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Abstract

In this paper, we prove some one level density results for the low-lying zeros of families of L-functions. More specifically, the families under consideration are that of L-functions of holomorphic Hecke eigenforms of level 1 and weight k twisted with quadratic Dirichlet characters and that of cubic and quartic Dirichlet L-functions.

1. Introduction

The density conjecture of Katz and Sarnak suggests that the distribution of zeros near the central point of a family of L-functions is the same as that of eigenvalues near 1 of a corresponding classical compact group. This has been confirmed for many families of L-functions, such as different types of Dirichlet L-functions [Gao, HR03, Mil08, OS99, Rub01], L-functions with characters of the ideal class group of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ [FI03], automorphic L-functions [DM09, HM07, ILS00, RR, Roy01], elliptic curve L-functions [BZ08, Bru92, Hea04, Mil04, You05], symmetric powers of GL(2) L-functions [DM06, Gul05b] and a family of GL(4) and GL(6) L-functions [DM06]. The literature in this direction is too numerous to be completely listed here. In this paper, we prove some one level density results for the low-lying zeros of families of L-functions of holomorphic Hecke eigenforms of level 1 and weight k twisted with quadratic Dirichlet characters and those of families of cubic and quartic Dirichlet L-functions.

Let χ be a primitive Dirichlet character and denote the non-trivial zeros of the Dirichlet L-function $L(s,\chi)$ by $\frac{1}{2}+i\gamma_{\chi,j}$. Without assuming the generalized Riemann hypothesis (GRH), we order them as

$$\cdots \leqslant \Re\gamma_{\chi,-2} \leqslant \Re\gamma_{\chi,-1} < 0 \leqslant \Re\gamma_{\chi,1} \leqslant \Re\gamma_{\chi,2} \leqslant \cdots . \tag{1.1}$$

For any primitive Dirichlet character χ of conductor q of size X, we set

$$\tilde{\gamma}_{\chi,j} = \frac{\gamma_{\chi,j}}{2\pi}\log X$$

and define for an even Schwartz class function ϕ,

$$S(\chi,\phi) = \sum_j \phi(\tilde{\gamma}_{\chi,j}). \tag{1.2}$$

In [OS99], Özluk and Snyder studied the family of quadratic Dirichlet L-functions. Consider the family of Dirichlet L-functions of the form $L(\chi_{8d},s)$ for d odd and square-free with $X \leqslant d \leqslant 2X$, where $\chi_{8d} = (\frac{8d}{\cdot})$ is the Kronecker symbol. Let $D(X)$ denote the set of such d.

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It is easy to see, as noted in [Gao], that

$$\#D(X) \sim \frac{4X}{\pi^2}.$$

Assuming the GRH for this family, it follows from the result of Özluk and Snyder [OS99] that for $W_{USp}(x) = 1 - (\sin(2\pi x)/2\pi x)$ we have

$$\frac{1}{\#D(X)} \sum_{d \in D(X)} S(\chi_{8d}, \phi) \sim \int_{\mathbb{R}} \phi(x) W_{USp}(x)\, dx \tag{1.3}$$

as $X \to \infty$ provided that the support of $\hat{\phi}$, the Fourier transform of ϕ, is contained in the interval $(-2, 2)$. The expression on the left-hand side of (1.3) is known as the one level density of the low-lying zeros for this family of L-functions under consideration.

The kernel of the integral W_{USp} in (1.3) is the same function which occurs on the random matrix theory side, when studying the eigenvalues of unitary symplectic matrices. This shows that the family of quadratic Dirichlet L-functions is a symplectic family. In [Rub01], Rubinstein studied all the n-level densities of the low-lying zeros of the families of quadratic twists of L-functions attached to a self-contragredient automorphic cuspidal representation, as well as the family of quadratic Dirichlet L-functions. He showed that they converge to the symplectic densities for test functions $\phi(x_1, \ldots, x_n)$ whose Fourier transforms $\hat{\phi}(u_1, \ldots, u_n)$ have their supports contained in the set

$$\left\{(u_1, \ldots, u_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |u_i| < 1\right\}.$$

The result of Rubinstein does not assume the GRH. In [Gao], assuming the truth of the GRH, the first-named author extended Rubinstein's result [Rub01] and showed that it holds for $\hat{\phi}(u_1, \ldots, u_n)$ with support in the set

$$\left\{(u_1, \ldots, u_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |u_i| < 2\right\}.$$

In this paper, we consider a few other families of L-functions. First, let f be a fixed holomorphic Hecke eigenform of level 1 and weight k. For $\Im(z) > 0$ we have a Fourier expansion of f,

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{(k-1)/2} e(nz),$$

where the coefficients $a_f(n)$ are real and normalized with $a_f(1) = 1$ and satisfy the Ramanujan–Petersson bound

$$|a(n)| \leqslant d(n),$$

with $d(n)$ being the divisor function. Let χ be a primitive Dirichlet character of conductor q. The L-function of the twist of f by χ is given by

$$L(f \times \chi, s) = \sum_{n=1}^{\infty} \frac{a_f(n)\chi(n)}{n^s} = \prod_p \left(1 - \frac{a_f(p)\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}}\right)^{-1} \tag{1.4}$$

for $\Re(s) > 1$. This L-function continues to an entire function and satisfies the functional equation

$$\left(\frac{q}{2\pi}\right)^s \Gamma\left(\frac{k}{2} + s\right) L\left(f, \frac{1}{2} + s\right) = \imath_\chi \left(\frac{q}{2\pi}\right)^{-s} \Gamma\left(\frac{k}{2} - s\right) L\left(f \times \overline{\chi}, \frac{1}{2} - s\right),$$

where $\imath_\chi = i^k\tau(\chi)^2/q$ and $\tau(\chi)$ is the Gauss sum associated to χ (thus $|\imath_\chi| = 1$); cf. [IK04, Proposition 14.20]. See [Iwa97, IK04] for detailed discussions of Hecke eigenforms.

For a fixed f, we consider the family of quadratic twists of L-functions $L(f \times \chi_{8d}, s)$ for d odd and square-free with $X \leqslant d \leqslant 2X$. For this family, Rubinstein [Rub01] has shown that the n-level densities of the low-lying zeros converge to the orthogonal densities for test functions $\phi(x_1, \ldots, x_n)$ whose Fourier transforms $\hat{\phi}(u_1, \ldots, u_n)$ are supported in the set

$$\left\{(u_1, \ldots, u_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |u_i| < \frac{1}{2}\right\}.$$

More precisely, we denote the zeros of $L(f \times \chi_{8d}, s)$ by $\frac{1}{2} + i\gamma_{f,8d,j}$, and order them in a manner similar to (1.1). Let $\tilde{\gamma}_{f,8d,j} = \gamma_{f,8d,j} 2\log X/(2\pi)$ and define, for an even Schwartz function ϕ,

$$D(d, f, \phi) = \sum_j \phi(\tilde{\gamma}_{f,8d,j}).$$

We set

$$W_{SO+}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x}, \quad W_{SO-}(x) = \delta(x) + 1 - \frac{\sin(2\pi x)}{2\pi x}$$

and

$$W_O(x) = \tfrac{1}{2}(W_{SO+}(x) + W_{SO-}(x)),$$

where $\delta_0(x)$ is the Dirac distribution at $x = 0$. Note that the three orthogonal densities are indistinguishable for test functions whose Fourier transforms are supported in $(-1, 1)$. The result of Rubinstein asserts that

$$\frac{1}{\#D(X)} \sum_{d \in D(X)} D(d, f, \phi) \sim \int_{\mathbb{R}} \phi(x)\left(1 + \frac{\sin(2\pi x)}{2\pi x}\right) dx$$

as $X \to \infty$ provided that $\hat{\phi}$ is supported on the interval $(-1/2, 1/2)$. This shows that the family $L(f \times \chi_{8d}, s)$ has orthogonal symmetry.

