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Arend Bayer and Charles Cadman

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## ABSTRACT

We give a construction of the moduli space of stable maps to the classifying stack  $B\mu_r$  of a cyclic group by a sequence of  $r$ th root constructions on  $\overline{M}_{0,n}$ . We prove a closed formula for the total Chern class of  $\mu_r$ -eigenspaces of the Hodge bundle, and thus of the obstruction bundle of the genus-zero Gromov–Witten theory of stacks of the form  $[\mathbb{C}^N/\mu_r]$ . We deduce linear recursions for genus-zero Gromov–Witten invariants.

## 1. Introduction

This paper combines two notions of stable maps, twisted [AV02] and weighted [Has03], to produce a formula for the genus-zero Gromov–Witten invariants of  $[\mathbb{C}^N/\mu_r]$ . More precisely, we derive a formula for the equivariant Euler class of the obstruction bundle on  $\overline{M}_{0,n}([\mathbb{C}^N/\mu_r])$  as a pull-back of classes on  $\overline{M}_{0,n}$ . Our definition of weighted twisted stable maps is *ad hoc*, applying only to genus-zero maps with target  $[\mathbb{C}^N/\mu_r]$ . Nevertheless, there is a notion of obstruction bundle on each of these spaces, and we have a wall-crossing formula relating their equivariant Euler classes. When all except one of the weights is small, the equivariant Euler class is easy to compute, so we deduce in this way our formula on  $\overline{M}_{0,n}([\mathbb{C}^N/\mu_r])$ .

The starting point of this work is Theorem 2.5.2, which provides an explicit construction of  $\overline{M}_{0,n}([\mathbb{C}^N/\mu_r])$  from  $\overline{M}_{0,n}$  via root constructions. This motivates the generalization to weighted stable maps. They are defined by applying root constructions to the space of weighted stable curves.

### 1.1 Introduction to $[\mathbb{C}^N/\mu_r]$

The quantum cohomology of stack quotients of the form  $[\mathbb{C}^N/G]$  has recently seen a lot of interest due to the crepant resolution conjecture (see [BG09]). However, they are also natural objects of study by themselves: whenever a smooth  $N$ -dimensional stack  $X$  has a local orbifold chart  $[U/G]$  where  $G$  acts with an isolated fixed point, part of the quantum cohomology of  $X$  will be described by the quantum cohomology of  $[\mathbb{C}^N/G]$ . Moreover, if  $X$  has a torus action which restricts to the natural torus action on the chart  $[\mathbb{C}^N/G]$ , then the equivariant quantum cohomology of  $[\mathbb{C}^N/G]$  is relevant for computing the quantum cohomology of  $X$  via localization.

The Chen–Ruan orbifold cohomology of  $[\mathbb{C}^N/\mu_r]$  has a natural basis  $h_e$  for  $e \in \mu_r$ . Consider the moduli space  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$  of twisted stable maps to the origin  $B\mu_r \cong [0/\mu_r] \subset [\mathbb{C}^N/\mu_r]$  in the sense of [AV02], where the branching behavior at the  $i$ th section is prescribed by  $e_i \in \mu_r$ . The non-trivial equivariant Gromov–Witten invariants of  $[\mathbb{C}^N/\mu_r]$  are given by integrals over these moduli spaces. The normal bundle to the origin is  $\mathbb{C}^N$ , understood as a vector bundle

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on  $B\mu_r$  via the given  $\mu_r$ -action. If we write the universal curve as  $\pi : \overline{C} \rightarrow \overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$ , and the universal map as  $f : \overline{C} \rightarrow B\mu_r$ , the obstruction bundle of the moduli space is  $R^1\pi_*f^*\mathbb{C}^N$ . The Gromov–Witten invariant for  $h_{e_1}, \dots, h_{e_n}$  is typically<sup>1</sup> given by the integral of the equivariant Euler class (with respect to the canonical action of the  $N$ -dimensional torus  $\mathcal{T}$  on  $\mathbb{C}^N$ ) of the obstruction bundle

$$\langle h_{e_1} \otimes \dots \otimes h_{e_n} \rangle^{[\mathbb{C}^N/\mu_r]} = \int_{\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)} e_{\mathcal{T}}([R^1\pi_*f^*\mathbb{C}^N]). \tag{1.1.1}$$

We call these integrals *generalized Hurwitz–Hodge integrals*, as the obstruction bundle is a direct sum of  $\mu_r$ -eigenspaces of the dual of the Hodge bundle, where the moduli space is to be understood as a compactification of the Hurwitz space of  $\mu_r$ -covers of  $\mathbb{P}^1$  by *admissible covers*.

