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Eigenvalues of partitioned hermitian matrices

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Let $C = \left(A_{ij}\right)_{1\le i,j\le t}$ be a hermitian matrix in partitioned form; here A_{ij} is an $n_i \times n_j$ block. The purpose of this paper is to obtain inequalities linking the eigenvalues of C to those of the main diagonal blocks $A_{11}, \ldots, A_{tt}$ of C. These inequalities include, as special cases, inequalities due to N. Aronszajn and A. Hoffman.

1. Introduction

Let

$$C = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

be an n-square hermitian matrix, with eigenvalues $\gamma_1 \ge \ldots \ge \gamma_n$. Let A be a-square with eigenvalues $\alpha_1 \ge \ldots \ge \alpha_a$, and let B be b-square with eigenvalues $\beta_1 \ge \ldots \ge \beta_b$. The inequality (for positive semidefinite C)

(1) $$\gamma_{i+j-1} \le \alpha_i + \beta_j , \quad 1 \le i \le a , \quad 1 \le j \le b$$

is due to N. Aronszajn [3]. This remarkable inequality is not as widely known as it should be. By applying (1) to

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$$C - \gamma_n I_n = \begin{bmatrix} A-\gamma_n I_a & X \\ & \\ X^* & B-\gamma_n I_b \end{bmatrix}$$

the following extension of (1) to arbitrary hermitian C is obtained:

(2) $$\gamma_{i+j-1} + \gamma_n \le \alpha_i + \beta_j \ , \quad 1 \le i \le a \ , \quad 1 \le j \le b \ .$$

(The inequality (2) may be found in [4].)

It is the purpose of this paper to secure a substantial generalization of (2).

Our investigation of Aronszajn's inequality was spurred by a generalization of (2) proved by A. Hoffman. We are greatly indebted to Dr Hoffman for showing us his result, a special case of our Theorem 2. Our techniques combine an improvement of Aronszajn's method with an extremal result of Amir-Moéz [1] and an extremal result of Fan [2]. It seems likely that sharpened versions of our inequalities will be obtained if proofs of them free of extremal considerations can be found. It seems likely also that many more results await discovery in which functions of the eigenvalues of A and B are linked to like functions of the eigenvalues of C. (One known result of this type is the Fischer inequality.)

2. Notation

The following notation will remain in force throughout §3. Let C be an n-square hermitian matrix, presented in partitioned form as

(3) $$C = (A_{ij})_{1 \le i,j \le t} \ ;$$

here A_{ij} is an $n_i \times n_j$ block, for $1 \le i, j \le t$. Let

(4) $$\gamma_1 \ge \dots \ge \gamma_n$$

be the eigenvalues of C, and let

(5) $$\alpha_{p1} \ge \alpha_{p2} \ge \dots \ge \alpha_{pn_p}$$

be the eigenvalues of the main diagonal block A_{pp} ; $p = 1, 2, \dots, t$.

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Following Amir-Moéz, if we are given a sequence of positive integers

$$(6) \qquad i_1 \le i_2 \le \dots \le i_k$$

with $1 \le i_1$, $i_k \le m$ and

$$(7) \qquad i_s \ge s \quad \text{for} \quad s = 1, \dots, k ,$$

we let

$$(1 \le)\ i'_1 < \dots < i'_k \quad (\le m)$$

be the largest strictly increasing sequence of positive integers bounded above termwise by $i_1, \dots, i_k$; this is defined recursively by $i'_k = i_k$, and $i'_s = \min(i_s, i'_{s+1}-1)$, for $s = k-1, \dots, 1$. This sequence exists because of (7). If, instead of (7), we have

$$(8) \qquad i_s \le m - k + s \quad \text{for} \quad s = 1, \dots, k ,$$

we let

$$(1 \le)\ i''_1 < \dots < i''_k \quad (\le m)$$

be the smallest strictly increasing sequence of positive integers bounded below termwise by $i_1, \dots, i_k$; it is defined recursively by $i''_1 = i_1$ and $i''_s = \max(i_s, i''_{s-1}+1)$ for $s = 2, 3, \dots, k$. This sequence exists because of (8).

All vectors in this paper are column vectors, and x^* denotes the conjugate transpose of x .