In this paper, we improve the above-mentioned result of Rubinstein by doubling the size of the allowable support of the Fourier transform of the test function in the case of the one level density. For technical reasons, we consider the average over the family by a smooth weight. Let $\Phi_X(t)$ be a non-negative smooth function supported on $(1, 2)$, satisfying $\Phi_X(t) = 1$ for $t \in (1 + 1/U, 2 - 1/U)$ with $U = \log X$ and such that $\Phi_X^{(j)}(t) \ll_j U^j$ for all integers $j \geqslant 0$. Our result is the following theorem.

THEOREM 1.1. *Suppose that the GRH is true. Let f be a fixed holomorphic Hecke eigenform of level 1 and weight k. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ has compact support in $(-1, 1)$. Then we have*

$$\lim_{X \to +\infty} \frac{1}{\#D(X)} \sum_{d \in D(X)} \Phi_X\left(\frac{d}{X}\right) D(d, f, \phi) = \int_{\mathbb{R}} \phi(x)\left(1 + \frac{\sin(2\pi x)}{2\pi x}\right) dx. \tag{1.5}$$

We shall describe briefly the proof of Theorem 1.1 in this paragraph. Using a modified version of the explicit formula, Lemma 2.2, $D(d, f, \phi)$ in (1.5), a sum over the zeros of the L-function under consideration, is converted into a sum over primes and prime powers. The most important among these will be the sums over primes (the one appearing in (2.3)) and prime squares, after showing that the contributions of the higher prime powers are negligible. The sum involving prime squares can be easily handled. Thus far, each of the sums can be disposed of for each

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individual L-function in question, without appealing to the additional averaging over d in (1.5). However, this additional sum over the family is needed in the treatment of the more difficult sum involving primes, (2.3). Following a method of Soundararajan [Sou00], after detecting the square-free condition of d using the Möbius function and applying the Poisson summation formula to the sum over d, we are led to consider sums involving quadratic Gauss sums, which is further split into pieces according to the size of the relevant parameters. The GRH is needed to estimate a component of these sums. Other estimates are quoted from [Mil08] to give the final estimate for the sum in (2.3). Theorem 1.1 follows after combining all estimates.

We note that the one level density of the low-lying zeros of the family of quadratic twists of Hecke L-functions evaluated in Theorem 1.1 is insufficient to determine which of the orthogonal symmetry types, SO(even), O or SO(odd), is attached to each family, due to the small support restriction on the Fourier transforms of test functions. However, as shown in [DM06, Mil04], the two level density allows one to distinguish between the three orthogonal symmetries for test functions whose Fourier transforms have arbitrarily small support. In fact, it follows from Rubinstein's result on the two level density of this family ([Rub01, Lemma 7] and also the formula on the top of p. 179 in [Rub01]) that the symmetry type attached to the family of quadratic twists of L-functions $L(f \times \chi_{8d}, s)$ for d odd and square-free with $X \leqslant d \leqslant 2X$ is SO(even).

In addition, we consider the one level density of the low-lying zeros of the families of cubic and quartic Dirichlet L-functions in this paper.

For a primitive cubic Dirichlet character χ of conductor q coprime to 3, it is shown in [BY10] that q must be square-free and a product of primes congruent to 1 modulo 3. It follows that χ is a product of primitive cubic characters modulo the prime divisors of q and for each prime divisor p of q there are exactly two primitive characters with conductor p, each being the square (also the complex conjugate) of the other.

We shall prove the following theorem.

Theorem 1.2. *Let $f(x)$ be an even Schwartz function whose Fourier transform $\hat{f}(u)$ has compact support in $(-3/7, 3/7)$; then*

$$\lim_{X\to+\infty} \frac{1}{\#C(X)} \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi^3=\chi_0}} S(\chi, f) = \int_{\mathbb{R}} f(x)\, dx.$$

Here the '$*$' on the sum over χ means that the sum is restricted to primitive characters and $C(X)$ denotes the set of primitive cubic characters of conductor q not divisible by 3 and $X \leqslant q \leqslant 2X$.

We point out here that Theorem 1.2 is analogous to a result of Güloğlu [Gul05a] and that his result depends on the GRH while Theorem 1.2 does not. Güloğlu [Gul05a] studied the one level density of the low-lying zeros of a family of Hecke L-functions associated with the cubic symbols $\chi_c = (\frac{\cdot}{c})_3$ with c square-free and congruent to 1 modulo 9, which are regarded as primitive ray class characters of the ray class group $h_{(c)}$. We recall here that, for any c, the ray class group $h_{(c)}$ is defined to be $I_{(c)}/P_{(c)}$, where $I_{(c)} = \{\mathcal{A} \in I, (\mathcal{A}, (c)) = 1\}$ and $P_{(c)} = \{(a) \in P, a \equiv 1 \pmod{c}\}$, with I and P denoting the group of fractional ideals in $K = \mathbb{Q}(\omega)$ and the subgroup of principal ideals, respectively. Here $\omega = \exp(2\pi i/3)$. The Hecke L-function associated with χ_c is defined for $\Re(s) > 1$ by

$$L(s, \chi_c) = \sum_{0\neq\mathcal{A}\subset O_K} \chi_c(\mathcal{A})(N(\mathcal{A}))^{-s},$$

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where $\mathcal{A}$ runs over all non-zero integral ideals in K and $N(\mathcal{A})$ is the norm of $\mathcal{A}$. As shown by E. Hecke [Hec20], $L(s, \chi_c)$ admits analytic continuation to an entire function and satisfies a functional equation. We refer the reader to [Gul05a, Luo04] for a more detailed discussion of these Hecke characters and L-functions. We denote non-trivial zeros of $L(s, \chi_c)$ by $\frac{1}{2} + i\gamma_{\chi_c,j}$ and order them in a fashion similar to (1.1). Let $C_{(9)}(X)$ stand for the set of χ_c with c square-free, congruent to 1 modulo 9 and $X \leqslant N(c) \leqslant 2X$. We define $S(\chi_c, f)$ similarly to $S(\chi, \phi)$ in (1.2).

For the family of Hecke L-functions considered by Güloğlu [Gul05a], we can apply our approach to Theorem 1.2 to obtain the following result without assuming the GRH.

