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Subspaces of de Branges Spaces Generated by Majorants

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Abstract. For a given de Branges space $\mathcal{H}(E)$ we investigate de Branges subspaces defined in terms of majorants on the real axis. If ω is a nonnegative function on $\mathbb{R}$, we consider the subspace

$$\mathcal{R}_\omega(E) = \operatorname{Clos}_{\mathcal{H}(E)} \left\{ F \in \mathcal{H}(E) : \text{ there exists } C > 0 : |E^{-1}F| \leq C\omega \text{ on } \mathbb{R} \right\}.$$

We show that $\mathcal{R}_\omega(E)$ is a de Branges subspace and describe all subspaces of this form. Moreover, we give a criterion for the existence of positive minimal majorants.

1 Introduction

The theory of Hilbert spaces of entire functions introduced by L. de Branges is an important branch of modern analysis. It is an intriguing example for a fruitful interplay of function theory and operator theory, which has deep applications in mathematical physics, namely in differential operators and scattering theory.

One of the striking features of a de Branges space is the structure of its de Branges subspaces (that is, subspaces which are themselves de Branges spaces) revealed by de Branges' Ordering Theorem. This theorem states, roughly speaking, that, for a given space, the set of all its de Branges subspaces "with the same real zeros" is totally ordered with respect to set-theoretic inclusion. However, given an individual de Branges space, there is no explicit way to determine the chain of its de Branges subspaces.

In a recent series of papers V. Havin and J. Mashreghi introduced the notion of admissible majorants for shift-coinvariant (model) subspaces of the Hardy space $H^2(\mathbb{C}^+)$. Since de Branges spaces are, essentially, particular model subspaces of the Hardy space, this notion is applicable. Of course, due to the rich structure of de Branges spaces, much more specific results than in the general setting can be expected.

It is the aim of our present work to show that admissible majorants give rise to de Branges subspaces and to study the structure of these subspaces. Our main results are a description of all subspaces which are induced by admissible majorants and a criterion for the existence of minimal majorants which are separated from zero.

As already indicated in the above abstract, an admissible majorant defines a de Branges subspace by means of a restriction on the growth along the real axis. It is an interesting observation that this concept is complementary to imposing growth

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conditions off the real axis. In a recent paper de Branges subspaces were defined by means of restriction on mean type [KW2]. We will see that the subspaces defined by majorants cannot be described by mean type conditions. Hence these two methods can, in conjunction, lead to a description of the whole chain of subspaces of a de Branges space. An elaboration of this idea will be subject of future work.

Let us describe the organization and content of this paper. In Section 2, we provide the necessary preliminaries concerning de Branges spaces and admissible majorants. Section 3 is devoted to the study of subspaces induced by majorants by means of Definition 3.1 and Proposition 3.2. The main result in this context is the characterization of those subspaces which can be realized in this way, given in Theorem 3.4 and Proposition 3.9. As corollaries we obtain a couple of conditions for density of "small" functions in a given de Branges space. Moreover, we give some rather general examples to illustrate these results. In Section 4 we turn to a thorough investigation of minimal majorants. Our main result is Theorem 4.2 where we relate minimal majorants to one-dimensional subspaces. In combination with Theorem 3.4 this leads to a characterization of existence of minimal majorants separated from zero (see Theorem 4.9). This result is closely related to the recent work in [BH].

2 Preliminaries

An entire function E is said to belong to the *Hermite-Biehler class* $\mathcal{H}B$, if it satisfies $|E(\bar{z})| < |E(z)|$, $z \in \mathbb{C}^+$. Sometimes in the literature it is only required that $|E(\bar{z})| \leq |E(z)|$. This, however, is no essential gain in generality. Throughout this paper we will, for any function F, denote by $F^\#$ the function $F^\#(z) := \overline{F(\bar{z})}$.

Definition 2.1 If $E \in \mathcal{H}B$, the *de Branges space* $\mathcal{H}(E)$ is defined as the set of all entire functions F which have the property that $E^{-1}F, E^{-1}F^\# \in H^2(\mathbb{C}^+)$. This space will be endowed with the norm

$$\|F\|_E := \Big(\int_{\mathbb{R}} \Big| \frac{F(t)}{E(t)} \Big|^2 dt \Big)^{1/2}, \qquad F \in \mathcal{H}(E).$$

It is shown in [deB, Theorem 21] that $\mathcal{H}(E)$ is a Hilbert space with respect to the norm $\|\cdot\|_E$.

Remark 2.2. The definition of $\mathcal{H}(E)$ given above can be reformulated. In fact, an entire function F belongs to $\mathcal{H}(E)$ if and only if $E^{-1}F, E^{-1}F^\# \in N(\mathbb{C}^+)$, $\operatorname{mt} E^{-1}F, \operatorname{mt} E^{-1}F^\# \leq 0$, and $E^{-1}F|_{\mathbb{R}} \in L^2(\mathbb{R})$.

Here $N(\mathbb{C}^+)$ denotes the set of all functions of bounded type in $\mathbb{C}^+$, and $\operatorname{mt} f$ denotes the mean type of a function $f \in N(\mathbb{C}^+)$, *i.e.*,

$$\operatorname{mt} f := \limsup_{y \to +\infty} y^{-1} \log |f(iy)|,$$

see *e.g.*, [RR]. In fact, this is the original definition given in [deB].

It is an important feature that de Branges spaces can be characterized axiomatically (see [deB, Problem 50, Theorem 23]). Let $\mathcal{H}$ be a nonzero Hilbert space whose elements are entire functions. Then $\mathcal{H}$ is equal to a space $\mathcal{H}(E)$ including equality of norms if and only if $\mathcal{H}$ satisfies the following:

(deB1) for every $\nu \in \mathbb{C}$ the point evaluation functional $\chi_\nu \colon F \mapsto F(\nu)$ is continuous on $\mathcal{H}$;
(deB2) if $F \in \mathcal{H}$, then also $F^\# \in \mathcal{H}$, and we have $\|F^\#\| = \|F\|$, $F \in \mathcal{H}$;
(deB3) if $F \in \mathcal{H}$ and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $F(z_0) = 0$, then also

$$\frac{z - \overline{z_0}}{z - z_0} F(z) \in \mathcal{H}, \quad \text{and} \quad \left\| \frac{z - \overline{z_0}}{z - z_0} F(z) \right\| = \|F\|.$$

Remark 2.3. If a Hilbert space $\mathcal{H}$ which satisfies (deB1)–(deB3) is given, the function $E \in \mathcal{H}B$ which realizes $\mathcal{H}$ as $\mathcal{H}(E)$ is not unique. In fact, if $E_1, E_2 \in \mathcal{H}B$, then $\mathcal{H}(E_1) = \mathcal{H}(E_2)$, including equality of norms, if and only if $(A_2, B_2) = (A_1, B_1)U$, where $A_k = \frac{1}{2}(E_k + E_k^\#)$, $B_k = \frac{i}{2}(E_k - E_k^\#)$, $k = 1, 2$, and where U is a 2×2-matrix with real entries and determinant 1. This result is contained in [deB]; an explicit proof can be found in [KW1, Corollary 6.2].

For an entire function G, let $\mathfrak{d}(G) \colon \mathbb{C} \to \mathbb{N}$ be the map which assigns to a point ν its multiplicity as a zero of G. For a de Branges space $\mathcal{H}$ we put $\mathfrak{d}(\mathcal{H})(\nu) := \min_{F \in \mathcal{H}} \mathfrak{d}(F)(\nu)$. Then for any $E \in \mathcal{H}B$ with $\mathcal{H} = \mathcal{H}(E)$ we have $\mathfrak{d}(\mathcal{H}(E))(t) = \mathfrak{d}(E)(t)$, $t \in \mathbb{R}$ (see [deB, Problem 45]). Note that by (deB3) we always have $\mathfrak{d}(\mathcal{H})|_{\mathbb{C} \setminus \mathbb{R}} = 0$.