3. The main result

THEOREM 1. *Let* C *be as described in* §2. *Let integers* $i_{p1}, \dots, i_{pk}$ *be given,* $p = 1, \dots, t$ *, such that*

$$(9) \qquad 1 \le i_{p1} \le \dots \le i_{pk} \le n_p ,$$
$$i_{ps} \ge s , \quad s = 1, \dots, k , \quad p = 1, \dots, t .$$

Then

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$$(10)\quad \sum_{s=1}^{k} \gamma_{(i_{1s}+i_{2s}+\cdots+i_{ts})'} + \sum_{s=1}^{(t-1)k} \gamma_s \geq \sum_{s=1}^{t} \left(\alpha_{si'_{s1}} + \cdots + \alpha_{si'_{sk}}\right) .$$

Proof. Our proof relies on the following lemma, due to Amir-Moéz [1]:

If H *is an* m*-square hermitian matrix with eigenvalues* $h_1 \geq \cdots \geq h_m$ *, and if integers* $i_1, \ldots, i_k$ $(1 \leq i_1,\ i_k \leq m)$ *are given satisfying* (6) *and* (7)*, then*

$$\max_{\substack{M_{i_1} \subseteq \cdots \subseteq M_{i_k} \\ \dim M_{i_p} = i_p}} \ \min_{\substack{x_p \in M_{i_p} \\ x_p \text{ orthonormal}}} \left(x_1^* H x_1 + \cdots + x_k^* H x_k\right) = h_{i'_1} + \cdots + h_{i'_k} .$$

Here, for a fixed nested chain $M_{i_1} \subseteq \cdots \subseteq M_{i_k}$ *of subspaces of column* m*-space (each subspace having dimension equal to its subscript), the* min *is taken over all sets of orthonormal vectors* $x_1, \ldots, x_k$ *with* $x_p \in M_{i_p}$ *for* $p = 1, \ldots, k$ *; and the* max *is then taken over all such nested chains of subspaces.*

Since $i_{1s} + i_{2s} + \cdots + i_{ts} \geq ts \geq s$, there exists a strictly increasing sequence bounded above by the sequence $i_{1p} + \cdots + i_{tp}$, $p = 1, \ldots, k$.

By the lemma, subspaces

$$(11)\qquad M_{pi_{p1}} \subseteq \cdots \subseteq M_{pi_{pk}}$$

in column n_p-space (each space having dimension equal to its second subscript) exist such that

$$\min_{\substack{x_q \in M_{pi_q} \\ x_q \text{ orthonormal},\ 1 \leq q \leq k \\ \dim M_{pi_{pq}} = i_{pq}}} \left(x_1^* A_{pp} x_1 + \cdots + x_k^* A_{pp} x_k\right) = \alpha_{pi'_{p1}} + \cdots + \alpha_{pi'_{pk}} .$$

In other words: given any orthonormal column n_p-tuples x_q with

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$x_q \in M_{pi_{pq}}$, for $q = 1, \ldots, k$, we have

(12) $$x_1^* A_{pp} x_1 + \ldots + x_k^* A_{pp} x_k \geq \alpha_{pi'_{p1}} + \ldots + \alpha_{pi'_{pk}} .$$

We now construct subspaces

(13) $$\tilde{M}_{pi_{p1}} \subseteq \ldots \subseteq \tilde{M}_{pi_{pk}}$$

of column n-space (each space having dimension equal to its second subscript) by placing $n_1 + \ldots + n_{p-1}$ zeros at the top and $n_{p+1} + \ldots + n_t$ zeros at the bottom of the vectors in the spaces (11). Let

(14) $$S_r = \left\langle \tilde{M}_{1i_{1r}}, \tilde{M}_{2i_{2r}}, \ldots, \tilde{M}_{ti_{tr}} \right\rangle$$

be the subspace of column n-space generated by the subspaces enclosed in the brackets $\langle \ \rangle$. Notice that for fixed r , $1 \leq r \leq k$, the subspaces

(15) $$\tilde{M}_{1i_{1r}}, \tilde{M}_{2i_{2r}}, \ldots, \tilde{M}_{ti_{tr}}$$

of column n-space are pairwise orthogonal. This is because a vector in

$$\tilde{M}_{pi_{pr}}$$

has its nonzero components confined to those positions j with $n_1 + \ldots + n_{p-1} < j \leq n_1 + \ldots + n_p$; and the vectors from the other subspaces (15) have only zeros in these positions. Consequently,