### 1.2 Our methods and results

The starting point of our work is the following explicit description of this moduli space of stable maps to  $B\mu_r$  via the  $r$ th root construction of [Cad07]. Given a divisor  $D$  on a scheme  $X$ , the  $r$ th root construction  $X_{D,r}$  is a stack over  $X$  that is isomorphic to  $X$  outside of  $D$ , but whose points over  $D$  are stacky with  $\mu_r$  as automorphism group. For every proper subset  $T \subset [n-1] := \{1, \dots, n-1\}$  having at least two elements, let  $r_T$  be the order  $\prod_{i \in T} e_i$ , and let  $D^T \subset \overline{M}_{0,n}$  be the divisor consisting of curves having a node which separates the marking labels  $1, \dots, n$  into  $T$  and  $[n] \setminus T$ .

**THEOREM.**  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$  is a  $\mu_r$ -gerbe over the stack constructed from  $\overline{M}_{0,n}$  by successively doing the  $r_T$ th root construction at the boundary divisor  $D^T \subset \overline{M}_{0,n}$  for all proper subsets  $T \subset [n-1]$  having at least two elements.

We prove this in Theorem 2.5.2, and we also give an explicit description of the universal curve and of the  $r$ -torsion line bundle defining the morphism to  $B\mu_r$ ; see Definitions 2.3.1 and 2.5.1. The root constructions along the boundary divisor introduce the additional automorphisms of curves with stacky nodes, called *ghost automorphisms*.

Now assume that  $\mu_r$  is acting linearly on  $\mathbb{C}^N$  with weights  $w_1, \dots, w_N$ . To determine a formula for the Chern class of the obstruction bundle, we use a reduction guided by the notion of weighted stable curves in [Has03] and weighted stable maps in [AG08, BM09, MM08]. Weighted stable curves with  $n$  marked points depend on weight data  $a_1, \dots, a_n$ , and yield many birational models of the moduli space  $\overline{M}_{0,n}$ . Particular choices of weight data lead to an explicit presentation of the moduli space  $\overline{M}_{0,n}$  by a series of blow-ups starting with  $\mathbb{P}^{n-3}$ , such that each intermediate blow-up step has an interpretation as a moduli space.

Motivated by this work and guided by Theorem 2.5.2, we make an *ad hoc* definition of a ‘moduli space of weighted stable maps to  $B\mu_r$ ’ in § 3.3. When the weights are chosen such that all fibers of the universal curve are irreducible, the obstruction bundle can easily be computed from general facts about the  $r$ th root construction; we do this in § 3.4 for the weight data that gives a moduli space isomorphic to  $\mathbb{P}^{n-3}$ .

By a careful analysis of the wall-crossing for changing weights in § 4, we can lift this to a closed formula for the equivariant top Chern class in (1.1.1) for the standard (non-weighted) stable maps. We will now state this formula in the case of  $N = 1$  and the standard representation of  $\mu_r$ .

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<sup>1</sup> The formula needs an additional factor in the case where there is a coordinate direction on which every  $e_i$  acts trivially.

For  $1 \leq i \leq n$ , let  $\delta_i \in [0, 1)$  be the age of  $e_i$ , i.e.  $e^{2\pi i \delta_i} = e_i$ . For all subsets  $T \subset [n]$ , let  $\delta_T = \sum_{i \in T} \delta_i$ . Let  $\langle x \rangle$  denote the fractional part of  $x$  if  $x$  is not an integer, and let  $\langle x \rangle = 1$  if  $x$  is an integer. For  $T \subset [n-1]$ ,  $|T| \geq 2$ , let  $\psi_T$  be any class in  $H^*(\overline{M}_{0,n})$  such that the restriction  $D^T \cdot \psi_T$  is the  $\psi$ -class of the node over  $D^T$  on the component corresponding to  $T$ ; e.g. we can set  $\psi_T := -\psi_n + \sum_{[n-1] \supseteq S \supseteq T} D^S$ .

**THEOREM.** *The equivariant Euler class of the obstruction bundle for  $[\mathbb{C}/\mu_r]$  is given as*

$$e_{\mathcal{T}}([R^1 \pi_* f^* \mathbb{C}]) = \prod_{p=\langle \delta_{[n-1]} \rangle}^{\delta_{[n-1]}-1} (t - p\psi_n) \cdot \prod_{\substack{T \subset [n-1] \\ 2 \leq |T|}} \prod_{p=\langle \delta_T \rangle}^{\delta_T-1} \left( 1 + \frac{pD^T}{t + p\psi_T} \right).$$

The case of different weights follows by adjusting the ages in the above formula, and the case of  $\mathbb{C}^N$  by multiplying the individual classes in  $H^*(\overline{M}_{0,n})$ ; the full formula is given in Theorem 5.1.1.