THEOREM 1.3. *Let $f(x)$ be an even Schwartz function whose Fourier transform $\hat{f}(u)$ has compact support in $(-3/5, 3/5)$; then*

$$\lim_{X\to+\infty} \frac{1}{\#C_{(9)}(X)} \sideset{}{^*}\sum_{\substack{c\equiv 1 \,(\mathrm{mod}\ 9)\\ X\leqslant N(c)\leqslant 2X}} S(\chi_c, f) = \int_{\mathbb{R}} f(x)\, dx.$$

Here the '$*$' on the sum over c means that the sum is restricted to square-free elements c of $\mathbb{Z}[\omega]$.

The support obtained under the GRH by Güloğlu [Gul05a] is $(-31/30, 31/30)$ in place of $(-3/5, 3/5)$ in our unconditional Theorem 1.3. In addition, the Güloğlu's result is a smoothed version of one level density and depends on other results that require this smoothness. We also note that, as

$$\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} f(x) W_U(x)\, dx \quad \text{with } W_U(x) = 1,$$

Theorems 1.2 and 1.3 show that the family of cubic Dirichlet L-functions as well as the family of Hecke L-functions associated with cubic symbols are unitary families, an observation made in [Gul05a].

Analogous to Theorem 1.2, we also study the one level density of the low-lying zeros of the family of quartic Dirichlet L-functions. We have the following theorem.

THEOREM 1.4. *Let $f(x)$ be an even Schwartz function whose Fourier transform $\hat{f}(u)$ has compact support in $(-3/7, 3/7)$; then*

$$\lim_{X\to+\infty} \frac{1}{\#Q(X)} \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi^4=\chi_0, \chi^2\neq\chi_0}} S(\chi, f) = \int_{\mathbb{R}} f(x)\, dx.$$

Here the '$*$' on the sum over χ means that the sum is restricted to primitive characters and $Q(X)$ denotes the set of primitive complex quartic characters with odd conductor q and $X \leqslant q \leqslant 2X$.

The proofs of Theorems 1.2–1.4 are similar. Therefore, we describe briefly here only the proof of Theorem 1.2. Similar to the proof of Theorem 1.1, we first convert the sum over zeros of the L-functions under consideration into sums over primes using the relevant versions of the explicit formula. This leads us to consider sums involving the cubic or quartic symbols over primes and prime squares. These are then rewritten as sums involving Hecke characters with new summation conditions. The summation conditions are handled using Möbius inversion and the final estimates are obtained using a Pólya–Vinogradov-type bound for these Hecke characters, see Lemma 4.1.

1.5 Notation

The following notation and conventions are used throughout the paper.

(i) $e(z) = \exp(2\pi i z) = e^{2\pi i z}$.

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(ii) $f=O(g)$ or $f \ll g$ means $|f| \leqslant cg$ for some unspecified positive constant c.

(iii) For $x \in \mathbb{R}$, $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$ denotes the distance between x and the closest integer.

2. Preliminaries

2.1 The explicit formula

Our approach in this paper relies on the following explicit formula, which essentially converts the sum over zeros of an L-function to the sum over primes.

Lemma 2.2. *Let $f(x)$ be an even Schwartz function whose Fourier transform $\hat{f}(u)$ is compactly supported. Then, for any primitive Dirichlet character χ, we have*

$$S(\chi, f) = \int_{-\infty}^{\infty} f(t)\,dt - \frac{1}{\log X}\sum_{p} \frac{\log p}{\sqrt{p}}\hat{f}\bigg(\frac{\log p}{\log X}\bigg)(\chi(p)+\overline{\chi}(p)) \\ - \frac{1}{\log X}\sum_{p} \frac{\log p}{p}\hat{f}\bigg(\frac{2\log p}{\log X}\bigg)(\chi(p^2)+\overline{\chi}(p^2)) + O\bigg(\frac{1}{\log X}\bigg). \tag{2.1}$$

Proof. We combine [RS96, (2.16)], the fact that $f(x)$ is rapidly decreasing and the Stirling formula, which gives that $\Gamma'/\Gamma(s) = \log s + O(1/|s|)$, uniformly for $|\arg s| \leqslant \pi - \delta$, $|s| \geqslant 1$ to replace the Γ'/Γ-terms in [RS96, (2.16)] by $O(1/\log X)$. Moreover, the terms $n = p^k$ for $k \geqslant 3$, p prime, in the sum on the right-hand side of [RS96, (2.16)] contribute

$$\ll \sum_{p^k, k\geqslant 3} \frac{\log p}{p^{k/2}} \ll 1.$$

The lemma follows from these observations. □

Similarly, we have [Gul05a, Lemma 4.1].

Lemma 2.3. *Let $f(x)$ be an even Schwartz function whose Fourier transform $\hat{f}(u)$ has compact support. Then, for any square-free $c \equiv 1 \pmod 9$ of $\mathbb{Z}[\omega]$, we have*

$$S(\chi_c, f) = \int_{-\infty}^{\infty} f(t)\,dt - \frac{1}{\log X}\sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{\sqrt{N(\mathfrak{p})}}\hat{f}\bigg(\frac{\log N(\mathfrak{p})}{\log X}\bigg)(\chi_c(\mathfrak{p})+\overline{\chi}_c(\mathfrak{p})) \\ - \frac{1}{\log X}\sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})}\hat{f}\bigg(\frac{2\log N(\mathfrak{p})}{\log X}\bigg)(\chi_c(\mathfrak{p}^2)+\overline{\chi}_c(\mathfrak{p}^2)) + O\bigg(\frac{1}{\log X}\bigg),$$

where $\mathfrak{p}$ runs over all non-zero integral ideals in $\mathbb{Q}(\omega)$.

For $\Re(s) > 1$, we can rewrite the Euler product (1.4) of $L(f\times\chi_{8d}, s)$ as

$$L(f\times\chi_{8d}, s) = \prod_{p}\bigg(1 - \frac{\alpha_f(p)\chi_{8d}(p)}{p^s}\bigg)^{-1}\bigg(1 - \frac{\alpha_f^{-1}(p)\chi_{8d}(p)}{p^s}\bigg)^{-1},$$

with $\alpha_f(p) + \alpha_f^{-1}(p) = a_f(p)$. Similar to Lemma 2.2, we have the following lemma.

Lemma 2.4. *Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ is compactly supported. Then, for $d \in D(X)$, we have*

$$D(d, f, \phi) = \int_{-\infty}^{\infty}\phi(t)\,dt + \frac{1}{2}\int_{-\infty}^{\infty}\hat{\phi}(u)\,du - S(f, d, X; \hat{\phi}) + O\bigg(\frac{\log\log 3X}{\log X}\bigg), \tag{2.2}$$

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with the implicit constant depending on ϕ and

$$S(f, d, X; \hat{\phi}) = \frac{1}{\log X} \sum_p \frac{a_f(p) \log p}{\sqrt{p}} \left(\frac{8d}{p}\right) \hat{\phi}\left(\frac{\log p}{2 \log X}\right). \tag{2.3}$$