(16) $$\dim S_r = i_{1r} + \ldots + i_{tr} , \quad r = 1, \ldots, k .$$

Owing to (13) we also have

(17) $$S_1 \subseteq S_2 \subseteq \ldots \subseteq S_k .$$

By the lemma, applied to C and using the spaces (17), we see that

$$\min_{\substack{x_q \in S_q \\ x_q \text{ orthonormal, } 1 \leq q \leq k}} \left(x_1^* C x_1 + \ldots + x_k^* C x_k\right) \leq \sum_{r=1}^{k} \gamma_{(i_{1r}+\ldots+i_{tr})'} .$$

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Thus there exists a set of orthonormal column n-tuples $y_1, \ldots, y_k$ such that:

$$y_1^* C y_1 + \ldots + y_k^* C y_k \leq \sum_{r=1}^{k} \gamma_{(i_{1r}+\ldots+i_{tr})'} \,, \tag{18}$$

with

$$y_r \in S_r \,, \quad r = 1, \ldots, k \,. \tag{19}$$

Because of (19) and (14), y_r may be expressed as

$$y_r = \sum_{s=1}^{t} w_{sr} \,, \quad r = 1, \ldots, k \,,$$

where

$$w_{sr} \in \tilde{M}_{si_{sr}} \,, \quad s = 1, \ldots, t \,, \quad r = 1, \ldots, k \,.$$

Since $\dim \tilde{M}_{si_{s1}} = i_{s1} \geq 1$, we may find a unit vector ξ_{s1} in $\tilde{M}_{si_{s1}}$ such that

$$w_{s1} \in \langle \xi_{s1} \rangle \,.$$

Since $\dim \tilde{M}_{si_{s2}} = i_{s2} \geq 2$, we may enlarge $\langle \xi_{s1} \rangle$ to a two dimensional subspace

$$\langle \xi_{s1}, \xi_{s2} \rangle \text{ in } \tilde{M}_{si_{s2}}$$

containing both w_{s1} and w_{s2}. Here ξ_{s1}, ξ_{s2} are to be orthonormal. Proceeding by induction, we pick a ν-dimensional subspace $\langle \xi_{s1}, \ldots, \xi_{s\nu} \rangle$ of $\tilde{M}_{si_{s\nu}}$ containing $w_{s\nu}$, $\xi_{s1}, \ldots, \xi_{s,\nu-1}$, where $\xi_{s1}, \ldots, \xi_{s,\nu-1}$, $\xi_{s\nu}$ are orthonormal. This is possible because $\dim \tilde{M}_{si_{s\nu}} = i_{s\nu} \geq \nu$. Do this for $\nu = 1, \ldots, k$. We thus obtain a set of vectors

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$$\xi_{11},\ \ldots,\ \xi_{1k}\ ;\ \ \xi_{21},\ \ldots,\ \xi_{2k}\ ;\ \ \ldots\ ;\ \ \xi_{t1},\ \ldots,\ \xi_{tk}\ . \tag{20}$$

Because of our construction and because

$$\tilde{M}_{pi_{pr}} \perp \tilde{M}_{qi_{qs}} \quad \text{if} \quad p \neq q\ ,$$

the vectors in the list (20) are orthonormal.

Thus y_r may be expressed in terms of the vectors (20), say as

$$\begin{aligned} y_r &= \sum_{s=1}^{t} w_{sr} \\ &= \sum_{s=1}^{t} \left(\sigma_{r,(s-1)k+1}\xi_{s1} + \ldots + \sigma_{r,sk}\xi_{sk}\right) \\ &= \sum_{s=1}^{t} \sum_{j=1}^{k} \sigma_{r,(s-1)k+j}\xi_{sj}\ , \quad r = 1,\ \ldots,\ k\ . \end{aligned} \tag{21}$$

Here $\sigma_{r,1},\ \ldots,\ \sigma_{r,tk}$ are scalars; $r = 1,\ \ldots,\ k$. (Because of the way the ξ's were chosen, many of the σ's are zero; this fact will not be needed.) Because $y_1,\ \ldots,\ y_k$ are orthonormal, and because the vectors (20) are orthonormal, the column tk-tuples

$$\left(\sigma_{r,1},\ \ldots,\ \sigma_{r,tk}\right)^T\ ,\quad r = 1,\ \ldots,\ k\ , \tag{22}$$

are orthonormal also.

