \int_{\mathbb{T}} \int_{\mathbb{T}} \log \frac{1}{|\zeta - \lambda|} d\nu_E(\zeta) d\nu_E(\lambda)$$

is finite; see *e.g.*, [17, Theorems 13 and 14].

The following theorem extends this result to Dirichlet spaces $\mathcal{D}(\mu)$.

Theorem 2.2 *Let E be a closed subset of $\mathbb{T}$; then there exists a unique measure $\nu_E \in \mathcal{M}^+(\mathbb{T})$ supported on E such that*

$$c_\mu(E) = \|p_{\nu_E}\|_\mu^2 = \nu_E(E),$$

where p_{ν_E} satisfies the following properties:

(i) $0 \le p_{\nu_E} \le 1$ *on* $\mathbb{T}$ *and* $\widehat{p_{\nu_E}} = 1$ c_μ*-q.e on* E;

(ii) $\langle p_{\nu_E}, v\rangle_\mu = \int_{\mathbb{T}} \widehat{v}(\zeta) d\nu_E(\zeta)$, *for each* $v \in \mathcal{D}^h(\mu)$;

(iii) $p_{\nu_E}(z) = \int_{\mathbb{T}} (2\,\mathrm{Re}\,\widehat{k^\mu(z,\lambda)} - 1) d\nu_E(\lambda)$, *for* $z \in \mathbb{D}$.

The function p_{ν_E} is called *the potential of the measure* ν_E.

Proof The proofs of assertions (i) and (ii), in a more general case, is given in [13, Theorems 2.1.5 and 2.2.2]. For the sake of completeness, we include the proofs here.

Fix a subset E of $\mathbb{T}$. Denote by

$$S(E) := \{f \in \mathcal{D}^h(\mu) : f \ge 0 \text{ and } \widehat{f} \ge 1\, c_\mu\text{-q.e. on } E\}.$$

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Note that $S(E)$ is a closed and convex subset of $\mathcal{D}^h(\mu)$. Then there exists a unique positive function $g_E \in \mathcal{D}^h(\mu)$ such that $\widehat{g_E} \geq 1$ c_μ-q.e. on E and $c_\mu(E) = \|g_E\|_\mu^2$. If $p_E = \min(g_E, 1)$, then $c_\mu(E) = \|p_E\|_\mu^2$, where $0 \leq p_E \leq 1$ and $\widehat{p_E} = 1$ c_μ-q.e. on E, that gives the assertion (i).

To prove (ii), let $v \in \mathcal{D}^h(\mu)$ be a positive function; then $p_E + \epsilon v \in \mathcal{D}^h(\mu)$, for each $\epsilon > 0$. By consequence, $2\langle p_E, v\rangle_\mu + \epsilon\|v\|_\mu^2 \geq 0$. Letting $\epsilon \to 0$, we obtain $\langle p_E, v\rangle_\mu \geq 0$ for any non-negative $v \in \mathcal{D}^h(\mu)$. This implies the existence of a unique positive Borel measure $\nu_E \in \mathcal{M}^+(\mathbb{T})$ supported on the closure $\overline{E}$ of E such that

$$\langle p_{\nu_E}, v\rangle_\mu := \langle p_E, v\rangle_\mu = \int_{\mathbb{T}} v(\zeta)\,d\nu_E(\zeta), \tag{2.1}$$

for each $v \in C(\mathbb{T})$. To extend (2.1) for $\mathcal{D}^h(\mu)$, let $v \in \mathcal{D}^h(\mu)$. There exists a sequence $v_n \in \mathcal{D}^h(\mu) \cap C(\mathbb{T})$ that is convergent to v and a subsequence v_{n_k} that converges c_μ-q.e. on $\mathbb{T}$ to $\widehat{v}$. We have from Fatou's lemma that

$$\begin{aligned}
\int_{\mathbb{T}} \big|\widehat{v}(\zeta) - v_n(\zeta)\big|\,d\nu_E(\zeta) &= \int_{\mathbb{T}} \liminf_{n_k\to+\infty} \big|v_{n_k}(\zeta) - v_n(\zeta)\big|\,d\nu_E(\zeta)\\
&\leq \liminf_{n_k\to+\infty} \int_{\mathbb{T}} \big|v_{n_k}(\zeta) - v_n(\zeta)\big|\,d\nu_E(\zeta)\\
&\leq \liminf_{n_k\to+\infty} \|p_{\nu_E}\|_\mu \|v_{n_k} - v_n\|_\mu\\
&= c_\mu(E)^{1/2} \liminf_{n_k\to+\infty} \|v_{n_k} - v_n\|_\mu.
\end{aligned}$$