Proof. We use [RS96, (2.16)] again here and identify $c_\pi(n)$ in [RS96, (2.16)] as $\Lambda(n)\chi_{8d}(n)a_f(n)$ with $a_f(p^k) = \alpha_f^k(p) + \alpha_f^{-k}(p)$ in our case. By Deligne's proof [Del74] of the Weil conjecture, we know that $|\alpha_f(p)| = 1$ so that the terms $n = p^k$ for $k \geqslant 3$, p prime, in the sum on the right-hand side of [RS96, (2.16)] contribute $O(1)$. Moreover, note that

$$\sum_{p|8d} \frac{\log p}{p} \ll \log\log 3X.$$

Thus the terms $n = p^2$ in the sum on the right-hand side of [RS96, (2.16)] contribute

$$\frac{1}{\log X} \sum_p \frac{a_f(p^2) \log p}{p} \hat{\phi}\left(\frac{\log p}{\log X}\right) + O\left(\frac{\log\log 3X}{\log X}\right).$$

Recall that

$$a_f(p^2) = a_f^2(p) - 2 \tag{2.4}$$

and it follows from [RS96, Proposition 2.3] that

$$\sum_{p \leqslant x} \frac{a_f^2(p) \log^2 p}{p} = \frac{\log^2 x}{2} + O(\log x). \tag{2.5}$$

Note also that we have Mertens' formula [Dav00, p. 57],

$$\sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1). \tag{2.6}$$

Combining (2.4)–(2.6), we deduce that

$$\sum_{p \leqslant x} \frac{a_f(p^2) \log^2 p}{p} = -\frac{\log^2 x}{2} + O(\log x).$$

From the above and partial summation, we get that

$$\begin{aligned}\frac{1}{\log X} \sum_p \frac{a_f(p^2) \log p}{p} \hat{\phi}\left(\frac{\log p}{\log X}\right) &= -\frac{1}{\log X} \int_1^\infty \hat{\phi}\left(\frac{\log t}{\log X}\right) \frac{dt}{t} + O\left(\frac{\log\log X}{\log X}\right) \\ &= -\frac{1}{2} \int_{-\infty}^\infty \hat{\phi}(t)\, dt + O\left(\frac{\log\log X}{\log X}\right).\end{aligned}$$

The assertion of the lemma follows from this easily. □

2.5 Poisson summation

We now fix f and X and henceforth write $\Phi(t)$ in place of $\Phi_X(t)$. We shall focus on finding the asymptotic expression of (with $n \geqslant 1$)

$$S(X, Y; \hat{\phi}, f, \Phi) := \sum_{\gcd(d,2)=1} \mu^2(d) \sum_{p \leqslant Y} \frac{a_f(p) \log p}{\sqrt{p}} \left(\frac{8d}{p}\right) \hat{\phi}\left(\frac{\log p}{2 \log X}\right) \Phi\left(\frac{d}{X}\right),$$

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where $\hat{\phi}(u)$ is smooth and has its support contained in the interval $(-1+\epsilon, 1-\epsilon)$ for any $\epsilon > 0$. To emphasize this condition, here and throughout, we shall set $Y = X^{2-2\epsilon}$ and write the condition $p \leqslant Y$ explicitly.

Let $Z > 1$ be a real parameter to be chosen later and write $\mu^2(d) = M_Z(d) + R_Z(d)$ where

$$M_Z(d) = \sum_{\substack{l^2 | d \\ l \leqslant Z}} \mu(l) \quad \text{and} \quad R_Z(d) = \sum_{\substack{l^2 | d \\ l > Z}} \mu(l).$$

Define

$$S_M(X, Y; \hat{\phi}, f, \Phi) = \sum_{\gcd(d,2)=1} M_Z(d) \sum_{p \leqslant Y} \frac{a_f(p) \log p}{\sqrt{p}} \left(\frac{8d}{p}\right) \hat{\phi}\left(\frac{\log p}{2 \log X}\right) \Phi\left(\frac{d}{X}\right)$$

and

$$S_R(X, Y; \hat{\phi}, f, \Phi) = \sum_{\gcd(d,2)=1} R_Z(d) \sum_{p \leqslant Y} \frac{a_f(p) \log p}{\sqrt{p}} \left(\frac{8d}{p}\right) \hat{\phi}\left(\frac{\log p}{2 \log X}\right) \Phi\left(\frac{d}{X}\right),$$

so that $S(X, Y; \hat{\phi}, f, \Phi) = S_M(X, Y; \hat{\phi}, f, \Phi) + S_R(X, Y; \hat{\phi}, f, \Phi)$.

Using standard techniques (see (3.3)), we can show that by choosing Z appropriately that $S_R(X, Y; \hat{\phi}, f, \Phi)$ is small. Hence the main term arises only from $S_M(X, Y; \hat{\phi}, f, \Phi)$. We write it as

$$S_M(X, Y; \hat{\phi}, f, \Phi) = \sum_{p \leqslant Y} \frac{a_f(p) \log p}{\sqrt{p}} \left(\frac{8}{p}\right) \hat{\phi}\left(\frac{\log p}{2 \log X}\right) \left(\sum_{\gcd(d,2)=1} M_Z(d) \left(\frac{d}{p}\right) \Phi\left(\frac{d}{X}\right)\right). \tag{2.7}$$

We now evaluate the inner sum above following a method of Soundararajan in [Sou00] by applying the Poisson summation formula to the sum over d. For all odd integers k and all integers m, we introduce the Gauss-type sums

$$\tau_m(k) := \sum_{a \pmod{k}} \left(\frac{a}{k}\right) e\left(\frac{am}{k}\right) =: \left(\frac{1+i}{2} + \left(\frac{-1}{k}\right)\frac{1-i}{2}\right) G_m(k).$$

We quote [Sou00, Lemma 2.3] which determines $G_m(k)$.

Lemma 2.6. *If $(k_1, k_2) = 1$ then $G_m(k_1 k_2) = G_m(k_1) G_m(k_2)$. Suppose that p^a is the largest power of p dividing m (put $a = \infty$ if $m = 0$). Then for $b \geqslant 1$ we have*

$$G_m(p^b) = \begin{cases} 0 & \text{if } b \leqslant a \text{ is odd,} \\ \phi(p^b) & \text{if } b \leqslant a \text{ is even,} \\ -p^a & \text{if } b = a+1 \text{ is even,} \\ \left(\dfrac{m/p^a}{p}\right) p^a \sqrt{p} & \text{if } b = a+1 \text{ is odd,} \\ 0 & \text{if } b \geqslant a+2. \end{cases}$$

For a Schwartz function F, we define

$$\tilde{F}(\xi) = \frac{1+i}{2}\hat{F}(\xi) + \frac{1-i}{2}\hat{F}(-\xi) = \int_{-\infty}^{\infty} (\cos(2\pi\xi x) + \sin(2\pi\xi x)) F(x) \, dx. \tag{2.8}$$

We quote [Sou00, Lemma 2.6] which determines the inner sum in (2.7).

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Lemma 2.7. *Let Φ be a non-negative, smooth function supported in $(1,2)$. For any odd integer k,*

$$\sum_{\gcd(d,2)=1} M_Z(d)\left(\frac{d}{k}\right)\Phi\left(\frac{d}{X}\right) = \frac{X}{2k}\left(\frac{2}{k}\right)\sum_{\substack{\alpha\leqslant Z\\ \gcd(\alpha,2k)=1}}\frac{\mu(\alpha)}{\alpha^2}\sum_m(-1)^m G_m(k)\tilde{\Phi}\left(\frac{mX}{2\alpha^2 k}\right),$$

where $\tilde{\Phi}$ is as defined in (2.8).