Combining (18) and (21), we obtain

$$\begin{aligned} &\sum_{r=1}^{k} \gamma\left(i_{1r}+\ldots+i_{tr}\right)' \\ &\quad \geq \sum_{r=1}^{k} y_r^* C y_r \\ &\quad = \sum_{r=1}^{k} \left\{\left(\sum_{s_1=1}^{t} \sum_{j_1=1}^{k} \bar{\sigma}_{r,(s_1-1)k+j_1}\xi_{s_1j_1}^*\right) C \left(\sum_{s_2=1}^{t} \sum_{j_2=1}^{k} \xi_{s_2j_2}\sigma_{r,(s_2-1)k+j_2}\right)\right\} \\ &\quad = \sum_{r=1}^{k} \left(\bar{\sigma}_{r,1},\ \ldots,\ \bar{\sigma}_{r,tk}\right) D \left(\sigma_{r,1},\ \ldots,\ \sigma_{r,tk}\right)^T\ ; \end{aligned} \tag{23}$$

here

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$$D = \left[\xi^*_{s_1 j_1} C \xi_{s_2 j_2}\right]_{\substack{1 \le s_1, s_2 \le t,\\ 1 \le j_1, j_2 \le k,}}$$

is a tk-square hermitian matrix in which the rows (and columns) are prescribed by postulating that the pairs (s, j), $1 \le s \le t$, $1 \le j \le k$ be ordered lexicographically, and then letting

$$\xi^*_{s_1 j_1} C \xi_{s_2 j_2}$$

be the entry of D in position $\big((s_1, j_1), (s_2, j_2)\big)$.

Because the vectors (22) are orthonormal tk-tuples, a result of Fan [2] asserts that the sum (23) is bounded below by the sum of the k lowest eigenvalues D. However, the sum of the k lowest eigenvalues of D equals trace $D - \big($the sum of the $k(t-1)$ highest eigenvalues of $D\big)$. Let the eigenvalues of D be $\delta_1 \ge \ldots \ge \delta_{tk}$. Thus we have

$$(24) \qquad \sum_{r=1}^{k} \gamma_{(i_{1r}+\ldots+i_{tr})'} \ge \sum_{s=1}^{t} \sum_{j=1}^{k} \xi^*_{sj} C \xi_{sj} - \big(\delta_1 + \ldots + \delta_{k(t-1)}\big) .$$

Now let

$$U = \begin{bmatrix} \xi_{11}, & \ldots, & \xi_{1k}, & \ldots, & \xi_{t1}, & \ldots, & \xi_{tk}, & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & & & & & & & \\ & & & & & & & \end{bmatrix}$$

be an n-square unitary matrix with $\xi_{11}, \ldots, \xi_{tk}$ as the first tk columns. Such a unitary U exists because the vectors (20) are orthonormal. Then D is the principal tk-square submatrix standing in the upper left corner of U^*CU. Since the eigenvalues of U^*CU are $\gamma_1, \ldots, \gamma_n$, the Cauchy inequalities tell us that

$$\gamma_1 \ge \delta_1, \gamma_2 \ge \delta_2, \ldots, \gamma_{(t-1)k} \ge \delta_{(t-1)k} .$$

Thus

$$(25) \qquad \delta_1 + \ldots + \delta_{(t-1)k} \le \gamma_1 + \ldots + \gamma_{(t-1)k} .$$

Inserting (25) into (24), we obtain

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$$(26) \qquad \sum_{r=1}^{k} \gamma_{(i_{1r}+\dots+i_{tr})'} + \sum_{r=1}^{(t-1)k} \gamma_r \ge \sum_{s=1}^{t} \left(\sum_{j=1}^{k} \xi_{sj}^* C \xi_{sj} \right) .$$