In the appendix, it is shown that this class can be expressed as a continuous, piecewise-analytic function from a real  $(n-1)$ -dimensional torus to  $H_T^*(\overline{M}_{0,n}, \mathbb{R})$  that encodes equivariant top Chern classes for all stacks  $[\mathbb{C}^N/\mu_r]$ , where  $N, n$ , and the weights  $w_1, \dots, w_N$  are fixed, and  $r$  and  $e_1, \dots, e_n$  are arbitrary. See the discussion after the proof of Lemma A.1.1.

By a generalized inclusion–exclusion principle, the Chern class formula leads to linear recursions for all Gromov–Witten invariants of  $[\mathbb{C}^N/\mu_r]$  by a sum over partitions, where every partition corresponds to a moduli space of comb curves. They are particularly nice for local Calabi–Yau 3-folds  $[\mathbb{C}^3/\mu_3]$ . We deduce an explicit formula for the non-equivariant invariants of  $[\mathbb{C}^3/\mu_3]$ . These invariants are the integrals in (1.1.1) for which  $N = r = 3$ ,  $n$  is a multiple of three, and all  $e_i = e^{2\pi i/3}$ . The recursion we discovered for these numbers is

$$\begin{aligned} \langle h_\omega^{\otimes n} \rangle_{0,n}^{[\mathbb{C}^3/\mu_3]} &= (-1)^{n+1} \left( \binom{n-4}{3}! \right)^3 \frac{1}{3} \\ &+ \sum_{p=1}^{(n-3)/3} \sum_{\underline{m}} \frac{(-1)^{p+1}}{|\text{Aut } \underline{m}|} \prod_{j=1}^k \left( \binom{m_j - 2}{3}! \right)^3 M(n-1, \underline{m}) \langle h_\omega^{\otimes n-3p} \rangle_{0,n}^{[\mathbb{C}^3/\mu_3]}, \end{aligned}$$

where the second sum is over all partitions  $\underline{m} = (m_1, \dots, m_k)$  of  $p$ ,  $M(n-1, \underline{m})$  is the multinomial coefficient

$$M(n-1, \underline{m}) = \binom{n-1}{3m_1+1, \dots, 3m_k+1, n-1-\sum_j (3m_j+1)},$$

and  $x! = \prod_{p=\langle x \rangle}^x p$ . The base case is  $\langle h_\omega^{\otimes 3} \rangle_{0,3}^{[\mathbb{C}^3/\mu_3]} = 1/3$ .

### 1.3 Relation to other work

The construction of the moduli space and the universal curve via  $r$ th roots has been described locally by Abramovich in [Abr08, § 3.5]. The global description along with the explicit description of the universal map to  $B\mu_r$  seems to be new.

Our approach is to describe maps to  $B\mu_r$  by  $r$ -torsion line bundles, which is particularly convenient in combination with the  $r$ th root constructions. Of course, one can instead work more geometrically with cyclic covers, which is the point of view adopted in [ACV03, CC09].

The Gromov–Witten theory of  $[\mathbb{C}^N/G]$  has recently generated a lot of interest due to the ‘crepant resolution conjecture’; we refer the reader to [BG09] for an introduction to the

conjecture and overview of the existing literature. When  $X$  is an orbifold that admits a crepant resolution  $\pi : Y \rightarrow X$ , Ruan first conjectured that the quantum cohomology rings of  $Y$  and  $X$  are isomorphic [Rua02], and suggested that the  $q$ -variables for  $\pi$ -exceptional divisor classes on  $Y$  need to be specialized to  $-1$  [Rua06] to recover the orbifold cohomology ring of  $X$ . In [BG09], the authors extended this conjecture: their claim can be formulated as a local linear isomorphism between the Frobenius manifolds of the quantum cohomology of  $X$  and  $Y$  (after analytic continuation). This isomorphism does not respect the natural origins of the two Frobenius manifolds, which corresponds to Ruan’s specialization of  $q$ -variables. When the action of  $G$  on  $\mathbb{C}^N$  leaves the volume form invariant, the stacks of the form  $[\mathbb{C}^N/G]$  yield many non-trivial test cases for the conjecture. In this form it is only expected to hold for orbifolds satisfying the strong Lefschetz theorem; a more general formulation can be found in [CIT09, § 5].