Note that, for any non-negative integer l,

$$\tilde{\Phi}^{(l)}(\xi)\ll 1, \quad |\xi|<1.$$

Also note, via integration by parts,

$$\tilde{\Phi}(\xi) = \frac{-1}{2\pi\xi}\left(\int_1^{1+1/U}+\int_{2-1/U}^{2}\right)\Phi'(x)(\sin(2\pi\xi x)-\cos(2\pi\xi x))\,dx \ll \frac{1}{|\xi|}.$$

Similarly, one can show that, for any $l\geqslant 0, j\geqslant 1$,

$$\tilde{\Phi}^{(l)}(\xi)\ll\frac{U^{j-1}}{|\xi|^j}.$$

2.8 Primitive cubic and quartic Dirichlet characters

The classification of all the primitive cubic characters of conductor q coprime to 3 is given in [BY10]. It is shown there that every such character is of the form $m\to(m/n)_3$ for some $n\in\mathbb{Z}[\omega]$, with $n\equiv 1 \pmod 3$, n square-free and not divisible by any rational primes, $N(n)=q$. Here the symbol $(\frac{\cdot}{n})_3$ is the cubic residue symbol in the ring $\mathbb{Z}[\omega]$. For a prime $\pi\in\mathbb{Z}[\omega]$ with $N(\pi)\neq 3$, the cubic character is defined for $a\in\mathbb{Z}[\omega]$, $\gcd(a,\pi)=1$ by $(a/\pi)_3\equiv a^{(N(\pi)-1)/3} \pmod{\pi}$, with $(a/\pi)_3\in\{1,\omega,\omega^2\}$. When $\pi|a$, set $(a/\pi)_3=0$. One then extends the cubic character to composite n with $\gcd(N(n),3)=1$ multiplicatively.

Similarly, one can give a classification of all the primitive complex quartic characters of conductor q coprime to 2. Every such character is of the form $m\to(m/n)_4$ for some $n\in\mathbb{Z}[i]$, with $n\equiv 1 \pmod{(1+i)^3}$, n square-free and not divisible by any rational primes and $N(n)=q$. Here the symbol $(\frac{\cdot}{n})_4$ is the quartic residue symbol in the ring $\mathbb{Z}[i]$. For a prime $\pi\in\mathbb{Z}[i]$ with $N(\pi)\neq 2$, the quartic character is defined for $a\in\mathbb{Z}[i]$, $\gcd(a,\pi)=1$ by $(a/\pi)_4\equiv a^{(N(\pi)-1)/4}$ $\pmod{\pi}$, with $(a/\pi)_4\in\{\pm1,\pm i\}$. When $\pi|a$, $(a/\pi)_4$ is defined to be zero. Then the quartic character can be extended to composite n with $\gcd(N(n),2)=1$ multiplicatively. Note that, in $\mathbb{Z}[i]$, every ideal coprime to 2 has a unique generator congruent to 1 modulo $(1+i)^3$.

3. Proof of Theorem 1.1

Note that, as $X\to\infty$,

$$\sum_{d\in D(X)}\Phi\left(\frac{d}{X}\right)\sim\#D(X).$$

Moreover, as $\hat{\phi}$ is supported in $(-1,1)$, we have

$$\int_{-\infty}^{\infty}\phi(t)\,dt+\frac{1}{2}\int_{-\infty}^{\infty}\hat{\phi}(u)\,du = \int_{-\infty}^{\infty}\phi(t)\left(1+\frac{\sin(2\pi t)}{2\pi t}\right)dt.$$

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Theorem 1.1 thus follows from (2.2), provided that we show, for any Schwartz function ϕ with $\hat{\phi}$ supported in $(-1, 1)$,

$$\lim_{X\to\infty} \frac{S(X, Y; \hat{\phi}, f, \Phi)}{X \log X} = 0. \tag{3.1}$$

As

$$S(X, Y; \hat{\phi}, f, \Phi) = S_M(X, Y; \hat{\phi}, f, \Phi) + S_R(X, Y; \hat{\phi}, f, \Phi),$$

the remainder of this section is therefore devoted to the evaluation of $S_R(X, Y; \hat{\phi}, f, \Phi)$ and $S_M(X, Y; \hat{\phi}, f, \Phi)$.

3.1 Estimation of $S_R(X, Y; \hat{\phi}, f, \Phi)$

In this section, we estimate $S_R(X, Y; \hat{\phi}, f, \Phi)$. We first seek a bound for

$$E(Y; \chi, \hat{\phi}, f) := \sum_{p\leqslant Y} \frac{a_f(p)\log p}{\sqrt{p}} \chi(p)\hat{\phi}\left(\frac{\log p}{2\log X}\right),$$

for any non-principal quadratic character χ with modulus q and $Y \leqslant X^{2-2\epsilon}$. For this we need the following result which follows from [IK04, Theorem 5.15].

Lemma 3.2. *Suppose that the GRH is true. For any Dirichlet character χ with modulus q, we have, for $x \geqslant 1$,*

$$\sum_{p\leqslant x} a_f(p)\chi(p)\log p \ll x^{1/2}\log^2(qx).$$

It follows from the above lemma and partial summation that

$$E(Y; \chi, \hat{\phi}, f) \ll \log^3(qX). \tag{3.2}$$

Now, on writing $d = l^2 m$, we obtain

$$\begin{aligned} S_R(X, Y; \hat{\phi}, f, \Phi) &= \sum_{\substack{l>Z\\ \gcd(l,2)=1}} \mu(l) \sum_{\gcd(m,2)=1} \Phi\left(\frac{l^2 m}{X}\right) E(Y; \chi_{8l^2m}, \hat{\phi}, f) \\ &\ll \sum_{l>Z} \sum_{X/l^2\leqslant m\leqslant 2X/l^2} \log^3(X) \ll \frac{X\log^3 X}{Z}. \end{aligned} \tag{3.3}$$

3.3 Estimation of $S_M(X, Y; \hat{\phi}, f, \Phi)$

Applying Lemma 2.7 to the inner sum of (2.7), we see that the sum in $S_M(X, Y; \hat{\phi}, f, \Phi)$ corresponding to the contribution of $m = 0$ is zero, as it follows directly from the definition that $G_0(k) = \varphi(k)$ if k is a square and $G_0(k) = 0$ otherwise.