But $\xi_{sj} \in \tilde{M}_{s,i_{sj}}$ and hence has $n_1 + \dots + n_{s-1}$ zeros at the top and $n_{s+1} + \dots + n_t$ zeros at the bottom. In fact,

$$(27) \qquad \xi_{sj} = \begin{bmatrix} 0 \\ \hat{\xi}_{sj} \\ 0 \end{bmatrix} \text{ where } \hat{\xi}_{sj} \in M_{s,i_{sj}} .$$

Because the ξ_{sj} , $j = 1, \dots, k$, are orthonormal, the $\hat{\xi}_{sj}$, $j = 1, \dots, k$, are also orthonormal. From (27) we see that

$$(28) \qquad \xi_{sj}^* C \xi_{sj} = \hat{\xi}_{sj}^* A_{ss} \hat{\xi}_{sj} .$$

Because of the property (12) possessed by spaces (11), we have

$$(29) \qquad \sum_{j=1}^{k} \hat{\xi}_{sj}^* A_{ss} \hat{\xi}_{sj} \ge \sum_{j=1}^{k} \alpha_{si'_{sj}} .$$

Putting (28) and (29) into (26), we obtain

$$\sum_{r=1}^{k} \gamma_{(i_{1r}+\dots+i_{tr})'} + \sum_{r=1}^{(t-1)k} \gamma_r \ge \sum_{s=1}^{t} \sum_{j=1}^{k} \alpha_{si'_{sj}} ,$$

which is just what we had to prove.

We next derive another version of the inequality of Theorem 1.

THEOREM 2. *Let* C *be as described in* §2. *Let integers* $j_{p1}, \dots, j_{pk}$ *be given,* $1 \le p \le t$ *, such that*

$$(30) \qquad \begin{gathered} 1 \le j_{p1} \le \dots \le j_{pk} \le n_p , \\ j_{ps} \le n_p - k + s , \quad s = 1, \dots, k , \quad p = 1, \dots, t . \end{gathered}$$

Then

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$$(31)\quad \sum_{p=1}^{k} \gamma_{(j_{1p}+\cdots+j_{tp}-(t-1))''} + \sum_{p=1}^{(t-1)k} \gamma_{n+1-p} \le \sum_{p=1}^{k} \left(\alpha_{pj''_{p1}} + \cdots + \alpha_{pj''_{pk}}\right) .$$

Proof. Define integers i_{ps} by

$$i_{ps} = n_p + 1 - j_{p,k+1-s} \;, \quad s = 1, \ldots, k \;, \quad p = 1, \ldots, t \;.$$

An easy calculation shows that $i_{ps} \ge s$. We claim that

$$(32)\qquad n_p + 1 - i'_{ps} = j''_{p,k+1-s}$$

and that

$$(33)\quad n + 1 - (i_{1s} + \cdots + i_{ts})' = (j_{1,k+1-s} + \cdots + j_{t,k+1-s} - (t-1))'' .$$

The verifications of (32) and (33) are a descending induction on s beginning with $s = k$, and are left to the reader. Then we obtain (31) by applying Theorem 1 to

$$-C = (-A_{ij}) .$$

As a special case of Theorem 1, let $r_1, \ldots, r_t$ be nonnegative integers with $r_p + k \le n_p$ for $p = 1, \ldots, t$. By setting $i_{p1} = \cdots = i_{pk} = r_p + k$, we obtain the following inequality involving sums of consecutive eigenvalues.

COROLLARY 1.

$$(34)\quad \sum_{s=1}^{k} \gamma_{r_1+\cdots+r_t+(t-1)k+s} + \sum_{s=1}^{(t-1)k} \gamma_s \ge \sum_{s=1}^{t} \left(\alpha_{s,r_s+1} + \alpha_{s,r_s+2} + \cdots + \alpha_{s,r_s+k}\right) .$$

As a special case of Theorem 2, let $\rho_1, \ldots, \rho_t$ be nonnegative integers with $\rho_s \le n_s - k$ for $s = 1, \ldots, t$. By setting $j_{s1} = \cdots = j_{sk} = \rho_s + 1$ for $s = 1, \ldots, t$, we obtain

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COROLLARY 2.

$$(35)\quad \sum_{s=1}^{k} \gamma_{\rho_1+\ldots+\rho_t+s} + \sum_{s=1}^{(t-1)k} \gamma_{n+1-s} \le \sum_{s=1}^{t} \left(\alpha_{s,\rho_s+1} + \ldots + \alpha_{s,\rho_s+k}\right) .$$

Hoffman's theorem is the special case $k = 1$ of (35).