The results so far have been obtained by the use of one of the following two techniques: either a combination of localization computations and use of the WDVV equations, or by using Tseng’s computation of the Chern *character* of the obstruction bundle and Givental’s framework for Gromov–Witten theory.

In [BGP08], the authors explicitly determined the genus-zero Gromov–Witten potential of  $[\mathbb{C}^2/\mu_3]$  and verified the crepant resolution conjecture. In [BG09], the case  $[\mathbb{C}^2/\mu_2]$  was derived from the Hodge integral computations of [FP00].

More generally, the case of  $A_n$ -singularities  $[\mathbb{C}^2/\mu_{n+1}]$  was shown in [CCIT07, CCIT09] based on the Chern character computation. Various other results have been announced in [BG09]. For  $[\mathbb{C}^3/\mu_3]$ , part of the potential was computed in [CCIT09], up to the problem of inversion of the ‘mirror map’. While their technique is completely different to ours, our results are surprisingly close, as explained in §§ 3.5 and 6.4; our recursion can be interpreted as a combinatorial inversion of the mirror map.

Different recursions for invariants of  $[\mathbb{C}^3/\mu_3]$  have been established by the second author and Cavalieri in [CC09], using localization on the space of twisted stable maps to  $\mu_3$ -gerbes over  $\mathbb{P}^1$ .

While our results are quite general, we make no attempt at verifying the crepant resolution conjecture.

### 1.4 Notation and conventions

We write  $[n]$  for the set  $[n] = \{1, 2, \dots, n\}$ . We write  $\langle x \rangle = x - [x] + 1 \in (0, 1]$  for the fractional part of  $x$ , set to 1 if  $x$  is integral. At various places we will write  $\prod_{p=\langle x \rangle}^x f(p)$  for the product  $\prod_{0 < p \leq x, \langle p \rangle = \langle x \rangle} f(p)$ . If  $x < 0$ , the notation  $\prod_{p=\langle x \rangle}^x f(p)$  means that  $\prod_{p=x+1}^{\langle x \rangle - 1} 1/f(p)$  (which is consistent with  $\prod_{p=\langle x \rangle}^x f(p) = f(x) \cdot \prod_{p=\langle x \rangle}^{x-1} f(p)$  for all  $x \in \mathbb{R}$ ).

For  $x > 0$ , we write  $x!$  for the fractional factorial  $x! = \prod_{p=\langle x \rangle}^x p$ .

We identify the rational Chow groups of the moduli stacks of twisted stable maps  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$  with those of their coarse moduli  $\overline{M}_{0,n}$  via pull-back, and similarly for all other moduli stacks we construct. In the appendix, we introduce and explain some non-standard notation for divisors on  $\overline{M}_{0,n}$  that is particularly well suited for our setting; most of it is only used in § 5.2, the exception being

$$\psi_T := -\psi_n + \sum_{[n-1] \supseteq S \supseteq T} D^S$$

for any  $T \subset [n - 1]$ .

**2. Moduli space of stable maps to  $B\mu_r$  via  $r$ th roots**

In this section, we show how to construct a component of the moduli space of genus-zero stable maps to  $B\mu_r$  from the moduli space  $\overline{M}_{0,n}$  of stable curves of genus zero by a series of  $r$ th root constructions.

**2.1 User’s guide to the  $r$ th root construction**

Given an effective Cartier divisor  $D$  of a Deligne–Mumford stack  $X$ , and a positive integer  $r$  which is invertible on  $X$ , the  $r$ th root construction of [Cad07] produces a DM stack  $X_{D,r}$  with the following properties.

- (1) There is a canonical map  $\pi : X_{D,r} \rightarrow X$  that is an isomorphism over  $X \setminus D$ .
- (2) Every point in  $X_{D,r}$  lying over  $D \subset X$  has stabilizer  $\mu_r$ .
- (3) The preimage of  $D$  is an infinitesimal neighborhood of the  $\mu_r$ -gerbe<sup>2</sup>  $\mathcal{D}$  over  $D$  parameterizing  $r$ th roots of the fibers of  $\mathcal{O}_X(D)|_D$ : this is the stack whose objects are triples  $(f : S \rightarrow D, L, \phi)$ , where  $f$  is a morphism,  $L$  is a line bundle on  $S$ , and  $\phi : L^r \rightarrow f^*\mathcal{O}_X(D)|_D$  is an isomorphism. (Only when  $\mathcal{O}_X(D)|_D$  is the  $r$ th power of a line bundle is  $\mathcal{D}$  isomorphic to  $D \times B\mu_r$ .)
- (4) On  $X_{D,r}$ , there is a line bundle  $\mathcal{O}_{X_{D,r}}(\mathcal{D})$  with a section  $s_{\mathcal{D}}$  and an isomorphism  $\phi : \mathcal{O}(\mathcal{D})^r \rightarrow \pi^*(\mathcal{O}(D))$  such that  $\phi(s_{\mathcal{D}}^r) = \pi^*(s_D)$ . (Here  $s_D \in \mathcal{O}(D)$  is the tautological section vanishing along  $D$ .)