This implies that $\widehat{\mathcal{D}^h(\mu)} \subset L^1(\nu_E)$, where $\widehat{\mathcal{D}^h(\mu)}$ is the set of the quasi-continuous functions belonging to $\mathcal{D}^h(\mu)$, and we have

$$\langle p_{\nu_E}, v\rangle_\mu = \int_{\mathbb{T}} \widehat{v}(\zeta)\,d\nu_E(\zeta), \quad v \in \mathcal{D}^h(\mu).$$

This gives (ii). To prove (iii), we need the following lemma.

Lemma 2.3 *Let $f \in \mathcal{D}^h(\mu)$, then $f(z) = \langle f, 2\operatorname{Re} k^\mu(z,\cdot) - 1\rangle_\mu$, $z \in \mathbb{D}$.*

Proof Let $\mathbf{P}$ be the Riesz projection of $\mathcal{D}^h(\mu)$ into $\mathcal{D}(\mu)$. Obviously, we have

$$\langle \mathbf{P}f, g\rangle_\mu = \langle f, \mathbf{P}g\rangle_\mu \tag{2.2}$$

for every f and g in $\mathcal{D}^h(\mu)$. Fix $f \in \mathcal{D}^h(\mu)$; we consider $f^+ := \mathbf{P}f$ and $f^- := f - \mathbf{P}f$. Using (2.2), we have

$$\begin{aligned}
f(z) &= f^+(z) + f^-(z)\\
&= \langle f^+, k^\mu(z,\cdot)\rangle_\mu + \langle f^-, \overline{k^\mu}(z,\cdot)\rangle_\mu\\
&= \langle f, k^\mu(z,\cdot)\rangle_\mu + \langle f, \overline{k^\mu}(z,\cdot)\rangle_\mu - \langle f, \mathbf{P}\overline{k^\mu}(z,\cdot)\rangle_\mu\\
&= \langle f, 2\operatorname{Re} k^\mu(z,\cdot) - 1\rangle_\mu.
\end{aligned}$$

The last equality holds, because $\mathbf{P}\overline{k^\mu}(z,\cdot) = 1$. ∎

Now we return to the proof of (iii). Since $p_{\nu_E} \in \mathcal{D}^h(\mu)$, by Lemma 2.3 we have

$$p_{\nu_E}(z) = \langle p_{\nu_E}, 2\operatorname{Re} k^\mu(z,\cdot) - 1\rangle_\mu = \int_{\mathbb{T}} \Big(2\operatorname{Re} \widehat{k^\mu(z,\lambda)} - 1\Big)\,d\nu_E(\lambda).$$

Finally, if E is closed, then ν_E is supported on E and we have

$$\|p_{\nu_E}\|_\mu^2 = \langle p_{\nu_E}, p_{\nu_E}\rangle_\mu = \int_{\mathbb{T}} \widehat{p_{\nu_E}}(\lambda)d\nu_E(\lambda) = \nu_E(E).$$

This completes the proof of Theorem 2.2. ∎

As an immediate consequence, we obtain the following corollary.

Corollary 2.4 *Let K be a closed subset of $\mathbb{T}$; then*

$$c_\mu(K) = \sup\left\{\nu(K) : \nu \in \mathcal{M}^+(\mathbb{T}), \operatorname{supp}\nu \subset K, \widehat{p_\nu} \le 1\, c_\mu\text{-q.e. on } \mathbb{T}\right\}.$$

Proof Denote by

$$M(K) := \left\{\nu \in \mathcal{M}^+(\mathbb{T}) : \operatorname{supp}\nu \subset K, \widehat{p_\nu} \le 1\, c_\mu\text{-q.e. on } \mathbb{T}\right\}.$$

By Theorem 2.2, there exists $\nu_K \in \mathcal{M}^+(\mathbb{T})$ such that supp $\nu_K \subset K$ and $\widehat{p_{\nu_K}} \le 1\ c_\mu$-q.e. on $\mathbb{T}$. This gives that $\nu_K \in M(K)$. Let $\nu \in M(K)$, and by Theorem 2.2, we have

$$\nu(K) = \int_K \widehat{p_{\nu_K}}(\zeta)d\nu(\zeta) = \langle p_{\nu_K}, p_\nu\rangle_\mu = \int \widehat{p_\nu}(\zeta)d\nu_K(\zeta) \le \nu_K(K) = c_\mu(K).$$ ∎

3 Proof of Theorem 1.1

In our proof we will use an analogue argument given in [2, 4].