The inequalities (34) and (35) have the special feature that two γ's with the same subscript do not appear on the left-hand side. This is not always so in (10) and (31).

4. Applications to singular values

An application of the inequalities (10) to singular values follows quite naturally with the aid of the following well-known fact: if H is an $m \times m$ matrix (not necessarily hermitian) with singular values $h_1 \ge \ldots \ge h_m$, then the eigenvalues of the $2m$-square hermitian matrix

$$\begin{bmatrix} 0 & H \\ H^* & 0 \end{bmatrix}$$

are $h_1 \ge \ldots \ge h_m \ge -h_m \ge \ldots \ge -h_1$.

Now let C, given by (3), be an n-square, not necessarily hermitian, matrix presented in partitioned form, where A_{ij} is an $n_i \times n_j$ block; $1 \le i, j \le t$. Let now (4) be the singular values of C, and let (5) be the singular values of the main diagonal block A_{pp}; $p = 1, \ldots, t$. We are going to prove (10). Thus we shall link the singular values of C to the singular values of the main diagonal blocks of C. To do this we consider the $2n$-square hermitian matrix

$$M = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}$$

whose eigenvalues are $\gamma_1 \ge \ldots \ge \gamma_n \ge -\gamma_n \ge \ldots \ge -\gamma_1$. We make a unitary similarity of M by a permutation matrix so as to produce

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$$(36)\qquad \begin{bmatrix} 0 & A_{pp} \\ & \\ A^*_{pp} & 0 \end{bmatrix}, \quad p = 1, \ldots, t ,$$

as main diagonal blocks. We do this by regarding M as a matrix of $2t$-block rows and $2t$-block columns, and then shuffling these block rows and block columns by taking them in the order

$$1,\ t+1,\ 2,\ t+2,\ 3,\ t+3,\ \ldots,\ t,\ 2t\ .$$

This shuffling has the effect of moving the blocks

$$\begin{matrix} 0 & A_{ii} \\ A^*_{ii} & 0 \end{matrix}$$

in block positions (i, i), $(i, t+i)$, $(t+i, i)$, $(t+i, t+i)$ respectively, into block positions $(2i-1, 2i-1)$, $(2i-1, 2i)$, $(2i, 2i-1)$, $(2i, 2i)$ respectively.

Let M_1 be the matrix obtained from M this way; it is hermitian and has the same eigenvalues as M because it is unitarily similar to M; and its main diagonal blocks are given by (36). We are now ready for the main result concerning singular values.

THEOREM 3. *Let C, given by* (3), *be an n-square partitioned matrix, not necessarily hermitian. Let* (4) *be the singular values of C, and let* (5) *be the singular values of A_{pp}, $p = 1, \ldots, t$. Let integers $i_{p1}, \ldots, i_{pk}$ satisfy* (9). *Then formula* (10) *is valid.*

Proof. We apply Theorem 1 to the $2n$-square hermitian matrix M_1 in which the p-th main diagonal block is the $2n_p$-square matrix (36). Because $i'_{pk} \le i_{pk} \le n_p$, $(t-1)k \le n$, and $\left(i_{1k} + i_{2k} + \ldots + i_{tk}\right)' \le n_1 + n_2 + \ldots + n_t = n$, none of the negative eigenvalues of M_1 or of the matrices (36) are involved in this application of Theorem 1, and hence (10) is established.

COROLLARY 3. *Assume that $C = \left(A_{ij}\right)$ is as presented in Theorem* 3. *Let $r_1, \ldots, r_t$ be nonnegative integers with $r_p + k \le n_p$ for*

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$p = 1, \ldots, t$. *Then* (34) *holds.*

Now let

$$C = \begin{bmatrix} A & X \\ & \\ Y & B \end{bmatrix}$$

where A is $m_1 \times m_2$ and B is $m_3 \times m_4$. Let the singular values of A (the square roots of the eigenvalues of AA^*) be $\alpha_1 \geq \ldots \geq \alpha_{m_1}$, those of X be $x_1 \geq \ldots \geq x_{m_1}$, those of Y be $y_1 \geq \ldots \geq y_{m_3}$, and finally those of B be $\beta_1 \geq \ldots \geq \beta_{m_3}$. Let the singular values of C be $\gamma_1 \geq \ldots \geq \gamma_{m_1+m_3}$. It is possible, by combining the results of §3 with known inequalities for the eigenvalues of a sum of hermitian matrices to obtain inequalities linking the squares of the singular values of A , X , Y , B to those of C .