Remark 2.4. Let $\nu \in \mathbb{R}$ and $F \in \mathcal{H}(E)$ with $F(\nu) = 0$ be given. If $\mathfrak{d}(F)(\nu) > \mathfrak{d}(E)(\nu)$, then $(z - \nu)^{-1} F(z) \in \mathcal{H}(E)$ see [deB, Problem 45].

By (deB1), a de Branges space $\mathcal{H}$ is a reproducing kernel Hilbert space of entire functions. This means that there exists a (unique) function $K(\nu, z)$, entire in z and in $\bar{\nu}$, such that for every fixed $\nu \in \mathbb{C}$, we have $K(\nu, \cdot) \in \mathcal{H}$ and $(F, K(\nu, \cdot)) = F(\nu)$, $F \in \mathcal{H}$. If $\mathcal{H}$ is realized as $\mathcal{H}(E)$ with some $E \in \mathcal{H}B$, the reproducing kernel of $\mathcal{H}$ can be written explicitly in terms of E. In fact, we have

$$K(\nu, z) = \frac{E(z)\overline{E(\nu)} - E^\#(z)E(\bar{\nu})}{2\pi i(\bar{\nu} - z)},$$

(see [deB, Theorem 19]).

Definition 2.5 A subset L of a de Branges space $\mathcal{H}$ is called a *deB-subspace* if it is itself, with the norm inherited from $\mathcal{H}$, a de Branges space. We shall denote the set of all deB-subspaces of a given space $\mathcal{H}$ by $\operatorname{Sub}(\mathcal{H})$.

In view of the above axiomatic characterization of de Branges spaces, a subset L of $\mathcal{H}$ is a deB-subspace if and only if

(Sub1) L is a closed linear subspace of $\mathcal{H}$;
(Sub2) if $F \in L$, then also $F^\# \in L$;
(Sub3) if $F \in L$ and $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $F(z_0) = 0$, then also $\frac{z - \bar{z}_0}{z - z_0} F(z) \in L$.

Those deB-subspaces L of a given de Branges space $\mathcal{H}$ with

(SubZ) $\mathfrak{d}(L) = \mathfrak{d}(\mathcal{H})$

are of particular importance. The set of all such deB-subspaces will be denoted by $\operatorname{Sub}^s(\mathcal{H})$. Note that, if $\mathcal{H}$ and L are written as $\mathcal{H}(E)$ and $\mathcal{H}(E_1)$, respectively, with some $E, E_1 \in \mathcal{H}B$, then the validity of (SubZ) just means that $\mathfrak{d}(E_1)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}}$.

One of the deepest and most fundamental results in the theory of de Branges spaces is the so-called de Branges' Ordering Theorem (see [deB, Theorem 35]).

Theorem 2.6 **(de Branges' Ordering Theorem)** *Let $\mathcal{H}$ be a de Branges space and let $\mathfrak{d}\colon \mathbb{R} \to \mathbb{N} \cup \{0\}$ be given. Then the set $\{L \in \operatorname{Sub}(\mathcal{H}) : \mathfrak{d}(L) = \mathfrak{d}\}$ is totally ordered with respect to set-theoretic inclusion.*

Example 2.7. An important example of a de Branges space is the classical *Paley–Wiener space* $\mathcal{P}W_a$, $a > 0$. It can be defined as the space of all entire functions of exponential type at most a, whose restrictions to the real axis belong to $L^2(\mathbb{R})$. The norm in the space $\mathcal{P}W_a$ is given by the usual L^2-norm, $\|F\|^2 := \int_{\mathbb{R}} |F(t)|^2 dt$, $F \in \mathcal{P}W_a$. It is a consequence of a theorem of M. G. Krein (see [RR, Examples/Addenda 2, p. 134]), that $\mathcal{P}W_a = \mathcal{H}(e^{-iaz})$. The chain $\operatorname{Sub}^s(\mathcal{P}W_a)$ is given as

$$\operatorname{Sub}^s(\mathcal{P}W_a) = \left\{ \mathcal{P}W_b : 0 < b \leq a \right\}.$$

The name of the space $\mathcal{P}W_a$ originates in the Paley–Wiener theorem, by which $\mathcal{P}W_a$ is the Fourier image of $L^2(-a, a)$.

Example 2.8. More general examples of de Branges spaces occur in the theory of canonical (or Hamiltonian) systems of differential equations (see *e.g.,* [deB, Theorems 37 and 38], [GK, HSW]). Let H be a 2×2-matrix valued function defined for $t \in [0, l]$, such that $H(t)$ is real and nonnegative, the entries of $H(t)$ belong to $L^1([0, l])$ and $H(t)$ does not vanish on any nonempty interval. We call an interval $(\alpha, \beta) \subseteq [0, l]$ H-indivisible if for some $\varphi \in \mathbb{R}$ and some scalar function $h(t)$ we have $H(t) = h(t)(\cos\varphi, \sin\varphi)^T(\cos\varphi, \sin\varphi)$, $t \in (\alpha, \beta)$ a.e.

Let $W(t, z)$ be the (unique) solution of the initial value problem

$$\frac{\partial}{\partial t} W(t, z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = zW(t, z)H(t),\ t \in [0, l], \qquad W(0, z) = I.$$

Put $(A_t(z), B_t(z)) := (1, 0)W(t, z)$, $t \in [0, l]$, and $E_t(z) := A_t(z) - iB_t(z)$. Then $E_t \in \mathcal{H}B$, $t \in (0, l]$, and $E_0 = 1$. If $0 < s \leq t \leq l$, then $\mathcal{H}(E_s) \subseteq \mathcal{H}(E_t)$ and the set-theoretic inclusion map is contractive. If s is not an inner point of an H-indivisible interval, it is actually isometric. Moreover, we have

$$\operatorname{Sub}^s(\mathcal{H}(E_l)) = \left\{ \mathcal{H}(E_t) : t \text{ not an inner point of } H\text{-indivisible interval} \right\}.$$

Paley–Wiener spaces can be realized in this way. In fact, if $H(t) = I$, $t \in [0, l]$, then $E_t(z) = e^{-itz}$.

Remark 2.9. If $E \in \mathcal{H}B$, then $\Theta_E := E^{-1}E^\#$ is an inner function in $\mathbb{C}^+$. The mapping $F \mapsto E^{-1}F$ is an isometric isomorphism of $\mathcal{H}(E)$ onto the *model subspace* $K_{\Theta_E} := H^2(\mathbb{C}^+) \ominus \Theta_E H^2(\mathbb{C}^+)$ (see *e.g.,* [HM1, Theorem 2.10]). Therefore, de Branges spaces are in a sense particular cases of model subspaces. However, de Branges spaces have many special properties, which have no analogs in the general theory of model subspaces, the first of which is the ordered structure of their subspaces.

Definition 2.10 Let $E \in \mathcal{H}B$. A nonnegative function ω on the real axis $\mathbb{R}$ is said to be an *admissible majorant* for the space $\mathcal{H}(E)$, if there exists a nonzero function $F \in \mathcal{H}(E)$ such that $|E(x)^{-1}F(x)| \le \omega(x)$, $x \in \mathbb{R}$. The set of all admissible majorants for $\mathcal{H}(E)$ is denoted by $\mathrm{Adm}(E)$.

Remark 2.11. If $E_1, E_2 \in \mathcal{H}B$ generate the same space, *i.e.*, $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ including equality of norms, then $\mathrm{Adm}(E_1) = \mathrm{Adm}(E_2)$. This follows from an elementary estimate using Remark 2.3.

Since for every $F \in \mathcal{H}(E)$, we have $E^{-1}F \in H^2(\mathbb{C}^+)$, a necessary condition for a function ω to be an admissible majorant for $\mathcal{H}(E)$ is the convergence of the logarithmic integral

$$\int_{\mathbb{R}} \frac{\log^- \omega(x)}{1+x^2} dx < \infty. \tag{2.1}$$

The description of admissible majorants for the Paley–Wiener spaces $\mathcal{P}W_a = \mathcal{H}(e^{-iaz})$ is a classical problem of harmonic analysis. By what we just said, any admissible majorant for a space $\mathcal{P}W_a$ must satisfy (2.1). The fact that this obvious, necessary condition is in many cases also sufficient is the content of the famous Beurling–Malliavin Multiplier Theorem (see [BM]).