Let E be a closed subset of $\mathbb{T}$ such that $c_\mu(E) = 0$; then there exists a decreasing sequence E_n of closed subsets of $\mathbb{T}$ such that

$$\sum_n c_\mu(E_n)^{1/2} < \infty.$$

By Theorem 2.2, for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}^+(\mathbb{T})$ such that $\operatorname{supp}(\nu_n) \subset E_n$ and $c_\mu(E_n) = \|p_{\nu_n}\|_\mu^2 = \nu_n(E_n)$. Set $\mu_n = \nu_n/\nu_n(E_n)$, and put $\phi_{\mu_n} = \mathbf{P}p_{\mu_n}$, where $\mathbf{P}$ is the Riesz projection of $\mathcal{D}^h(\mu)$ into $\mathcal{D}(\mu)$. We have $\|\phi_{\mu_n}\|_\mu^2 \le \|p_{\mu_n}\|_\mu^2 = 1/c_\mu(E_n)$. Now consider

$$\phi(z) = \sum_n c_\mu(E_n)\phi_{\mu_n}(z), \quad z \in \mathbb{D}.$$

Since $|\phi_{\mu_n}(z)| \le \frac{1}{\sqrt{c_\mu(E_n)}}\|k_z^\mu\|_\mu$, for any $z \in \mathbb{D}$, ϕ is well defined and

$$|\phi(z)| \le \sum_n c_\mu(E_n)^{1/2}\|k_z^\mu\|_\mu.$$

Furthermore,

$$\Big\|\sum_n c_\mu(E_n)\phi_{\mu_n}\Big\|_\mu \le \sum_n c_\mu(E_n)^{1/2} < \infty.$$

Then $\phi \in \mathcal{D}(\mu)$. Now set $f(z) = \exp(-\phi(z))$, for each $z \in \mathbb{D}$. Clearly, $f \in \mathcal{D}(\mu)$. On the other hand, by Theorem 2.2 we have $\widehat{p_{\mu_n}}(\zeta) = \frac{1}{c_\mu(E_n)}\ c_\mu$-q.e. on E_n, for each

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$n \in \mathbb{N}$. Using this fact with (1.2), we obtain

$$\text{(3.1)} \qquad \begin{aligned} \operatorname{Re}(\phi_{\mu_n}(z)) &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} \operatorname{Re}\big(\phi_{\mu_n}(\zeta)\big)\, dm(\zeta) \\ &\geq \frac{1}{2}\int_{E_n} \frac{1-|z|^2}{|\zeta - z|^2} p_{\mu_n}(\zeta) dm(\zeta) \\ &\geq \frac{\varpi(z, E_n, \mathbb{D})}{2c_\mu(E_n)}, \quad z \in \mathbb{D}, \quad n \in \mathbb{N}, \end{aligned}$$

where $\varpi(z, E_n, \mathbb{D})$ denotes the harmonic measure of E_n at z. Therefore,

$$|f(z)| \leq e^{-\sum_n \varpi(z, E_n, \mathbb{D})}.$$

It is well known that $\lim_{z\to\xi} \varpi(z, E, \mathbb{D}) = 1$, for each $\xi \in E$; it follows that f vanishes on E.

Now we will modify the construction presented above to obtain the continuity on $\overline{\mathbb{D}} \setminus E$. From (3.1), we can choose a sequence (r_n) increasing to 1 such that

$$\operatorname{Re}(\phi_{\mu_n}(r_n\zeta)) \geq \frac{1}{2c_\mu(E_n)}, \quad \zeta \in E.$$

Now set $\phi_n(z) = \phi_{\mu_n}(r_n z)$, for $z \in \mathbb{D}$, $\phi = \sum_n c_\mu(E_n)\phi_n$, and $f = \exp(-\phi)$. Since $\mathcal{D}_\mu(\phi_n) \leq \mathcal{D}_\mu(\phi_{\mu_n})$ (see [9, Lemma 7.3.2]), we obtain $f \in \mathcal{D}(\mu)$. Hence, for large N, we get that

$$\liminf_{z\to\zeta} \operatorname{Re}(\phi(z)) \geq \sum_{n=1}^{N} c_\mu(E_n) \operatorname{Re}(\phi_n)(\zeta) \geq \sum_{n=1}^{N} \frac{1}{2} = \frac{N}{2}, \quad \zeta \in E.$$

So $E \subset Z(f)$.