For simplicity, we do not give the most general form of these inequalities.

THEOREM 4. *Let* r_1, r_2, r_3, r_4 *be nonnegative integers with* $r_1 + k \leq m_1$, $r_2 + k \leq m_1$, $r_3 + k \leq m_3$, $r_4 + k \leq m_3$, *and with* $r_1 + r_2 + k - m_1 \geq 0$, $r_3 + r_4 + k - m_3 \geq 0$. *Then*

$$\sum_{p=1}^{k} \alpha^2_{r_1+p} + \sum_{p=1}^{k} x^2_{r_2+p} + \sum_{p=1}^{k} y^2_{r_3+p} + \sum_{p=1}^{k} \beta^2_{r_4+p}$$
$$\leq \sum_{p=1}^{k} \gamma^2_{r_1+r_2+r_3+r_4+3k-m_1-m_3+p} + \sum_{p=1}^{k} \gamma^2_p .$$

Proof. Let $\lambda_1(M) \geq \lambda_2(M) \geq \ldots$ indicate the eigenvalues of a hermitian matrix M . Then

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$$\sum_{p=1}^{k} \alpha^2_{r_1+p} + \sum_{p=1}^{k} x^2_{r_2+p} + \sum_{p=1}^{k} y^2_{r_3+p} + \sum_{p=1}^{k} \beta^2_{r_4+p}$$
$$\leq \sum_{p=1}^{k} \lambda_{r_1+r_2+p+k-m_1}(AA^*+XX^*) + \sum_{p=1}^{k} \lambda_{r_3+r_4+p+k-m_3}(BB^*+YY^*)$$
$$\leq \sum_{p=1}^{k} \gamma^2_{r_1+r_2+r_3+r_4+3k-m_1-m_3+p} + \sum_{p=1}^{k} \gamma^2_p .$$

Here we first used known inequalities linking the eigenvalues of $AA^* + XX^*$ and $BB^* + YY^*$ to those of AA^*, XX^*, BB^*, YY^*; then we applied Corollary 1 to CC^*, which has $AA^* + XX^*$, $BB^* + YY^*$ as the block diagonal.

THEOREM 5. *Let* $\rho_1, \rho_2, \rho_3, \rho_4$ *be nonnegative integers with* $\rho_1 + \rho_2 \leq m_1 - k$, *and* $\rho_3 + \rho_4 \leq m_3 - k$. *Then*

$$\sum_{p=1}^{k} \alpha^2_{\rho_1+p} + \sum_{p=1}^{k} x^2_{\rho_2+p} + \sum_{p=1}^{k} y^2_{\rho_3+p} + \sum_{p=1}^{k} \beta^2_{\rho_4+p}$$
$$\geq \sum_{p=1}^{k} \gamma^2_{\rho_1+\rho_2+\rho_3+\rho_4+p} + \sum_{p=1}^{k} \gamma^2_{n+1-p} .$$

Proof.

$$\sum_{p=1}^{k} \alpha^2_{\rho_1+p} + \sum_{p=1}^{k} x^2_{\rho_2+p} + \sum_{p=1}^{k} y^2_{\rho_3+p} + \sum_{p=1}^{k} \beta^2_{\rho_4+p}$$
$$\geq \sum_{p=1}^{k} \lambda_{\rho_1+\rho_2+p}(AA^*+XX^*) + \sum_{p=1}^{k} \lambda_{\rho_3+\rho_4+p}(BB^*+YY^*)$$
$$\geq \sum_{p=1}^{k} \gamma^2_{\rho_1+\rho_2+\rho_3+\rho_4+p} + \sum_{p=1}^{k} \gamma^2_{n+1-p} .$$

Here we use Corollary 2 and a different set of inequalities involving the eigenvalues of sums of hermitian matrices.

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