Theorem 2.12 **(Beurling–Malliavin Multiplier Theorem)** *Let ω be a positive function on $\mathbb{R}$ satisfying* (2.1)*, and assume that the function* $\log \omega$ *is Lipschitz on $\mathbb{R}$. Then ω is an admissible majorant for every space $\mathcal{P}W_a$, $a > 0$.*

This is one of the deepest results of harmonic analysis, and several different proofs of it are known (see *e.g.*, [HJ, MNH, K]). It is referred to as *Multiplier Theorem* since it means that for any $a > 0$ there exists a nonzero multiplier $f \in \mathcal{P}W_a$ such that $f\omega^{-1} \in L^\infty(\mathbb{R})$.

Admissible majorants for general de Branges spaces (and even in a more general setting of the model subspaces of the Hardy class) were studied for the first time by Havin and Mashreghi in [HM1, HM2], where a complete parametrization of the class $\mathrm{Adm}(E)$ is found and a number of conditions sufficient for admissibility are obtained. Further applications of this approach may be found in [BH, BBH] and in [MNH] where a new and essentially simpler proof of the Beurling–Malliavin theorem is given.

All cited papers are concerned with existence of an individual function majorized by a given ω. A novel feature of this paper is that we study the whole class of functions majorized by ω, which may be quite large.

Definition 2.13 Let $E \in \mathcal{H}B$. We say that an admissible majorant ω for $\mathcal{H}(E)$ is *separated from zero* if each point $x \in \mathbb{R}$ has a neighbourhood $U(x) \subseteq \mathbb{R}$ such that $\inf\{\omega(t) : t \in U(x)\} > 0$. The set of all admissible majorants for $\mathcal{H}(E)$ that are separated from zero will be denoted by $\mathrm{Adm}^+(E)$.

Example 2.14. Examples of admissible majorants can be obtained from elements of $\mathcal{H}(E)$. For $F \in \mathcal{H}(E) \setminus \{0\}$, consider the function $\omega_F(x) := |E(x)^{-1}F(x)|$, $x \in \mathbb{R}$.

Then, by definition, $|E(x)^{-1}F(x)| \leq \omega_F(x)$, and hence ω_F is an admissible majorant for $\mathcal{H}(E)$. Clearly, in this situation, we have $\omega_F \in \operatorname{Adm}^+(E)$ if and only if $\mathfrak{d}(F)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}}$.

3 Subspaces Generated by Majorants

Throughout this paper we will use the following notation. We write $f \lesssim g$ if there exists a positive constant C such that $f \leq Cg$ for all admissible values of variables. Moreover, we write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

The relation $\lesssim$ is reflexive and transitive, hence it induces an order on equivalence classes of functions modulo the equivalence relation $\asymp$. In particular, given $E \in \mathcal{H}B$, we obtain an order on the set $\operatorname{Adm}(E)/_{\asymp}$ as well as on $\operatorname{Adm}^+(E)/_{\asymp}$. Clearly, if $\omega \in \operatorname{Adm}(E)$, or $\omega \in \operatorname{Adm}^+(E)$, and $\omega_1 \asymp \omega$, then also $\omega_1 \in \operatorname{Adm}(E)$, or $\omega_1 \in \operatorname{Adm}^+(E)$, respectively.

Admissible majorants give rise to deB-subspaces of $\mathcal{H}(E)$.

Definition 3.1 For $E \in \mathcal{H}B$ and $\omega \in \operatorname{Adm}(E)$ define

$$R_\omega(E) := \left\{ F \in \mathcal{H}(E) : |E(x)^{-1}F(x)| \lesssim \omega(x), x \in \mathbb{R} \right\},$$

and $\mathcal{R}_\omega(E) := \operatorname{Clos}_{\mathcal{H}(E)} R_\omega(E)$.

Proposition 3.2 *Let $E \in \mathcal{H}B$ and let $\omega \in \operatorname{Adm}(E)$. Then the space $\mathcal{R}_\omega(E)$ is a deB-subspace of $\mathcal{H}(E)$. The assignment $\mathfrak{R}\colon \omega \mapsto \mathcal{R}_\omega(E)$ defines a monotone map of $\operatorname{Adm}(E)/_{\asymp}$ into $\operatorname{Sub}(\mathcal{H}(E))$. Moreover, $\omega \in \operatorname{Adm}^+(E)$ if and only if $\mathcal{R}_\omega(E) \in \operatorname{Sub}^s(\mathcal{H}(E))$.*

Proof By its definition $\mathcal{R}_\omega(E)$ is a closed linear subspace of $\mathcal{H}(E)$. Clearly, $R_\omega(E)$ is invariant under the map $F \mapsto F^\#$. Since this map is continuous with respect to the norm of $\mathcal{H}(E)$, (Sub2) follows.

Let $F \in R_\omega(E)$ and $\nu \in \mathbb{C} \setminus \mathbb{R}$ with $F(\nu) = 0$ be given. Then also $\frac{z-\bar{\nu}}{z-\nu}F(z) \in R_\omega(E)$, i.e. $R_\omega(E) \cap \ker \chi_\nu$, where χ_ν is the point evaluation functional at ν, is mapped into $R_\omega(E)$ by the map $\Phi\colon F(z) \mapsto \frac{z-\bar{\nu}}{z-\nu}F(z)$. Note that, in particular, one can always find an element $G \in R_\omega(E)$ with $G(\nu) = 1$.

Since Φ maps $\ker \chi_\nu$ isometrically and, thus, continuously into $\mathcal{H}(E)$, it follows that

$$\Phi\Big(\operatorname{Clos}_{\mathcal{H}(E)}(R_\omega(E) \cap \ker \chi_\nu)\Big) \subseteq \operatorname{Clos}_{\mathcal{H}(E)} R_\omega(E) = \mathcal{R}_\omega(E).$$

Let $F \in \mathcal{R}_\omega(E) \cap \ker \chi_\nu$, and choose $F_n \in R_\omega(E)$ such that $F_n \to F$. Moreover, choose $G \in R_\omega(E)$ with $G(\nu) = 1$. Since $F_n(\nu) \to F(\nu) = 0$, we have $F_n - F_n(\nu)G \to F$. Hence $\mathcal{R}_\omega(E) \cap \ker \chi_\nu \subseteq \operatorname{Clos}_{\mathcal{H}(E)}(R_\omega(E) \cap \ker \chi_\nu)$, and (Sub3) follows.

If $\omega_1, \omega_2 \in \operatorname{Adm}(E)$, $\omega_1 \lesssim \omega_2$, then, clearly, $R_{\omega_1}(E) \subseteq R_{\omega_2}(E)$ and, therefore, $\mathcal{R}_{\omega_1}(E) \subseteq \mathcal{R}_{\omega_2}(E)$. It follows that $\mathcal{R}_\omega(E)$ depends only on the equivalence class $\omega/_{\asymp}$ and is monotone.