The universality of the data in the property (4) is the defining property: giving a morphism  $f : S \rightarrow X_{D,r}$  is equivalent to giving a quadruple  $(g, L, s, \phi)$ , where  $g = \pi \circ f$  is a morphism to  $X$ ,  $L$  is a line bundle on  $S$ ,  $s$  is a section, and  $\phi : L^r \rightarrow g^*\mathcal{O}(D)$  is an isomorphism sending  $s^r$  to  $g^*(s_D)$ .

Locally, when  $X = \text{Spec } A$  is affine and the divisor  $D$  is given by an equation  $(x = 0)$ , the  $r$ th root construction is given by a stack quotient  $[(\text{Spec } A[u]/(u^r - x))/\mu_r]$  of the cyclic  $\mu_r$ -cover branched at  $D$ , but of course globally such a cover may not exist.

To the best of our knowledge, the  $r$ th root construction was originally due to A. Vistoli and spread as a rumor for quite some time. His notation is  $\sqrt[r]{(X, D)}$ .

We call  $\mathcal{O}_{X_{D,r}}(\mathcal{D})$  the tautological line bundle of the  $r$ th root construction at  $D$ . The zero stack  $Z \subset X_{D,r}$  of  $s_{\mathcal{D}}^r$  is the preimage of  $D \subset X$ . The zero stack of  $s_{\mathcal{D}}$  is the gerbe  $\mathcal{D}$ . To simplify notation, we write  $\mathcal{O}(1/r)D$  to refer to  $\mathcal{O}(\mathcal{D})$ . More generally, if  $d \in (1/r)\mathbb{Z}$ , we write  $\mathcal{O}(dD)$  for  $\mathcal{O}(\mathcal{D})^{\otimes dr}$ . This notation is particularly nice to describe the push-forward of line bundles along  $\pi$ :

$$\pi_*\mathcal{O}(dD) = \mathcal{O}(\lfloor d \rfloor D). \tag{2.1.1}$$

(This follows from [Cad07, Theorem 3.1.1].)

If  $X$  is an algebraic space, then the coarse moduli space of  $X_{D,r}$  is  $X$ . When  $X$  is smooth and  $D \subset X$  is smooth, then  $X_{D,r}$  is smooth. The construction commutes with base change for a morphism  $f : Y \rightarrow X$  such that  $f^{-1}(D)$  is a Cartier divisor. (The construction can be generalized a little to make it compatible with arbitrary base change: see  $X_{(L,s,r)}$  in [Cad07].)

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<sup>2</sup> A gerbe over  $D$  is a stack  $\mathcal{D}$  over  $D$  which étale locally admits a section and has the property that any two local sections are locally 2-isomorphic. A gerbe  $\mathcal{D} \rightarrow D$  is a  $\mu_r$ -gerbe if  $\mu_r$  acts as the 2-automorphism group of every section in a compatible way. In particular,  $\mathcal{D}$  is étale locally over  $D$  isomorphic to the trivial stack quotient  $[D/\mu_r]$ , but not necessarily globally.

If  $\mathbb{D} = (D_1, \dots, D_n)$  is an  $n$ -tuple of Cartier divisors and  $\vec{r} = (r_1, \dots, r_n)$ , we can iterate the root constructions to obtain a stack denoted by  $X_{\mathbb{D}, \vec{r}}$ . This stack can also be realized as the  $n$ -fold fiber product over  $X$  of the root stacks  $X_{D_i, r_i}$ . If  $X$  is smooth, each individual  $D_i$  is smooth, and the  $D_i$  have normal crossing, then  $X_{\mathbb{D}, \vec{r}}$  is smooth, too. If  $d_i \in (1/r_i)\mathbb{Z}$ , we extend the above notation by writing  $\mathcal{O}(\sum_i d_i D_i)$  for the tensor product of the line bundles  $\mathcal{O}(D_i)^{d_i r_i}$ , where  $\mathcal{O}(D_i)$  is the tautological bundle corresponding to the root construction along  $D_i$ .