The function f is continuous in $\overline{\mathbb{D}} \setminus E$. Indeed, let $z_0 \in \overline{\mathbb{D}} \setminus E$; then $d(z_0, E) > 0$, it follows that for n sufficiently large,

$$d(r_n z, E_n) \geq \delta > 0$$

for all points z in some open disk centered at z_0 denoted by $D(z_0)$. By Theorem 2.2(ii) and (1.3), we obtain

$$\begin{aligned} \big|\phi_n(z)\big| &= \big|\langle p_{\mu_n}, k^\mu(r_n z, \cdot)\rangle_\mu\big| = \Big|\int_{\mathbb{T}} k^\mu\widehat{(r_n z, \lambda)} d\mu_n(\lambda)\Big| \\ &\leq \int_{\mathbb{T}} \big|k^\mu\widehat{(r_n z, \lambda)}\big| d\mu_n(\lambda) \leq \frac{2}{d(r_n z, E_n)}. \end{aligned}$$

Therefore,

$$c_\mu(E_n)|\phi_n(z)| \leq \frac{2c_\mu(E_n)}{\delta}.$$

From Weierstrass' test the series $\sum_n c_\mu(E_n)\phi_n$ converges uniformly to ϕ on $D(z_0)$. Then ϕ is continuous on $D(z_0)$. And thus we deduce that $Z(f) = E$.

Finally, we prove that the function f is cyclic. Let

$$f_n = \exp\Big(-\sum_{i\geq n} c_\mu(E_i)\phi_i\Big).$$

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Then we have $\mathcal{D}_\mu(f_n) \le (\sum_{i\ge n} c_\mu(E_i)^{1/2})^2 \to 0$, as $n \to +\infty$. Also, f_n converges pointwise to 1 as $n \to +\infty$, and $f_n/f = \exp(\sum_{i=1}^n c_\mu(E_i)\phi_i)$ is a bounded function. So $f_n \in [f]_{\mathcal{D}(\mu)}$. We deduce that $1 \in [f]_{\mathcal{D}(\mu)}$. The proof is complete. ∎

4 Proof of Theorem 1.2

The proof of Theorem 1.2 is based on an adaptation of a technique due to El-Fallah, Kellay, and Ransford in [10, 12]. The first key of the proof is the following converse of the strong-type inequality (1.5).

Theorem 4.1 *Let E be a closed subset of $\mathbb{T}$, and let $h\colon (0,\pi] \to (0,+\infty)$ be a continuous and decreasing function such that $h(0) = +\infty$. Then there exists a real function $f \in \mathcal{D}^h(\mu)$ such that*

$$\liminf_{z\to\zeta} f(z) \ge h\big(d(\zeta,E)\big), \quad \zeta \in \mathbb{T}, \tag{4.1}$$

if and only if the function h satisfies

$$\int_0 c_\mu(E_t)\big|dh^2(t)\big| < \infty. \tag{4.2}$$

To prove Theorem 4.1, we need the following elementary lemma.

Lemma 4.2 *Let $(H, \|\cdot\|)$ be a Hilbert space, and let $(\psi_n)_n$ be a sequence of H such that $\psi_n - \psi_m \perp \psi_m$, for all $n \ge m$. Then $\sum_{n\ge1} \psi_n/\|\psi_n\|^2$ belongs to H if and only if $\sum_{n\ge1} n/\|\psi_n\|^2$ is finite.*

Proof See [9, Lemma 3.4.4] and [10]. ∎

Proof of Theorem 4.1 Suppose that there exists a real function $f \in \mathcal{D}^h(\mu)$ satisfying (4.1). By [5, Theorem 1.3], we have $\lim_{z\to\zeta} f(z)$ exists c_μ-q.e. then $f(\zeta) \ge h(t)\ c_\mu - q.e$ on E_t. So

$$\int_0^\pi c_\mu(E_t)\big|dh^2(t)\big| \le \int_0^\pi c_\mu(|f| \ge h(t))\big|dh^2(t)\big| = \int_{h(\pi)}^{+\infty} c_\mu(|f| \ge s)ds^2 < \infty.$$

The last integral is finite, because the c_μ-capacity satisfies the strong-type inequality (1.5).