We come to the proof of the last assertion. Let $\omega \in \operatorname{Adm}(E)$. Assume first that $\mathcal{R}_\omega(E) \in \operatorname{Sub}^s(\mathcal{H}(E))$ and let $t \in \mathbb{R}$ be given. Choose $F \in \mathcal{R}_\omega(E)$ with $\mathfrak{d}(F)(t) = \mathfrak{d}(E)(t)$. Then, by continuity, there exists $\delta > 0$ and a compact neighbourhood $U(t)$

of t such that $|E(x)^{-1}F(x)| \geq \delta$, $x \in U(t)$. Choose a sequence $G_n \in R_\omega(E)$ such that $G_n \to F$ in the norm of $\mathcal{H}(E)$. Then G_n also converges to F locally uniformly. Since $\mathfrak{d}(G_n)(x) \geq \mathfrak{d}(E)(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, by the Maximium Modulus Principle, $E^{-1}G_n \to E^{-1}F$ locally uniformly on $\mathbb{C} \setminus \{v \in \mathbb{C}^- : E(v) = 0\}$. Hence there exists $n \in \mathbb{N}$ such that $|E(x)^{-1}G_n(x)| \geq \delta/2$, $x \in U(t)$. Let $C > 0$ be such that $|E(x)^{-1}G_n(x)| \leq C\omega(x)$, $x \in \mathbb{R}$, then $\inf_{x\in U(t)} \omega(x) \geq \frac{\delta}{2C} > 0$. It follows that $\omega \in \operatorname{Adm}^+(E)$.

Conversely, assume that $\omega \in \operatorname{Adm}^+(E)$. Let $t \in \mathbb{R}$ be given and choose $F \in R_\omega(E) \setminus \{0\}$. Put $n := \mathfrak{d}(F)(t) - \mathfrak{d}(E)(t)$, then $n \in \mathbb{N} \cup \{0\}$, the function $(z-t)^{-n}F(z)$ belongs to $\mathcal{H}(E)$, and $\mathfrak{d}((z-t)^{-n}F(z))(t) = \mathfrak{d}(E)(t)$. Let $U(t)$ be a compact neighbourhood of t such that $\inf_{x\in U(t)} \omega(x) > 0$. Then, by continuity, $[(x-t)^nE(x)]^{-1}F(x)$ is bounded on $U(t)$. Thus

$$|[(x-t)^nE(x)]^{-1}F(x)| \lesssim \omega(x), \quad x \in U(t).$$

Since $\frac{1}{|x-t|}$ is bounded for $x \notin U(t)$, clearly,

$$|[(x-t)^nE(x)]^{-1}F(x)| \lesssim |E(x)^{-1}F(x)| \lesssim \omega(x), \quad x \notin U(t).$$

Hence $\frac{F(z)}{(z-t)^n} \in R_\omega(E)$, and it follows that $\mathcal{R}_\omega(E) \in \operatorname{Sub}^s(E)$. ∎

Remark 3.3. Taking the closure $\operatorname{Clos}_{\mathcal{H}(E)}$ in the definition of $\mathcal{R}_\omega(E)$ is actually necessary in order to obtain de Branges subspaces. Although the linear space $R_\omega(E)$ always satisfies (Sub2) and (Sub3), it will, in general, not be closed. In fact, if one assumes that $\omega \in L^2(\mathbb{R})$, then the linear space $R_\omega(E)$ is not closed unless it is finite-dimensional. This is seen by an application of a theorem of Grothendieck with the probability measure

$$d\mu(x) := \frac{\omega^2(x)}{\int_{\mathbb{R}} \omega^2(t)dt} dx$$

(see [R, Theorem 5.2]). The assumption $\omega \in L^2(\mathbb{R})$ is not too restrictive; for example, it is met by every admissible majorant of the form ω_F, $F \in \mathcal{H}(E) \setminus \{0\}$ (see Example 2.14).

In the next theorem we characterize the deB-subspaces of a given space $\mathcal{H}(E)$ which are of the form $\mathcal{R}_\omega(E)$. This is the first main result of this paper.

Theorem 3.4 *Let $E, E_1 \in \mathcal{H}B$ be given, such that $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$. Then $\mathcal{H}(E_1) \in \mathfrak{R}(\operatorname{Adm}(E))$ if and only if* $\operatorname{mt} \frac{E_1}{E} = 0$.

Remark 3.5. The mean type condition in this theorem does not depend on the choice of E and E_1. In fact, by Remark 2.3, if $\mathcal{H}(E_1) = \mathcal{H}(E_2)$ with equality of norms, then $\operatorname{mt} \frac{E_1}{E_2} = \operatorname{mt} \frac{E_2}{E_1} = 0$.

In the proof of Theorem 3.4 we will use a class of deB-subspaces defined by a growth condition (see [KW2]). If $\mathcal{H}(E)$ is a de Branges space and $\beta_+, \beta_- \leq 0$, denote by $\mathcal{H}(E)_{(\beta_+,\beta_-)}$ the linear subspace

$$\mathcal{H}(E)_{(\beta_+,\beta_-)} := \left\{ F \in \mathcal{H}(E) : \operatorname{mt} \frac{F}{E} \leq \beta_+, \operatorname{mt} \frac{F^\#}{E} \leq \beta_- \right\}.$$

Then the space $\mathcal{H}(E)_{(\beta_+,\beta_-)}$ is closed. Moreover, if $\beta_+ = \beta_-$, it actually belongs to $\operatorname{Sub}^s(\mathcal{H}(E)) \cup \{0\}$ (see [KW2, Lemma 2.6, Corollary 5.2]).

Lemma 3.6 *Let $\mathcal{H}(E)$ be a de Branges space, $\beta < 0$, and assume $\mathcal{H}(E)_{(\beta,\beta)} \neq \{0\}$. Then* $\dim\left(\mathcal{H}(E)_{(\beta',\beta')}\big/\mathcal{H}(E)_{(\beta,\beta)}\right) = \infty$, $\beta' \in (\beta, 0]$.

Proof It is enough to show that for all β with $\mathcal{H}(E)_{(\beta,\beta)} \neq \{0\}$ and $\beta' \in (\beta, 0]$ we have $\mathcal{H}(E)_{(\beta,\beta)} \neq \mathcal{H}(E)_{(\beta',\beta')}$. To see this, choose $F \in \mathcal{H}(E)_{(\beta,\beta)} \setminus \{0\}$ and put $\alpha := \operatorname{mt}\frac{F}{E}$. Then the function $G(z) := e^{i(\alpha-\beta')z}F(z)$ belongs to $\mathcal{H}(E)$ (see [KW2, Lemma 2.6]), and satisfies $\operatorname{mt}\frac{G}{E} = \beta'$. Since $\alpha \leq \beta \leq \beta'$, we have $\operatorname{mt}\frac{G^\#}{E} = \alpha - \beta' + \frac{F^\#}{E} \leq \alpha - \beta' + \beta \leq \beta'$. Hence $G \in \mathcal{H}_{(\beta',\beta')} \setminus \mathcal{H}_{(\beta,\beta)}$. ∎

Lemma 3.7 *Let $E, E_1 \in \mathcal{H}B$, $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$, and $\beta < 0$ be given. Then $\mathcal{H}(E_1) \subseteq \mathcal{H}(E)_{(\beta,\beta)}$ if and only if $\operatorname{mt}\frac{E_1}{E} \leq \beta$.*

Proof Assume that $\mathcal{H}(E)_{(\beta,\beta)} \neq \{0\}$. Then [KW2, Lemma 5.5] implies that $\mathcal{H}(E)_{(\beta,\beta)} = \mathcal{H}(E_\beta)$ with $E_\beta \in \mathcal{H}B$ and $\operatorname{mt}\frac{E_\beta}{E} = \beta$. Hence, if $\mathcal{H}(E_1) \subseteq \mathcal{H}(E)_{(\beta,\beta)}$, we get

$$\operatorname{mt}\frac{E_1}{E} = \underbrace{\operatorname{mt}\frac{E_1}{E_\beta}}_{\leq 0} + \operatorname{mt}\frac{E_\beta}{E} \leq \beta.$$

Conversely, if $\operatorname{mt}\frac{E_1}{E} \leq \beta$, we obtain

$$\operatorname{mt}\frac{F}{E} = \operatorname{mt}\left(\frac{F}{E_1}\cdot\frac{E_1}{E}\right) = \operatorname{mt}\frac{F}{E_1} + \operatorname{mt}\frac{E_1}{E} \leq \beta$$

for every $F \in \mathcal{H}(E_1) \setminus \{0\}$. Hence $F \in \mathcal{H}(E)_{(\beta,0)}$. Since with F also $F^\#$ belongs to $\mathcal{H}(E_1)$, the same argument will show that $F \in \mathcal{H}(E)_{(0,\beta)}$ and, therefore, $F \in \mathcal{H}(E)_{(\beta,\beta)}$. ∎

Proof of Theorem 3.4 Let $E, E_1 \in \mathcal{H}B$, $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$, be given.