It is instructive (and important for the construction of  $\overline{M}_{0,n}(B\mu_r)$  later on) to compare the stacks  $X_{\mathbb{D}, \vec{r}}$  and  $X_{D,r}$  in the case where  $n = 2$ ,  $r = r_1 = r_2$ , and  $D = D_1 \cup D_2$ . On  $X_{\mathbb{D}, \vec{r}}$ , the line bundle  $\mathcal{O}((1/r)(D_1 + D_2))$  with section  $s_{D_1} \cdot s_{D_2}$  defines an  $r$ th root of  $D$ , and thus there is a natural map

$$X_{\mathbb{D}, \vec{r}} \rightarrow X_{D,r}. \tag{2.1.2}$$

However, this is not an isomorphism if the divisors intersect, and one way to see this is by looking at stabilizer groups of points of the two stacks. If  $x \in D_1 \cap D_2$ , then the stabilizer group of the point in  $X_{\mathbb{D}, \vec{r}}$  lying over  $x$  is  $\mu_r \times \mu_r$ . On the other hand, the stabilizer group of any point in the preimage of  $D$  in  $X_{D,r}$  is  $\mu_r$ . If  $X$  is smooth and  $D_1, D_2$  are smooth with normal crossings, then these stacks are also distinguished by the fact that  $X_{\mathbb{D}, \vec{r}}$  is smooth, while  $X_{D,r}$  is singular over  $D_1 \cap D_2$ .

### 2.2 Stable maps to $B\mu_r$

Consider the moduli space  $\overline{M}_{0,n}(B\mu_r)$  of balanced twisted stable maps of genus zero to  $B\mu_r$  in the sense of [AV02], where we work over  $\mathbb{C}$ . Such a map over a scheme  $S$  can be described by the following data:

- a stacky nodal curve  $\mathcal{C}$  over  $S$ , with  $n$  divisors  $\Sigma_1, \dots, \Sigma_n$  in the smooth locus of  $\mathcal{C}$ ; and
- a line bundle  $L$  on  $\mathcal{C}$  together with an isomorphism  $\phi: L^r \rightarrow \mathcal{O}_{\mathcal{C}}$ .

These have to satisfy various properties:

- the stacky curve  $\mathcal{C}$  is a scheme away from its nodes and the divisors  $\Sigma_i$ , and its nodes are *balanced*;
- each  $\Sigma_i$  is a cyclotomic gerbe over  $S$ ;
- if  $C$  is the coarse moduli space of  $\mathcal{C}$ , then the image of every divisor  $\Sigma_i \subset \mathcal{C}$  in  $C$  is isomorphic to the image of a section  $x_i: S \rightarrow C$ , so that  $(C, x_1, \dots, x_n)$  becomes a stable curve of genus zero with  $n$  marked points;
- the map  $\mathcal{C} \rightarrow B\mu_r$  induced by  $(L, \phi)$  is representable.

The line bundle  $L$  is the pull-back of the line bundle on  $B\mu_r$  given by the canonical one-dimensional representation of  $\mu_r$ . Every point  $x \in \Sigma_i$  has an automorphism group isomorphic to  $\mu_p$  for some  $p$  dividing  $r$ . This identification is canonical if the representation of  $\mu_p$  corresponding to the fiber of the normal bundle  $\mathcal{O}_{\Sigma_i}(\Sigma_i)$  at  $x$  equals the standard representation. Let  $\omega$  be the primitive  $r$ th root of unity  $\omega = e^{2\pi i/r}$ . Then  $\omega^{r/p}$  acts on the fiber  $L_{x_i}$  as multiplication by  $e_i$  for some  $e_i \in \mu_r$ . Equivalently, the map of stabilizer groups  $\mu_p \rightarrow \mu_r$  (which is injective by representability of  $\mathcal{C} \rightarrow B\mu_r$ ) sends the canonical generator of  $\mu_p$  to  $e_i$ . These group elements  $e_1, \dots, e_n$  are constant on every connected component of  $\overline{M}_{0,n}(B\mu_r)$ .

From now on, we assume that we are given  $e_1, \dots, e_n \in \mu_r$  and restrict our attention to the connected component  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$ . There is a natural map  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r) \rightarrow \overline{M}_{0,n}$  induced by the coarse moduli space of the universal curve. Our theorem will describe this

map explicitly via a series of root constructions. One explanation for these root constructions is that twisted curves have ghost automorphisms for each twisted node (cf. [Abr08, § 3.5]).