To prove the converse, we first observe that

$$\begin{aligned}\int_0^\pi c_\mu(E_t)|dh^2(t)| &\ge \sum_{n=n_0+1}^{+\infty} \int_{\delta_n}^{\delta_{n-1}} c_\mu(E_{\delta_n})|dh^2(t)| \\ &= \sum_{n=n_0+1}^{+\infty} c_\mu(E_{\delta_n})(n^2-(n-1)^2) \\ &\asymp \sum_{n=n_0+1}^{+\infty} nc_\mu(E_{\delta_n}),\end{aligned}$$

where $n_0 \in \mathbb{N}$ with $n_0 \ge h(\pi)$ and $\delta_n := h^{-1}(n)$ for $n \ge n_0$. By (4.2), we have

$$\sum_{n=n_0+1}^{+\infty} nc_\mu(E_{\delta_n}) < \infty.$$

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Otherwise, according to Theorem 2.2, for each $n \ge n_0$, there exists a measure $\nu_n \in \mathcal{M}^+(\mathbb{T})$, such that $\operatorname{supp}(\nu_n) \subset E_{\delta_n}$ and $c_\mu(E_{\delta_n}) = \nu_n(\mathbb{T})$.

Now, taking $\mu_n := \nu_n/\nu_n(\mathbb{T})$, we have $\|p_{\mu_n}\|_\mu^2 = \frac{1}{c_\mu(E_{\delta_n})}$, for $n \ge n_0$ and

$$\begin{aligned}\langle p_{\mu_n} - p_{\mu_m}, p_{\mu_m}\rangle_\mu &= \langle p_{\mu_n}, p_{\mu_m}\rangle_\mu - \langle p_{\mu_m}, p_{\mu_m}\rangle_\mu \\ &= \int_{\mathbb{T}} \widehat{p_{\mu_m}}(\zeta)\, d\mu_n(\zeta) - \frac{1}{c_\mu(E_{\delta_m})} = 0.\end{aligned}$$

The last equality holds, because $\widehat{p_{\mu_m}} = \frac{1}{c_\mu(E_{\delta_m})}$ c_μ-q.e. on E_{δ_m} and $E_{\delta_n} \subset E_{\delta_m}$, for $n \ge m$. Therefore, by Lemma 4.2, we get that the function

$$f(z) := n_0 + \sum_{n \ge n_0} c_\mu(E_{\delta_n}) p_{\mu_n}(z), \quad z \in \mathbb{D},$$

belongs to $\mathcal{D}^h(\mu)$.

Finally, we will prove (4.1). If $d(\zeta, E) \ge \delta_{n_0}$, then

$$\liminf_{z \to \zeta} f(z) \ge n_0 = h(\delta_{n_0}) \ge h(d(\zeta, E)).$$

Otherwise, let $N \in \mathbb{N}$, with $\delta_{N+1} < d(\zeta, E) \le \delta_N$. We have

$$\widehat{f} \ge n_0 + N + 1 - n_0 = N + 1, \quad c_\mu\text{-q.e. on } E_{\delta_N}.$$

Thus,

$$f(z) \ge h(d(\zeta, E))\varpi(z, E_{\delta_N}, \mathbb{D}), \quad \zeta \in E_{\delta_N}.$$

Letting $z \to \zeta$, we obtain conclusion (4.1), and this completes the proof. ∎

Theorem 4.3 *Let $f \in \mathcal{D}(\mu) \cap A(\mathbb{D})$ be an outer function and $E = \{\zeta \in \mathbb{T} : f(\zeta) = 0\}$. If there exists a function $g \in \mathcal{D}(\mu)$ such that $|g(z)| \lesssim d(z, E)^4$, $z \in \mathbb{D}$, then $g \in [f]_{\mathcal{D}(\mu)}$.*