Sufficiency: Assume that $\operatorname{mt}\frac{E_1}{E} = 0$. Since $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$, we have $\mathfrak{d}(E_1)|_{\mathbb{R}} \geq \mathfrak{d}(E)|_{\mathbb{R}}$. Define ω as

$$\omega(x) := \frac{|E_1(x)|}{(1+|x|)|E(x)|}, \quad x \in \mathbb{R},$$

then ω is a continuous and nonnegative function on $\mathbb{R}$. Let $\nu \in \mathbb{C} \setminus \mathbb{R}$ and consider the reproducing kernel

$$K_1(\nu, z) = \frac{E_1(z)\overline{E_1(\nu)} - E_1^\#(z)E_1(\bar\nu)}{2\pi i(\bar\nu - z)}$$

of $\mathcal{H}(E_1)$. Then for $x \in \mathbb{R}$ we have

$$\begin{aligned}|K_1(\nu,x)| &= \frac{1}{2\pi}\left|\frac{E_1(x)\overline{E_1(\nu)} - E_1^\#(x)E_1(\bar\nu)}{\bar\nu - x}\right| \\ &\leq \frac{1}{\pi}\max\{|E_1(\nu)|, |E_1(\bar\nu)|\}\cdot\max_{t\in\mathbb{R}}\frac{1+|t|}{|t-\bar\nu|}\cdot\frac{|E_1(x)|}{1+|x|} = C\omega(x)|E(x)|,\end{aligned}$$

where

$$C := \frac{1}{\pi} \max\{|E_1(\nu)|, |E_1(\bar\nu)|\} \max_{t\in\mathbb{R}} \frac{1+|t|}{|t-\bar\nu|}.$$

Hence $|E(x)^{-1}K_1(\nu,x)| \lesssim \omega(x)$, $E(x) \neq 0$, and by continuity this inequality holds for all $x \in \mathbb{R}$. Hence $\omega \in \operatorname{Adm}(E)$ and $K_1(\nu,\cdot) \in R_\omega(E)$. Since the linear span of the reproducing kernels $K_1(\nu,\cdot)$, $\nu \in \mathbb{C}\setminus\mathbb{R}$, is dense in $\mathcal{H}(E_1)$, we conclude that $\mathcal{H}(E_1) \subseteq \mathfrak{R}(\omega)$.

Conversely, let $F \in R_\omega(E)$. Then $F \in \mathcal{H}(E)$ and, by [KW2, §2], $E_1 \in \mathcal{H}(E) + z\mathcal{H}(E)$. Thus $E^{-1}F, E^{-1}E_1 \in N(\mathbb{C}^+)$, and it follows that $E_1^{-1}F \in N(\mathbb{C}^+)$. Moreover, by our assumption that $\operatorname{mt}\frac{E_1}{E} = 0$, we have

$$\operatorname{mt}\frac{F}{E_1} = \operatorname{mt}\frac{F}{E} + \operatorname{mt}\frac{E}{E_1} = \operatorname{mt}\frac{F}{E} \leq 0.$$

Since $F^\#$ also belongs to $\mathcal{H}(E)$ whenever F does, this argument also applies to $F^\#$ and we obtain $E_1^{-1}F^\# \in N(\mathbb{C}^+)$, $\operatorname{mt}(E_1^{-1}F^\#) \leq 0$.

Since $F \in R_\omega(E)$, *i.e.*, $|F(x)| \lesssim \omega(x)|E(x)|$, $x \in \mathbb{R}$, we have $|E_1(x)^{-1}F(x)| \lesssim (1+|x|)^{-1} \in L^2(\mathbb{R})$. It follows that $F \in \mathcal{H}(E_1)$ for any $F \in R_\omega(E)$. Thus, also $\mathfrak{R}(\omega) \subseteq \mathcal{H}(E_1)$.

Necessity: Assume that $\mathcal{H}(E_1) = \mathfrak{R}(\omega)$ for some $\omega \in \operatorname{Adm}(E)$. Let us assume on the contrary that $\operatorname{mt}\frac{E_1}{E} = \beta < 0$. Then, by Lemma 3.7, $\mathcal{H}(E_1) \subseteq \mathcal{H}_{(\beta,\beta)}$. Consider the map

$$\Phi : \begin{cases} \mathcal{H}_{(\beta,\beta)}, & \rightarrow \quad \mathcal{H}(E), \\ F(z), & \mapsto \quad e^{i\beta z}F(z) \end{cases}$$

(see [KW2, Lemma 2.6]). This map is isometric and, therefore, continuous. Since $R_\omega(E) \subseteq \mathcal{H}_{(\beta,\beta)}$ it follows that $\Phi(R_\omega(E)) \subseteq \mathcal{H}(E)$. Clearly, we have $|\Phi(F)(x)| = |F(x)|$, $x \in \mathbb{R}$. Thus, $\Phi(R_\omega(E)) \subseteq R_\omega(E)$ and, consequently, $\Phi(\mathcal{R}_\omega(E)) \subseteq \mathcal{R}_\omega(E)$. Hence, if $F \in \mathcal{R}_\omega(E)$, then for every $n \in \mathbb{N}$ we have $\Phi^n(F) \in \mathcal{R}_\omega(E) \subseteq \mathcal{H}(E)$. However,

$$\operatorname{mt}\frac{\Phi^n(F)}{E} = \operatorname{mt}\frac{e^{in\beta z}F(z)}{E(z)} = -n\beta + \operatorname{mt}\frac{F}{E}.$$

If $F \neq 0$ and n is chosen sufficiently large, we have a contradiction since, due to the inclusion $\Phi^n(F) \in \mathcal{H}(E)$, always $\operatorname{mt}\frac{\Phi^n(F)}{E} \leq 0$ must hold. ■

As a byproduct of the proof of Theorem 3.4 we obtain a result that will be of importance in our further investigation of the structure of $\operatorname{Adm}(E)$:

Corollary 3.8 *Let $E \in \mathcal{H}B$ and $\omega \in \operatorname{Adm}(E)$. Then there exists $F \in \mathcal{H}(E)\setminus\{0\}$ such that $\mathfrak{R}(\omega) = \mathfrak{R}(\omega_F)$.*

Proof Choose $E_1 \in \mathcal{H}B$ such that $\mathfrak{R}(\omega) = \mathcal{H}(E_1)$. Then $\operatorname{mt}\frac{E_1}{E} = 0$, and, as we have seen in the proof of sufficiency of Theorem 3.4,

$$\mathcal{H}(E_1) = \mathfrak{R}\Big(\frac{|E_1(x)|}{(1+|x|)|E(x)|}\Big).$$

Let K_1 be the reproducing kernel of $\mathcal{H}(E_1)$ and fix $\nu \in \mathbb{C}^+$. It is easy to see that $|K_1(\nu,x)| \asymp (1+|x|)^{-1}|E_1(x)|$, and we conclude that $\mathcal{H}(E_1) = \mathfrak{R}(\omega_{K_1(\nu,\cdot)})$. ■

The deB-subspaces of highest interest are those which satisfy (SubZ), *i.e.*, the elements of $\operatorname{Sub}^s(\mathcal{H}(E))$. Correspondingly, the admissible majorants of highest interest are the elements of $\operatorname{Adm}^+(E)$. From Theorem 3.4 we deduce a characterization of $\mathfrak{R}(\operatorname{Adm}^+(E))$.