Now, the sums in $S_M(X, Y; \hat{\phi}, f, \Phi)$ corresponding to the contribution of $m \neq 0$ can be written as $XR/2$, where

$$R = \sum_{\substack{p\leqslant Y\\ \gcd(2,p)=1}} \frac{a_f(p)\log p}{p}\hat{\phi}\left(\frac{\log p}{2\log X}\right) \sum_{\substack{\alpha\leqslant Z\\ \gcd(\alpha,2p)=1}} \frac{\mu(\alpha)}{\alpha^2} \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \left(\frac{m}{p}\right)(-1)^m \tilde{\Phi}\left(\frac{mX}{2\alpha^2 p}\right).$$

We recast the condition $\gcd(p, 2\alpha) = 1$ as $\chi_{4\alpha^2}(p)$ and use estimation (3.2) to deduce by partial summation (note that we will take Z to be smaller than some power of X so that $\log\alpha \ll \log X$)

that

$$\sum_{\substack{p\leqslant Y\\ \gcd(2\alpha,p)=1}} \frac{a_f(p)\log p}{p}\hat{\phi}\Big(\frac{\log p}{2\log X}\Big)\tilde{\Phi}\Big(\frac{mX}{2\alpha^2 p}\Big)\Big(\frac{m}{p}\Big) = \int_1^Y \frac{1}{\sqrt{V}}\tilde{\Phi}\Big(\frac{mX}{2\alpha^2 V}\Big)\, dE(V;\chi_{4\alpha^2 m},\hat{\phi},f)$$

$$\ll \log^3(X(|m|+2))\bigg(\frac{1}{\sqrt{Y}}\Big|\tilde{\Phi}\Big(\frac{mX}{2\alpha^2 Y}\Big)\Big| + \int_1^Y \frac{1}{V^{3/2}}\Big|\tilde{\Phi}\Big(\frac{mX}{2\alpha^2 V}\Big)\Big|\, dV + \int_1^Y \frac{X}{\alpha^2 V^{5/2}}\Big|m\tilde{\Phi}'\Big(\frac{mX}{2\alpha^2 V}\Big)\Big|\, dV\bigg).$$

Hence we have

$$R \ll \sum_{\alpha\leqslant Z}\frac{1}{\alpha^2}(R_1+R_2+R_3),$$

where

$$R_1 = \frac{1}{\sqrt{Y}}\sum_{m\neq 0}\log^3(X(|m|+2))\Big|\tilde{\Phi}\Big(\frac{mX}{2\alpha^2 Y}\Big)\Big|,$$

$$R_2 = \int_1^Y \frac{1}{V^{3/2}}\sum_{m\neq 0}\log^3(X(|m|+2))\Big|\tilde{\Phi}\Big(\frac{mX}{2\alpha^2 V}\Big)\Big|\, dV$$

and

$$R_3 = \int_1^Y \frac{X}{\alpha^2 V^{5/2}}\sum_{m\neq 0}\log^3(X(|m|+2))\Big|m\tilde{\Phi}'\Big(\frac{mX}{2\alpha^2 V}\Big)\Big|\, dV.$$

We now gather the estimates in [Mil08, Appendix C] for R_1, R_2 and R_3 (but be aware that the sum over p in [Mil08] does not exist in our situation here):

$$R_1+R_2 \ll \frac{U\alpha^2\sqrt{Y}\log^7 X}{X} \quad\text{and}\quad R_3 \ll \frac{U\alpha^2\sqrt{Y}\log^7 X}{X} + \frac{U^2\alpha^4 Y^{3/2}\log^7 X}{X^{2008}}. \tag{3.4}$$

Some of the estimates quoted above from [Mil08] have their origins in [Gao]. The estimates from [Mil08] would suffice for our purpose, but the improved estimates from [Gao] for R_1+R_2 are neater to use here. Combining these estimates in (3.4), we obtain

$$R \ll \frac{UZ\sqrt{Y}\log^7 X}{X}.$$

Thus we conclude that the contribution of $m\neq 0$ is bounded by

$$UZ\sqrt{Y}\log^7 X. \tag{3.5}$$

3.4 Conclusion

We now combine the bounds (3.3), (3.5) and take $Y = X^{2-2\epsilon}$, $Z=\log^3 X$ (recall that $U=\log X$) with any fixed $\epsilon>0$ to obtain

$$S(X,Y;\hat{\phi},f,\Phi) \ll \frac{X\log^3 X}{Z} + UZ\sqrt{Y}\log^7 X = o(X\log X),$$

which implies (3.1) and this completes the proof of Theorem 1.1.

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4. Proof of Theorem 1.2

Let $N_d(X)$ be the number of primitive cubic characters of conductor $q \leqslant X$ with $\gcd(q,d)=1$. It is shown in [DFK04] that $N_d(X) \sim c(d)X$ as $X \to \infty$ for some constant $c(d)$. It follows from this that $\#C(X) \sim cX$ for some constant c as $X \to \infty$. Combining this with (2.1), we see that, in order to establish Theorem 1.2, it suffices to show that, for any Schwartz function f with $\hat{f}$ supported in $(-3/7, 3/7)$,

$$\lim_{X\to\infty} \frac{1}{X\log X} \sum_p \frac{\log p}{\sqrt{p}} \hat{f}\left(\frac{\log p}{\log X}\right) \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi^3=\chi_0}} (\chi(p)+\overline{\chi}(p)) = 0 \tag{4.1}$$

and

$$\lim_{X\to\infty} \frac{1}{X\log X} \sum_p \frac{2\log p}{p} \hat{f}\left(\frac{2\log p}{\log X}\right) \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi^3=\chi_0}} (\chi(p^2)+\overline{\chi}(p^2)) = 0. \tag{4.2}$$

As both χ and $\bar{\chi}$ are primitive cubic characters, it is enough to consider the two limits for χ only. The term $p=3$ in each sum above is $O(X)$. Hence we may assume $p \neq 3$ in the sums above and we apply the Cauchy–Schwarz inequality to see that

$$\begin{aligned} &\sum_{p\neq 3} \frac{\log p}{\sqrt{p}} \hat{f}\left(\frac{\log p}{\log X}\right) \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi^3=\chi_0}} \chi(p) \\ &\quad \ll \left(\sum_{p\leqslant X^{1/5}} \frac{\log^2 p}{p}\right)^{1/2} \left(\sum_{3\neq p\leqslant X^{1/5}} \left| \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi^3=\chi_0}} \chi(p)\right|^2\right)^{1/2} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\sum_{p\neq 3} \frac{\log p}{p} \hat{f}\left(\frac{2\log p}{\log X}\right) \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi^3=\chi_0}} \chi(p^2) \\ &\quad \ll \left(\sum_{p\leqslant X^{1/5}} \frac{\log^2 p}{p^2}\right)^{1/2} \left(\sum_{3\neq p\leqslant X^{1/5}} \left| \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi^3=\chi_0}} \chi(p^2)\right|^2\right)^{1/2}. \end{aligned} \tag{4.4}$$

It is easy to see that

$$\sum_{p\leqslant X^{1/5}} \frac{\log^2 p}{p} \ll \log^2 X \quad \text{and} \quad \sum_{p\leqslant X^{1/5}} \frac{\log^2 p}{p^2} \ll 1.$$

Moreover, note that, for a primitive cubic character χ, $\chi(p^2)=\bar{\chi}(p)$ which implies that the values of the sums on the right-hand sides of (4.3) and (4.4) are the same. Hence, it remains to estimate

$$\sum_{3\neq p\leqslant Y} \left| \sum_{X\leqslant q\leqslant 2X} \sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi^3=\chi_0}} \chi(p)\right|^2. \tag{4.5}$$