Theorem 4.3 is a $\mathcal{D}(\mu)$-analogue of [11, Theorem 3.1] and [12, Theorem 2.1]. We will use the same basic technique here. First, we introduce some notation. Let Γ be a Borel subset of $\mathbb{T}$. We denote by $\partial\Gamma$ the boundary of Γ in $\mathbb{T}$. We associate with a given outer function f the function f_Γ defined by

$$f_\Gamma(z) := \exp\Big(\int_\Gamma \frac{\zeta + z}{\zeta - z} \log|f(\zeta)|\, dm(\zeta)\Big).$$

Lemma 4.4 *Let f be a bounded outer function. For every Borel set $\Gamma \subset \mathbb{T}$, we have*

$$|f_\Gamma'(z)| \lesssim |f'(z)| + d(z, \partial\Gamma)^{-4}, \quad z \in \mathbb{D}.$$

Proof See [12, Lemma 2.2]. ∎

Proof of Theorem 4.3 Let $(I_i)_{i \ge 1}$ be the complete set of components of $\mathbb{T} \setminus E$ and set $J_n := \bigcup_{j=1}^n I_j$. We claim that

(i) $f_{\mathbb{T} \setminus J_n} g$ converges pointwise to g, as $n \to +\infty$,
(ii) $\liminf_{n \to +\infty} \mathcal{D}_\mu(f_{\mathbb{T} \setminus J_n} g) < \infty$,
(iii) $f_{\mathbb{T} \setminus J_n} g \in [f]_{\mathcal{D}(\mu)}$, for all n,

and thus the theorem is proved.

The assertion (i) is obvious. To prove (ii) by Lemma 4.4, we get that

$$\begin{aligned}\mathcal{D}_\mu(f_{\mathbb{T}\setminus J_n}g) &\lesssim \mathcal{D}_\mu(f)\|g\|_\infty^2 + \|f\|_\infty^2\mathcal{D}_\mu(g) + \int_{\mathbb{D}}\Big(\frac{|g(z)|}{d(z,\partial\mathbb{T}\setminus J_n)^4}\Big)^2 P[\mu](z)dA(z)\\ &\lesssim \mathcal{D}_\mu(f)\|g\|_\infty^2 + \|f\|_\infty^2\mathcal{D}_\mu(g) + \mathcal{D}_\mu(z).\end{aligned}$$

Then

$$\liminf_{n\to+\infty}\mathcal{D}_\mu(f_{\mathbb{T}\setminus J_n}g) < \infty.$$

To check (iii) it is sufficient to show that $f_{\mathbb{T}\setminus I}g \in [f]_{\mathcal{D}(\mu)}$, where I is a connected component of $\mathbb{T}\setminus E$, say $I = (e^{ia}, e^{ib})$. Let $\rho > 1$, define

$$\psi_\rho(z) = (z-1)^4/(z-\rho)^4 \quad\text{and}\quad \phi_\rho(z) = \psi_\rho(e^{-ia}z)\psi_\rho(e^{-ib}z),$$

let $\epsilon > 0$, and set $I_\epsilon = (e^{i(a+\epsilon)}, e^{i(b-\epsilon)})$, and

$$\phi_{\rho,\epsilon}(z) = \psi_\rho(e^{-i(a+\epsilon)}z)\psi_\rho(e^{-i(b-\epsilon)}z).$$

By Lemma 4.4 again, we have

$$\mathcal{D}_\mu(\phi_{\rho,\epsilon}f_{\mathbb{T}\setminus I_\epsilon}) \lesssim \mathcal{D}_\mu(\phi_{\rho,\epsilon}) + \|\phi_{\rho,\epsilon}\|_\infty^2\mathcal{D}_\mu(f) + \mathcal{D}_\mu(z).$$

Then

$$\liminf_{\epsilon\to 0}\mathcal{D}_\mu(\phi_{\rho,\epsilon}f_{\mathbb{T}\setminus I_\epsilon}) \lesssim \mathcal{D}_\mu(\phi_\rho) + \|\phi_\rho\|_\infty^2\mathcal{D}_\mu(f) + \mathcal{D}_\mu(z).$$