Proposition 3.9 *Let $E, E_1 \in \mathcal{H}B$ be such that $\mathcal{H}(E_1) \in \operatorname{Sub}(\mathcal{H}(E))$. Then the following are equivalent:*

(i) $\mathcal{H}(E_1) \in \mathfrak{R}(\operatorname{Adm}^+(E))$*;*
(ii) $\mathcal{H}(E_1) \in \operatorname{Sub}^s(\mathcal{H}(E))$ *and* $\operatorname{mt} \frac{E_1}{E} = 0$*;*
(iii) $\mathcal{H}(E_1) \in \operatorname{Sub}^s(\mathcal{H}(E))$ *and* $\mathcal{H}(E_1) \supseteq \bigcup_{\beta<0} \mathcal{H}(E)_{(\beta,\beta)}$.

Proof Combining Theorem 3.4 and Proposition 3.2 we immediately see that (i) is equivalent to (ii).

Assume that (iii) does not hold, *i.e.*, there exists $\beta < 0$ with $\mathcal{H}(E)_{(\beta,\beta)} \nsubseteq \mathcal{H}(E_1)$. Since $\operatorname{Sub}^s(\mathcal{H}(E)) \cup \{0\}$ is totally ordered, this implies that $\mathcal{H}(E)_{(\beta,\beta)} \supseteq \mathcal{H}(E_1)$. We obtain from Lemma 3.7 that $\operatorname{mt} \frac{E_1}{E} \leq \beta$, and see that (ii) does not hold.

Conversely, assume that $\beta_0 := \operatorname{mt} \frac{E_1}{E} < 0$. Then $\mathcal{H}(E_1) \subseteq \mathcal{H}(E)_{(\beta_0,\beta_0)}$. Since $\mathcal{H}(E_1) \neq \{0\}$, it follows that $\mathcal{H}(E)_{(\beta,\beta)} \supsetneq \mathcal{H}(E)_{(\beta_0,\beta_0)}$, $\beta \in (\beta_0, 0]$. ■

From this result we obtain a criterion for density of a set $R_\omega(E)$ in $\mathcal{H}(E)$. Results of this type are of interest since density of $R_\omega(E)$ means that all elements of $\mathcal{H}(E)$ can be approximated by functions F satisfying $|E^{-1}F| \lesssim \omega$ on the real axis, *i.e.*, in a certain sense, by "small" functions.

Corollary 3.10 *Let $E \in \mathcal{H}B$.*

(i) *If the linear space $\mathcal{L}_0 := \bigcup_{\beta<0} \mathcal{H}(E)_{(\beta,\beta)}$ is dense in $\mathcal{H}(E)$, then for every $\omega \in \operatorname{Adm}^+(E)$ the linear space $R_\omega(E)$ is dense in $\mathcal{H}(E)$. Unless $\dim \mathcal{H}(E) = 1$, the converse also holds.*
(ii) *Assume that $\operatorname{Clos}_{\mathcal{H}(E)} \mathcal{L}_0 = \mathcal{H}(E)$ and let $F_0 \in \mathcal{H}(E)$, $\mathfrak{d}(F_0)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}}$. Then the set $\{F \in \mathcal{H}(E) : |F(x)| \lesssim |F_0(x)|, x \in \mathbb{R}\}$ is dense in $\mathcal{H}(E)$.*

Proof The asserted implication in (i) follows immediately from Proposition 3.9, (i)⇒(iii). To prove the converse, let $\dim \mathcal{H}(E) > 1$ and assume that $\mathcal{L}_0$ is not dense in $\mathcal{H}(E)$. If $\mathcal{L}_0 = \{0\}$, let L be any element of $\operatorname{Sub}^s(\mathcal{H}(E)) \setminus \{\mathcal{H}(E)\}$. Note that this set is nonempty since $\dim \mathcal{H}(E) > 1$. If $\mathcal{L}_0 \neq \{0\}$, put $L := \operatorname{Clos}_{\mathcal{H}(E)} \mathcal{L}_0$. Since, clearly, $\mathcal{L}_0$ satisfies (Sub2), (Sub3), and (SubZ), the same argument as in Proposition 3.2 shows that $L \in \operatorname{Sub}^s(E)$. By Proposition 3.9, we have $L = \mathfrak{R}(\omega_0)$ for some $\omega_0 \in \operatorname{Adm}^+(E)$. We see that $R_{\omega_0}(E)$ is not dense in $\mathcal{H}(E)$.

To establish the assertion (ii), apply (i) with the majorant ω_{F_0}. ■

We would like to illustrate the above statements with some examples. First let us make explicit two extreme cases.

Example 3.11. Assume that $\tau_E := \operatorname{mt} \frac{E^\#}{E} < 0$. Then $\mathcal{R}_\omega(E) = \mathcal{H}(E)$ for all $\omega \in \operatorname{Adm}^+(E)$. Indeed, in this situation, by [KW2, Theorem 2.7(ii)], we have

$$\operatorname{Clos}_{\mathcal{H}(E)} \bigcup_{\beta<0} \mathcal{H}(E)_{(\beta,\beta)} = \mathcal{H}(E).$$

This result applies, in particular, to the Paley–Wiener space $\mathcal{P}W_a = \mathcal{H}(e^{-iaz})$.

Corollary 3.12 *For any $\omega \in \operatorname{Adm}^+(e^{-iaz})$ (in particular, if $\omega(x) = |F_0(x)|$, where $F_0 \in \mathcal{P}W_a$ has no real zeros, or if $\omega\colon \mathbb{R} \to (0,\infty)$ satisfies (2.1) and $\log\omega$ is Lipschitz on $\mathbb{R}$) the set $R_\omega(E)$ is dense in $\mathcal{P}W_a$.*

Example 3.13. Assume that E is of zero exponential type. Then, by [KW2, Lemma 5.6], we have $\mathcal{H}(E)_{(\beta,\beta)} = \{0\}$ for all $\beta < 0$. Hence every element $L \in \operatorname{Sub}^s(E)$ can be written as $L = \mathfrak{R}(\omega)$ for some $\omega \in \operatorname{Adm}^+(E)$.

Next, we give an example where some, but not all, deB-subspaces can be realized as $\mathfrak{R}(\omega)$. This example also shows that the concepts of deB-subspaces defined by majorants on the one hand and by mean type conditions on the other, are in a way complementary.

Example 3.14. Consider a canonical system on $[0,l]$ with Hamiltonian H (see Example 2.8). Then we have $1 \in \mathcal{H}(E_t) + z\mathcal{H}(E_t)$, $t \in (0,l]$. The function E_t belongs to $N(\mathbb{C}^+)$ and $\tau(t) := \operatorname{mt} E_t = \int_0^t \sqrt{\det H(s)}ds$. Note that τ is a continuous and nondecreasing function on $[0,l]$. We obtain from Proposition 3.9 that a space $\mathcal{H}(E_t)$, where $t \in (0,l]$ is not an inner point of an indivisible interval, belongs to $\mathfrak{R}(\operatorname{Adm}^+(E_l))$ if and only if $\tau(t) = \tau(l)$. On the other hand, by Lemma 3.7, we have for $\beta \le 0$

$$\mathcal{H}(E_l)_{(\beta,\beta)} = \begin{cases} \mathcal{H}(E_{s(\beta)}), & s(\beta) > 0, \\ \{0\}, & \text{otherwise}, \end{cases}$$

where $s(\beta) := \sup\left\{ t \in [0,l] : \tau(t) = \tau(l) + \beta \right\}$.

4 Minimal Majorants

In this section we will have a closer look at the order structure of $\operatorname{Adm}(E)/_{\asymp}$ and $\operatorname{Adm}^+(E)/_{\asymp}$, respectively. It turns out that the question of existence of minimal elements is an intriguing matter.

Definition 4.1 An admissible majorant ω is said to be *minimal* if its equivalence class $\omega/_{\asymp}$ is a minimal element of $\operatorname{Adm}(E)/_{\asymp}$. This means that for every admissible majorant $\tilde\omega$ with $\tilde\omega \lesssim \omega$, we must have $\tilde\omega \asymp \omega$.