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Here, Y is a parameter independent of X. From our discussion in § 2.8, we can recast (4.5) in terms of the cubic residue symbol as

$$\sum_{3\neq p\leqslant Y}\Bigg|\sideset{}{'}\sum_{\substack{N(n)\in\mathcal{I}(X)\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}\left(\frac{p}{n}\right)_3\Bigg|^2,$$

where the inner sum runs over square-free elements n of $\mathbb{Z}[\omega]$ that have no rational prime divisor and $\mathcal{I}(Z)$ henceforth denotes the dyadic interval

$$\mathcal{I}(Z) := [Z, 2Z]$$

for any real number Z. We now regard $(\frac{p}{\cdot})_3$ as a ray class group character ξ on $h_{(3p)}$ where we define $\xi((n)) = (p/n)_3$. Now we remove the condition that n has no rational prime divisor by using Möbius inversion (note that one can uniquely express any $n\in\mathbb{Z}[\omega]$ as $n = n_1n_2$, where $n_1\in\mathbb{N}$, and n_2 has no rational prime divisor), obtaining

$$\sideset{}{'}\sum_{\substack{N(n)\in\mathcal{I}(X)\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}\left(\frac{p}{n}\right)_3=\sum_{\substack{d\in\mathbb{Z},d^2\leqslant 2X\\ d\equiv 1 \ (\mathrm{mod}\ 3)\\ \gcd(d,p)=1}}\mu_{\mathbb{N}}(|d|)\sideset{}{''}\sum_{\substack{N(n)\in\mathcal{I}(X/d^2)\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}\left(\frac{p}{n}\right)_3,$$

where the double prime indicates that nd is square-free (viewed as an element of $\mathbb{Z}[\omega]$). Here $\mu_{\mathbb{N}}$ is the usual Möbius function defined on $\mathbb{N}$. Note as d and p are coprime rational integers, it follows from the corollary to [IR90, Proposition 9.3.4] that $(p/d)_3 = 1$.

Since d is automatically square-free (as an element of $\mathbb{Z}[\omega]$), nd being square-free simply means that n is square-free and $\gcd(n, d) = 1$. Now use Möbius inversion again (writing μ_ω for the Möbius function on $\mathbb{Z}[\omega]$) to detect the condition that n is square-free, getting

$$\sideset{}{'}\sum_{\substack{X\leqslant N(n)\leqslant 2X\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}\left(\frac{p}{n}\right)_3=\sum_{\substack{d\in\mathbb{Z},d^2\leqslant 2X\\ d\equiv 1 \ (\mathrm{mod}\ 3)\\ \gcd(d,p)=1}}\mu_{\mathbb{N}}(|d|)\sum_{\substack{N(l)^2\leqslant 2X/d^2\\ \gcd(l,d)=1\\ l\equiv 1 \ (\mathrm{mod}\ 3)}}\mu_\omega(l)\left(\frac{p}{l^2}\right)_3\sum_{\substack{N(n)\in\mathcal{I}(X/(N(l)d)^2)\\ \gcd(n,d)=1\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}\left(\frac{p}{n}\right)_3. \tag{4.6}$$

Here we changed variables via $n = l^2n'$ and fixed l up to a unit by the condition $l\equiv 1 \pmod 3$ (note that in $\mathbb{Z}[\omega]$, every ideal coprime to 3 has a unique generator congruent to 1 modulo 3).

We now apply a further Möbius inversion to remove the condition $\gcd(n, d) = 1$ to get

$$\sum_{\substack{N(n)\in\mathcal{I}(X/(N(l)d)^2)\\ \gcd(n,d)=1\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}\left(\frac{p}{n}\right)_3=\sum_{\substack{e\mid d\\ e\equiv 1 \ (\mathrm{mod}\ 3)}}\mu_\omega(e)\left(\frac{p}{e}\right)_3\sum_{\substack{N(n)\in\mathcal{I}(X/N(e)N^2(l)d^2)\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}\left(\frac{p}{n}\right)_3. \tag{4.7}$$

Now we need the following lemma, which establishes a Pólya–Vinogradov-type inequality for the cubic symbols.

Lemma 4.1. *Let $p\neq 3$ be a rational prime. Then we have*

$$\sum_{\substack{N(n)\leqslant X\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}\left(\frac{p}{n}\right)_3\ll X^{1/3}p^{2/3}\log^2 p, \tag{4.8}$$

where the sum runs over elements $n\in\mathbb{Z}[\omega]$.

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Proof. As we mentioned above, we regard $(\frac{p}{\cdot})_3$ as a ray class group character ξ on $h_{(3p)}$ so that we can recast the sum in (4.8) as

$$\sum_{\substack{N(n)\leqslant X\\ n\equiv 1 \pmod 3}} \left(\frac{p}{n}\right)_3 = \sum_{\substack{N(I)\leqslant X\\ \gcd(I,3)=1}} \xi(I), \tag{4.9}$$

where the sum above runs over non-zero integral ideals $I \in \mathbb{Z}[\omega]$. It is easy to see that ξ is induced by a primitive character of conductor (ap) for some $a|3$. Therefore, the condition $\gcd(I,3)=1$ imposed on the sum on the right-hand side of (4.9) implies that the said sum remains unchanged if ξ is replaced by ξ^* (say), the primitive character that induces ξ. We may therefore assume without loss of generality that ξ is primitive and we further use Möbius inversion to detect the condition $\gcd(I,3)=1$ in the second sum in (4.9) while noting that the only ideals dividing 3 are (1), $(1-\omega)$ and (3) to get

$$\sum_{\substack{N(n)\leqslant X\\ n\equiv 1 \pmod 3}} \left(\frac{p}{n}\right)_3 = \sum_{\substack{N(I)\leqslant X\\ \gcd(I,3)=1}} \xi(I) = \sum_{h=1,1-\omega,3} \mu_\omega(h)\xi(h) \sum_{N(I)\leqslant X/N(h)} \xi(I).$$

Now we quote a result of Landau [Lan18] (see also [Sun72, Theorem 2]), which states that, for an algebraic number field K of degree $n \geqslant 2$, for ξ any primitive ideal character of K with conductor $\mathfrak{f}$, and $k = |N(\mathfrak{f})\cdot d_K|$ with d_K being the discriminant of K, we have, for $X \geqslant 1$,

$$\sum_{N(I)\leqslant X} \xi(I) \leqslant k^{1/(n+1)} \log^n(k)\cdot X^{(n-1)/(n+1)},$$

where I runs over integral ideas of K.