### 2.3 Construction of the moduli space via root constructions

Let  $r$  and  $e_i \in \mu_r$ ,  $i = 1, \dots, r$  be given. For convenience, we allow  $e_i = 1$ , i.e. untwisted points; then the universal curve is given by the forgetful morphism

$$\pi_{0,(e_1,\dots,e_n)} : \overline{M}_{0,n}(e_1, \dots, e_n, 1; B\mu_r) \rightarrow \overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r).$$

The component is empty unless  $\prod_i e_i = 1$ .

Consider the universal stable curve of genus zero  $\pi_{0,n} : \overline{M}_{0,n+1} \rightarrow \overline{M}_{0,n}$ . The boundary divisors of  $\overline{M}_{0,n}$  are indexed by subsets  $T \subset [n]$  such that  $2 \leq |T| \leq n - 2$  and  $n \notin T$  (see the appendix). For every such  $T$ , let  $r_T$  be the order of  $\prod_{i \in T} e_i$ .

DEFINITION 2.3.1. Let  $\overline{M}^{(1)}$  be the stack constructed from  $\overline{M}_{0,n}$  by doing the  $r_T$ th root construction at every boundary divisor  $D^T$ . We construct  $\overline{C}^{(1)}$  from  $\overline{M}_{0,n+1}$  in the same way after setting  $e_{n+1} = 1$ .

(In particular, we take the  $r_i$ th root construction at every section  $s_i = D^{\{i,n+1\}}$ , where  $r_i$  is the order of  $e_i$ .)

LEMMA 2.3.2. *There is a canonical map  $\pi^{(1)} : \overline{C}^{(1)} \rightarrow \overline{M}^{(1)}$ .*

*Proof.* Equivalently, we construct a map to the fiber product

$$\overline{C}^{(0)} = \overline{M}_{0,n+1} \times_{\overline{M}_{0,n}} \overline{M}^{(1)};$$

then  $\overline{C}^{(0)}$  will be the relative coarse moduli space. Since  $\pi_{0,n}^{-1}(D^T) = D^T \cup D^{T \cup \{n+1\}}$  and the  $r$ th root construction is compatible with such a base change, this fiber product can be constructed from  $\overline{M}_{0,n+1}$  by the  $r_T$ th root constructions at all divisors  $D^T \cup D^{T \cup \{n+1\}}$  for  $T \subset [n]$ . To construct  $\overline{C}^{(1)}$ , we instead took the  $r_T$ th root construction at  $D^T$  and  $D^{T \cup \{n+1\}}$  separately; hence, the morphism  $\overline{C}^{(1)} \rightarrow \overline{C}^{(0)}$  is given by forgetting the root construction along all sections, followed by a composition of morphisms as in (2.1.2) above.  $\square$

Note that  $\overline{C}^{(1)}$  has additional automorphisms along the nodes of the curves lying over  $D^T$ . When we restrict this family to the  $\mu_{r_T}$ -gerbe in  $\overline{M}^{(1)}$  lying over  $D^T$ , the fibers become stacky curves with a twisted node. The node is balanced because after base change to a scheme over the base, the remaining automorphism group is the kernel of the multiplication  $\mu_{r_T} \times \mu_{r_T} \rightarrow \mu_{r_T}$ , which acts with opposite weights on the two branches. The so-called ‘ghost automorphisms’ are accounted for by the additional automorphism introduced in the moduli space. We have thus proved the following proposition.

PROPOSITION 2.3.3. *The morphism  $\pi^{(1)} : \overline{C}^{(1)} \rightarrow \overline{M}^{(1)}$  is a family of balanced twisted curves.*

Note that we adapted [AV02, Definition 4.1.2] to a family over a Deligne–Mumford stack: all conditions have to be checked after étale base change to a scheme covering  $\overline{M}^{(1)}$ .

Each fiber of  $\pi^{(1)}$  admits a morphism to  $B\mu_r$  having the correct restrictions to  $\Sigma_i$  (given by  $e_1, \dots, e_n$ ). However, these morphisms do not in general glue to a morphism  $\overline{C}^{(1)} \rightarrow B\mu_r$ . They will glue precisely when the  $\mu_r$ -gerbe of Definition 2.5.1 is trivial.



**2.4 The universal line bundle**

For  $T \subset [n]$ , we will write always  $T^C = [n] \setminus T$  for its complement.