Our investigation is based on the following result, which shows that minimal admissible majorants correspond to one-dimensional deB-subspaces of $\mathcal{H}(E)$.

Theorem 4.2 *Let $E \in \mathcal{H}B$. If $\omega \in \operatorname{Adm}(E)$ is minimal in $\operatorname{Adm}(E)/_{\asymp}$, then $\dim \mathcal{R}_\omega(E) = 1$. Conversely, if $\omega \in \operatorname{Adm}(E)$ and $\dim \mathcal{R}_\omega(E) = 1$, then there exists $\omega_0 \in \operatorname{Adm}(E)$, which is minimal in $\operatorname{Adm}(E)/_{\asymp}$, such that $\mathcal{R}_\omega(E) = \mathcal{R}_{\omega_0}(E)$.*

Proof

Step 1. Let $\omega \in \operatorname{Adm}(E)$ and assume that $\dim \mathcal{R}_\omega(E) > 1$. We show that ω is not minimal in $\operatorname{Adm}(E)/_{\asymp}$.

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Since $\dim \mathcal{R}_\omega(E) > 1$, we also have $\dim R_\omega(E) > 1$. Choose linearly independent elements F_1, F_2 of $R_\omega(E)$. Fix $\nu \in \mathbb{C} \setminus \mathbb{R}$ and choose $\alpha_1, \alpha_2 \in \mathbb{C}$, not both zero, such that $\alpha_1 F_1(\nu) + \alpha_2 F_2(\nu) = 0$. Put

$$F(z) := \frac{\alpha_1 F_1(z) + \alpha_2 F_2(z)}{z - \nu}.$$

Then $F \in R_\omega(E)$ and does not vanish identically. Hence $\omega_F \lesssim \omega$. However, we have $|E(x)^{-1}F(x)| \lesssim (1 + |x|)^{-1}\omega(x)$, $x \in \mathbb{R}$, and hence $\inf_{x\in\mathbb{R}} \frac{\omega_F(x)}{\omega(x)} = 0$. Thus, $\omega \not\lesssim \omega_F$. It follows that ω is not minimal in $\operatorname{Adm}(E)/_{\asymp}$.

Step 2. Let $F \in \mathcal{H}(E) \setminus \{0\}$ and $\dim \mathcal{R}_{\omega_F}(E) = 1$. Then ω_F is minimal in $\operatorname{Adm}(E)/_{\asymp}$.

Let $\omega \in \operatorname{Adm}(E)$ be given such that $\omega \lesssim \omega_F$, and choose $G \in R_\omega(E) \setminus \{0\}$. Then G also belongs to $R_{\omega_F}(E)$. Our assumption that $\dim \mathcal{R}_{\omega_F}(E) = 1$ implies $F = \lambda G$ for some $\lambda \in \mathbb{C}$. It follows that, for some appropriate constant $C > 0$,

$$\omega_F(x) = \left|\frac{F(x)}{E(x)}\right| = |\lambda| \left|\frac{G(x)}{E(x)}\right| \leq C|\lambda|\omega(x).$$

Hence $\omega_F \lesssim \omega$, and we see that ω_F is minimal in $\operatorname{Adm}(E)/_{\asymp}$.

Step 3. Let $\omega \in \operatorname{Adm}(E)$ and $\dim \mathcal{R}_\omega(E) = 1$. Then for every $F \in \mathcal{R}_\omega(E) \setminus \{0\}$ we have $\omega_F \lesssim \omega$ and $\mathcal{R}_\omega(E) = \mathcal{R}_{\omega_F}(E)$.

Fix $F \in \mathcal{R}_\omega(E) \setminus \{0\}$, and consider ω_F. Since $\mathcal{R}_\omega(E)$ is finite-dimensional, we have $\mathcal{R}_\omega(E) = R_\omega(E)$. Thus, $\omega_F \lesssim \omega$ and so $\mathcal{R}_{\omega_F}(E) \subseteq \mathcal{R}_\omega(E)$. Since $\dim \mathcal{R}_\omega(E) = 1$, this implies that $\dim \mathcal{R}_{\omega_F}(E) = 1$, and thus, clearly, also $\mathcal{R}_{\omega_F}(E) = \mathcal{R}_\omega(E)$.

Step 4. The proof of the theorem is now easily completed. Assume that ω is minimal, then by Step 1 we must have $\dim \mathcal{R}_\omega(E) = 1$. Assume conversely that $\dim \mathcal{R}_\omega(E) = 1$. Choose $F \in \mathcal{R}_\omega(E) \setminus \{0\}$, then, by Step 3, $\mathcal{R}_\omega(E) = \mathcal{R}_{\omega_F}(E)$ and, by Step 2, ω_F is minimal. ■

Remark 4.3. An analogous result is actually true in the model subspaces setting (see [BH, Proposition 5.6]). However, in contrast to the general situation, we obtain in the case of de Branges spaces a one-to-one correspondence between minimal majorants and one-dimensional de Branges subspaces.

Corollary 4.4 *Let* $\omega \in \operatorname{Adm}(E)$. *Then* ω *is minimal in* $\operatorname{Adm}(E)/_{\asymp}$ *if and only if* $\dim \mathcal{R}_\omega(E) = 1$ *and* $\omega \asymp \omega_F$ *for some* $F \in \mathcal{H}(E) \setminus \{0\}$. *In this case* $\omega \asymp \omega_F$ *for any* $F \in \mathcal{R}_\omega(E) \setminus \{0\}$.

Proof Assume that $\omega \in \operatorname{Adm}(E)$ is minimal in $\operatorname{Adm}(E)/_{\asymp}$. By the above theorem we have $\dim \mathcal{R}_\omega(E) = 1$. By Step 3 of its proof, for $F \in R_\omega(E) \setminus \{0\}$, the majorant ω_F satisfies $\omega_F \lesssim \omega$. By minimality of ω, this implies $\omega_F \asymp \omega$. The converse is just Step 2 of the above proof. ■

Remark 4.5. It should be emphasized that if $\omega \in \operatorname{Adm}(E)$ has the property that $\dim \mathcal{R}_\omega(E) = 1$, it does not necessarily follow that ω itself is minimal.

For example, let $E(z) := (z + i)(z + 2i)$. Then $\mathcal{H}(E) = \operatorname{span}\{1, z\}$ and we see that $\omega(x) := |E(x)|^{-1}$ and $\omega_1(x) := |E(x)|^{-1}\sqrt{|x| + 1}$ belong to $\operatorname{Adm}^+(E)$ and that $\mathcal{R}_\omega(E) = \mathcal{R}_{\omega_1}(E) = \operatorname{span}\{1\}$. However, clearly, ω is essentially smaller than ω_1.

Corollary 4.6 *Let $E \in \mathcal{H}B$ and assume that* $\dim \mathcal{H}(E) > 1$. *Then the set* $\operatorname{Adm}(E)/_{\asymp}$ *contains uncountably many minimal elements.*

Proof Let $x_0 \in \mathbb{R}$ and consider the function $S_\alpha(z) := e^{i\alpha}E(z) - e^{-i\alpha}E^\#(z)$, $\alpha \in [0,\pi)$. Then there exists at most one number $\alpha \in [0,\pi)$ such that $S_\alpha \in \mathcal{H}(E)$. Hence all but at most countably many real numbers t are not zeros of a function S_α belonging to $\mathcal{H}(E)$. Then, by [deB, Theorem 22], for such t the space $\mathcal{R}_{\omega_{K(t,\cdot)}}(E)$ is one-dimensional; in fact

$$\mathcal{R}_{\omega_{K(t,\cdot)}}(E) = \operatorname{span}\{K(t,\cdot)\}.$$

By Corollary 4.4, $\omega_{K(t,\cdot)}$ is minimal in $\operatorname{Adm}(E)/_{\asymp}$.