We now identify $K=\mathbb{Q}(\omega)$ with $n=2$ and $k=3ap^2$, where $a=1$, 3 or 9, to see that the sum on the right-hand side of (4.9) is

$$O(X^{1/3}p^{2/3}\log^2 p).$$

This now completes the proof of the lemma. □

Applying Lemma 4.1, we can majorize the left-hand side expression in (4.7) as

$$\sum_{\substack{N(n)\in\mathcal{I}(X/(N(l)d)^2)\\ \gcd(n,d)=1\\ n\equiv 1 \pmod 3}} \left(\frac{p}{n}\right)_3 \ll \frac{X^{1/3}p^{2/3}\log^2 p}{(N(l)d)^{2/3}} \sum_{e|d} \frac{1}{\sqrt{N(e)}} \ll \frac{X^{1/3}p^{2/3}\log^2 p}{(N(l))^{2/3}d^{2/3-2\epsilon}}, \tag{4.10}$$

for any $\epsilon > 0$. The last bound follows since we have $\#\{e\in\mathbb{Z}[\omega] : e|d\} \ll N(d)^\epsilon$. From (4.10) and (4.6), we have

$$\sideset{}{'}\sum_{\substack{N(n)\in\mathcal{I}(X)\\ n\equiv 1 \pmod 3}} \left(\frac{p}{n}\right)_3 \ll X^{1/3}p^{2/3}\log^2 p \sum_{\substack{d\in\mathbb{N}\\ d^2\leqslant 2X}} \frac{1}{d^{2/3-2\epsilon}} \sum_{\substack{N(l)^2\leqslant 2X/d^2\\ l\equiv 1 \pmod 3}} \frac{1}{(N(l))^{2/3}}.$$

Note that it follows from [Sun72, Theorem 2] that

$$\sum_{\substack{N(l)\leqslant X\\ l\equiv 1 \pmod 3}} 1 \ll X.$$

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We then deduce by partial summation that

$$\sum_{\substack{N(l)^2 \leqslant 2X/d^2 \\ l \equiv 1 \pmod 3}} \frac{1}{(N(l))^{2/3}} \ll \left(\frac{X}{d^2}\right)^{1/6}.$$

Therefore

$$\sideset{}{'}\sum_{\substack{N(n) \in \mathcal{I}(X) \\ n \equiv 1 \pmod 3}} \left(\frac{p}{n}\right)_3 \ll X^{1/2} p^{2/3} \log^2 p \sum_{\substack{d \in \mathbb{N} \\ d^2 \leqslant 2X}} \frac{1}{d^{1-2\epsilon}} \ll X^{1/2+3\epsilon} p^{2/3} \log^2 p.$$

Hence, we obtain

$$\sum_{3 \neq p \leqslant Y} \left| \sideset{}{'}\sum_{\substack{N(n) \in \mathcal{I}(X) \\ n \equiv 1 \pmod 3}} \left(\frac{p}{n}\right)_3 \right|^2 \ll X^{1+6\epsilon} \sum_{p \leqslant Y} p^{4/3} \log^4 p \ll X^{1+6\epsilon} Y^{7/3} \log^3 Y.$$

Applying the above bound in the estimations (4.3) and (4.4), we find that (4.1) and (4.2) hold so long as $Y^{7/6} \leqslant X^{1/2-7\epsilon/2}$ and, as ϵ is arbitrary, the proof of Theorem 1.2 is completed.

5. Proof of Theorems 1.3 and 1.4

The proofs of both Theorems 1.3 and 1.4 are similar to that of Theorem 1.2 so we shall skip most of the details. For the proof of Theorem 1.3, one can show, similar to the proof of [Gul05a, Lemma 4.2], that

$$\#C_{(9)}(X) \sim c' X \quad \text{as } X \to \infty$$

for some constant c'. Recall that $C_{(9)}$ denotes the set of cubic symbols $\chi_c = (\frac{\cdot}{c})_3$ with c square-free, congruent to 1 modulo 9 and $X \leqslant N(c) \leqslant 2X$. We then proceed as in the proof of Theorem 1.2 to see that it suffices to show, for any fixed $\epsilon > 0$, we have

$$\sum_{\substack{\mathfrak{p} \\ 3 \neq N(\mathfrak{p}) \leqslant Y}} \left| \sideset{}{^*}\sum_{\substack{N(c) \in \mathcal{I}(X) \\ c \equiv 1 \pmod 9}} \left(\frac{\mathfrak{p}}{c}\right)_3 \right|^2 \ll X^{1+\epsilon} Y^{5/3} \log^3 Y, \tag{5.1}$$

where Y is a parameter independent of X. We now regard $(\frac{\mathfrak{p}}{\cdot})_3$ as a ray class group character ξ on $h_{(3)\mathfrak{p}}$ where we set $\xi((c)) = (\mathfrak{p}/c)_3$. Using the ray class characters on $h_{(3)\mathfrak{p}}$ to detect the condition $c \equiv 1 \pmod 9$ in the inner sum on the left-hand side of (5.1), we get

$$\sideset{}{^*}\sum_{\substack{N(c) \in \mathcal{I}(X) \\ c \equiv 1 \pmod 9}} \left(\frac{\mathfrak{p}}{c}\right)_3 = \frac{1}{\#h_{(9)}} \sum_{\psi \pmod 9} \sideset{}{^*}\sum_{\substack{N(c) \in \mathcal{I}(X) \\ c \equiv 1 \pmod 3}} \psi((c))\xi((c)).$$

The estimation in (5.1) follows, after using Möbius inversion to detect the condition that c is square-free, from the following estimation:

$$\sum_{\substack{N(c) \leqslant X \\ c \equiv l \pmod 9}} \psi((c))\xi((c)) \ll X^{1/3} N(\mathfrak{p})^{1/3} \log^2 N(\mathfrak{p})$$

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where $l \equiv 1 \pmod 3 \in \mathbb{Z}[\omega]$. One more application of the ray class characters on $h_{(9)}$ shows that the above estimation follows from

$$\sum_{\substack{N(I)\leqslant X \\ \gcd(I,3)=1}} \psi(I)\psi'(I)\xi(I) \ll X^{1/3}N(\mathfrak{p})^{1/3}\log^2 N(\mathfrak{p}), \tag{5.2}$$

where the sum above runs over non-zero integral ideals $I \in \mathbb{Z}[\omega]$. The character $\psi\psi'\xi$ can be viewed as a ray class group character on $h_{(9)\mathfrak{p}}$ and our definition of ξ implies that it is induced from a primitive character on $h_{(a)\mathfrak{p}}$ with $a|9$. Consequently, as the condition $\gcd(I,3)=1$ is imposed on the summation in (5.2), the value of this sum remains unaltered if $\psi\psi'\xi$ is replaced by the primitive character that induces it. We may therefore without loss of generality assume that $\psi\psi'\xi$ is primitive and an application of the Möbius inversion function as in the proof of Lemma 4.1 allows us to obtain the desired bound in (5.2).

For the proof of Theorem 1.4, one can show, following the approach in [DFK04], that

$$\#Q(X) \sim dX \quad \text{as } X \to \infty$$

for some constant d as $X \to \infty$. The rest of the proof goes in a similar fashion to that of Theorem 1.2.

6. Notes

We remark here that it is conceivable that results along the lines of Theorems 1.2–1.4 can be proved using a simpler approach involving mean-value estimates for sums of characters of a fixed order. This method was used in [Mil08]. The afore-mentioned mean-value estimate for quadratic character sums is due to Jutila [Jut81, Lemma 5], but the analogous results for characters of orders higher than two which would be needed here do not seem to be available. Moreover, if good mean-value estimates can be obtained for cubic and quartic characters, one would expect that the support of $\hat{f}$ in Theorems 1.2–1.4 can be significantly widened.

It would also be interesting to consider the one level density of low-lying zeros of families of Dirichlet L-functions for characters of orders larger than 4. However, the relation between higher order residue symbols and nth order primitive Dirichlet characters would be more difficult to describe.

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