On the other hand, it follows from boundedness of $|\phi_{\rho,\epsilon}f_{\mathbb{T}\setminus I_\epsilon}|/|f|$ on $\mathbb{T}$ that $\phi_{\rho,\epsilon}f_{\mathbb{T}\setminus I_\epsilon} \in [f]_{\mathcal{D}(\mu)}$. Since $\phi_{\rho,\epsilon}f_{\mathbb{T}\setminus I_\epsilon}$ converges pointwise to $\phi_\rho f_{\mathbb{T}\setminus I}$, as $\epsilon \to 0$, we have $\phi_\rho f_{\mathbb{T}\setminus I} \in [f]_{\mathcal{D}(\mu)}$. We multiply by g. As $g \in \mathcal{D}(\mu)\cap H^\infty$, $\phi_\rho f_{\mathbb{T}\setminus I}g \in [f]_{\mathcal{D}(\mu)}$. Again, according to Lemma 4.4, we have

$$\mathcal{D}_\mu(f_{\mathbb{T}\setminus I}\phi_\rho g) \lesssim \mathcal{D}_\mu(f)\|g\|_\infty^2 + \mathcal{D}_\mu(z) + \|f\|_\infty^2\mathcal{D}_\mu(\phi_\rho g).$$

Using $|g(z)| \lesssim d(z,E)^4$, it is easy to check that $\mathcal{D}_\mu(\phi_\rho g)$ is bounded as $\rho \to 1$. These imply that $f_{\mathbb{T}\setminus I}g \in [f]_{\mathcal{D}(\mu)}$. As a similar argument to that above gives $f_{\mathbb{T}\setminus J_n}g \in [f]_{\mathcal{D}(\mu)}$. ■

The last ingredient of the proof of Theorem 1.2 is the following theorem due to Richter and Sundberg [19, Theorem 4.3].

Theorem 4.5 *Let f be an outer function and $\gamma > 0$. If $f, f^\gamma \in \mathcal{D}(\mu)$, then $[f^\gamma]_{\mathcal{D}(\mu)} = [f]_{\mathcal{D}(\mu)}$.*

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 To see that f is cyclic, we prove that $1 \in [f]_{\mathcal{D}(\mu)}$.

Using Theorem 4.1 for $h(t) = \log(1/t)$, the condition $\int_0 c_\mu(E_t)\frac{\log(1/t)}{t}dt < \infty$ implies that there exists a real function $v \in \mathcal{D}^h(\mu)$ such that $\liminf_{z\in\zeta} v(z) \geq h(d(\zeta,E))$, $\zeta \in \mathbb{T}$. Now set $u := v + i\widetilde{v}$, where $\widetilde{v}$ is the harmonic conjugate for v. By the Cauchy–Riemann equations, we get that $|u'(z)| = |\nabla v(z)|$, for all $z \in \mathbb{D}$. Hence, $\mathcal{D}_\mu(u) = 2\mathcal{D}_\mu(v)$, then $u \in \mathcal{D}(\mu)$. We put $g_\lambda(z) = e^{-\lambda u(z)}$, for $\lambda \geq 0$. We have $|g_\lambda(z)| = e^{-\lambda v(z)} \leq 1$ and $|g'_\lambda(z)| \leq \lambda|u'(z)|$. Then $\mathcal{D}_\mu(g_\lambda) \leq \lambda^2\mathcal{D}_\mu(u)$. Hence, $\liminf_{\lambda\to 0}\mathcal{D}_\mu(g_\lambda) = 0$ and $g_\lambda \in \mathcal{D}(\mu)$ for all $\lambda \geq 0$. Since g_λ converges pointwise to

1, as $\lambda \to 0$, we obtain that g_λ converges weakly to 1. Otherwise, for almost all $\zeta \in \mathbb{T}$, we have

$$|g_\lambda(\zeta)| = e^{-\lambda v(\zeta)} \le e^{-\lambda h(d(\zeta,E))} = d(\zeta,E)^\lambda.$$

That gives $|g_\lambda(z)| \le (\pi/2)^\lambda d(z,E)^\lambda$, for all $\lambda \ge 0$. Using Theorem 4.3 we obtain $g_4 \in [f]_{\mathcal{D}(\mu)}$, and by Theorem 4.5 we get that $[g_\lambda]_{\mathcal{D}(\mu)} = [g_4]_{\mathcal{D}(\mu)}$, for all $\lambda \ge 0$. Hence $g_\lambda \in [f]_{\mathcal{D}(\mu)}$, for all $\lambda \ge 0$. Then $1 \in [f]_{\mathcal{D}(\mu)}$. ∎

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