Since $\dim \mathcal{H}(E) > 1$, no two of the elements $K(t,\cdot)$, $t \in \mathbb{R}$, are linearly dependent. Hence no two of the spaces $\mathcal{R}_{\omega_{K(t,\cdot)}}(E)$ coincide. Thus no two of the majorants $\omega_{K(t,\cdot)}$ define the same equivalence class in $\operatorname{Adm}(E)/_{\asymp}$. ∎

For admissible majorants separated from zero the situation is significantly different. Below we will show that the set $\operatorname{Adm}^+(E)/_{\asymp}$ need not necessarily contain minimal elements and give a criterion for the existence of minimal elements in $\operatorname{Adm}^+(E)/_{\asymp}$.

Lemma 4.7 *Let $\omega \in \operatorname{Adm}^+(E)$ be given. Then $\omega/_{\asymp}$ is a minimal element in $\operatorname{Adm}^+(E)/_{\asymp}$ if and only if it is minimal in $\operatorname{Adm}(E)/_{\asymp}$.*

Proof Let $\omega/_{\asymp}$ be a minimal element of $\operatorname{Adm}^+(E)/_{\asymp}$ and assume that $\omega_1 \in \operatorname{Adm}(E)$ is such that $\omega_1 \leq \omega$ and $\inf_{\mathbb{R}} \omega_1/\omega = 0$. It is elementary to see that, since $\omega \in \operatorname{Adm}^+(E)$, there exists a function ω_2 separated from zero and such that $\omega_1 \leq \omega_2 \leq \omega$, $\inf_{\mathbb{R}} \omega_2/\omega = 0$. Thus, $\omega_2 \in \operatorname{Adm}^+(E)$, $\omega_2 \lesssim \omega$, but $\omega_2 \not\gtrsim \omega$, which contradicts the minimality of $\omega/_{\asymp}$ in $\operatorname{Adm}^+(E)/_{\asymp}$. ∎

Lemma 4.8 *Let $E \in \mathcal{H}B$. Then the space $\mathcal{H}(E)$ contains a real function S with*

$$\mathfrak{d}(S)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}} \quad \textit{and} \quad \mathfrak{d}(S)|_{\mathbb{C}\setminus\mathbb{R}} = 0, \tag{4.1}$$

if and only if there exists $L \in \operatorname{Sub}^s(\mathcal{H}(E))$ such that $\dim L = 1$. *In this case there exists, up to constant real multiples, exactly one real function $S \in \mathcal{H}(E)$ which satisfies* (4.1).

Proof If $S = S^\#$ and (4.1) holds, then, clearly, $L := \operatorname{span}\{S\}$ satisfies (Sub1)–(Sub3) and (SubZ). Conversely, assume that $L \in \operatorname{Sub}^s(\mathcal{H}(E))$ is one-dimensional. By (Sub2) there exists $S = S^\# \in L \setminus \{0\}$. Since, for a zero ν of S, the functions $S(z)$ and $\frac{S(z)}{z-\nu}$ are linearly independent, it follows from (Sub3) and Remark 2.4 that S must satisfy (4.1).

If S_1, S_2 are real elements of $\mathcal{H}(E)$ which both satisfy (4.1), then $\operatorname{span}\{S_1\}$ and $\operatorname{span}\{S_2\}$ are one-dimensional elements of $\operatorname{Sub}^s(\mathcal{H}(E))$. Hence, by the Ordering Theorem, $\operatorname{span}\{S_1\} = \operatorname{span}\{S_2\}$. ∎

Combining Theorem 4.2 with Theorem 3.4 leads to the following.

Theorem 4.9 *Let $E \in \mathcal{H}B$. Then there exists a minimal element in $\operatorname{Adm}^+(E)/_{\asymp}$ if and only if the following hold:*

(i) *there exists* $L \in \operatorname{Sub}^s(\mathcal{H}(E))$ *with* $\dim L = 1$;

(ii) *for all* $\beta < 0$ *we have* $\mathcal{H}(E)_{(\beta,\beta)} = \{0\}$.

In this case there exists exactly one minimal element in $\operatorname{Adm}^+(E)/_{\asymp}$, *namely* $\omega_S/_{\asymp}$ *where* S *is the (up to scalar multiples unique) real element of* $\mathcal{H}(E)$ *with* $\mathfrak{d}(S)|_{\mathbb{R}} = \mathfrak{d}(E)|_{\mathbb{R}}$, $\mathfrak{d}(S)|_{\mathbb{C}\setminus\mathbb{R}} = 0$.

Proof Assume that conditions (i) and (ii) hold. Let L be the one-dimensional element of $\operatorname{Sub}^s(\mathcal{H}(E))$, and let S be as in Lemma 4.8. By Proposition 3.9 there exists $\omega \in \operatorname{Adm}^+(E)$ such that $L = \mathfrak{R}(\omega)$. By Step 3 of the proof of Theorem 4.2, we have $\mathcal{R}_{\omega_S}(E) = L$, and ω_S is minimal by Step 2. Since S satisfies (4.1), we have $\omega_S \in \operatorname{Adm}^+(E)$.

Assume that ω is a minimal element of $\operatorname{Adm}^+(E)/_{\asymp}$. By Lemma 4.7 and Theorem 4.2, we have $\dim \mathcal{R}_\omega(E) = 1$. Hence (i) holds. Moreover, by Theorem 3.4, we must have $\mathcal{R}_\omega(E) \supseteq \mathcal{H}_{(\beta,\beta)}$ for all $\beta < 0$. Thus, $\dim \mathcal{H}_{(\beta,\beta)} \in \{0,1\}$ for all $\beta < 0$. If for some $\beta < 0$ we have $\dim \mathcal{H}_{(\beta,\beta)} = 1$, we would have $\mathcal{H}_{(\beta,\beta)} = \mathcal{H}_{(\beta',\beta')}$ for all $\beta' \in (\beta, 0)$, which contradicts Lemma 3.6.

Let ω_1, ω_2 be minimal elements of $\operatorname{Adm}^+(E)/_{\asymp}$. By Lemma 4.7 and Theorem 4.2 we have $\dim \mathcal{R}_{\omega_j}(E) = 1$, $j = 1, 2$. Since $\mathcal{R}_{\omega_j}(E) \in \operatorname{Sub}^s(\mathcal{H}(E))$, it follows that $\mathcal{R}_{\omega_1}(E) = \mathcal{R}_{\omega_2}(E) = \operatorname{span}\{S\}$ where S is as in Lemma 4.8. By Corollary 4.4 we have $\omega_j \asymp \omega_S$, $j = 1, 2$. ■

Remark 4.10. The present Theorem 4.9 is a (slight) generalization of a result of Havin and Mashreghi (see [HM1] or [B, BH]), which states the following: *Assume that* $E \in \mathcal{H}B$ *is of zero exponential type. Then there exists a positive and continuous minimal majorant in* $\operatorname{Adm}(E)$ *if and only if* $1 \in \mathcal{H}(E)$. *Moreover, in case of existence, this majorant is given by* $\omega = |E|^{-1}$, *and any other continuous positive minimal majorant* $\omega_1 \in \operatorname{Adm}(E)$ *satisfies* $\omega_1 \asymp |E|^{-1}$.

Note that, in the present setting, the inclusion $1 \in \mathcal{H}(E)$ is equivalent to $|E|^{-1} \in L^2(\mathbb{R})$. A number of conditions sufficient for the inclusion $1 \in \mathcal{H}(E)$ may be found in [B, KW3].

Example 4.11. Consider a canonical system defined on $[0, l]$ with Hamiltonian H. Then $\operatorname{Adm}^+(E_l)$ contains a minimal element if and only if for some $\epsilon > 0$ the interval $(0, \epsilon)$ is indivisible and $\det H(t) = 0$, a.e. $t \in [0, l]$. In this case the minimal majorant is given by $|E_l(x)|^{-1}$.

In particular, for the Paley–Wiener space $\mathcal{P}W_a = \mathcal{H}(e^{-iaz})$ there are no minimal majorants.

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