LEMMA 2.4.1. *Let  $C$  be a geometric fiber of  $\pi^{(1)}$  and let*

$$\mathcal{L} = \mathcal{O}_{\overline{C}^{(1)}} \left( \frac{1}{r_T} D^{T \cup \{n+1\}} \right).$$

- (1) *If there is no node  $x \in C$  such that one of the two connected components of  $C \setminus \{x\}$  contains exactly the markings of  $T$  (and the other those of  $T^C$ ), then  $\mathcal{L}|_C$  is trivial.*
- (2) *Otherwise, let  $C_3, C_4$  be the two connected components of  $C$  after normalization at  $x$ , such that  $C_3$  contains all the markings of  $T$ , and  $C_4$  those of  $T^C$ , and let  $C_1 \subset C_3$  and  $C_2 \subset C_4$  be the two irreducible components of  $C$  meeting at  $x$ . Then*

$$\mathcal{L}|_{C_1} \cong \mathcal{O}_{C_1} \left( -\frac{1}{r_T} x \right), \tag{2.4.1}$$

$$\mathcal{L}|_{C_2} \cong \mathcal{O}_{C_2} \left( \frac{1}{r_T} x \right), \tag{2.4.2}$$

$$\mathcal{L}|_{C'} \cong \mathcal{O}_{C'}, \tag{2.4.3}$$

where  $C'$  is any irreducible component of  $C$  other than  $C_1, C_2$ .

*Proof.* The first statement is obvious, as  $C$  does not meet the divisor  $D^T$  in that case.

In the second case,  $C_3 = C \cap D^{T \cup \{n+1\}}$ , and so (2.4.2) is obvious, as is (2.4.3) for all  $C' \subset C_4$ . The claim then follows by symmetry and the fact that the restriction of  $\mathcal{O}((1/r_T)(D^T + D^{T \cup \{n+1\}}))$  to  $C$  is trivial, as it is the pull-back of the tautological line bundle  $\mathcal{O}((1/r_T)D^T)$  on  $\overline{M}^{(1)}$ . □

Now choose  $d_i \in (1/r) \cdot \mathbb{Z}$  such that  $e^{2\pi i d_i} = e_i$  and  $\sum_{i=1}^n d_i = 0$  (which is possible since  $\prod_{i=1}^n e_i = 1$ ). For  $T \subset [n]$ , let  $d_T = \sum_{i \in T} d_i$ .

LEMMA 2.4.2. *Define the line bundle  $L_1$  on  $\overline{C}^{(1)}$  as*

$$L_1 := \mathcal{O} \left( \sum_{i=1}^n d_i s_i + \sum_{\substack{T \subset [n] \\ 2 \leq |T| \leq n-2 \\ n \notin T}} d_T D^{T \cup \{n+1\}} \right). \tag{2.4.4}$$

Then  $L_1^r$  is the pull-back of a line bundle on  $\overline{M}^{(1)}$ :

$$L_1^r = (\pi^{(1)})^*(L_2).$$

We write  $n \notin T$  to stress that this is for now just an arbitrary way to pick exactly one of  $T, T^C$  for all subsets  $T$ .

*Proof.* First note that  $L_1^r$  is pulled back from the coarse moduli space  $\overline{M}_{0,n+1}$ , since  $d_T \in (1/r)\mathbb{Z}$  for all  $T$ . A line bundle on a family of nodal curves of genus zero over a smooth scheme is pulled back from the base if and only if its degree on any irreducible component of every fiber is zero (in which case it is the pull-back of its own push-forward to the base). Hence, it is sufficient to check that the degree of  $L_1^r$  (or, equivalently, the degree of  $L_1$ ) is zero on any irreducible component  $C_0$  of any fiber  $C$  of  $\pi^1$  (in which case  $L_1^r$  is even pulled back from  $\overline{M}_{0,n}$ ).

Let  $x_1, \dots, x_m$  be the nodes of  $C$  contained in  $C_0$ . Let  $T_i \subset \{1, \dots, n\}, 1 \leq i \leq m$ , be the markings contained in the irreducible components which are connected to  $C_0$  via the node  $x_i$ , and let  $T_0$  be the markings contained in  $C_0$ . For every  $j$  with  $1 \leq j \leq m$ , exactly one of  $\mathcal{O}(d_{T_j} D^{T_j \cup \{n+1\}})$  and  $\mathcal{O}(d_{T_j^c} D^{T_j^c \cup \{n+1\}})$  will appear on the right-hand side of (2.4.4) defining  $L_1$ ; by the previous lemma and  $d_T = -d_{T^c}$ , both restrict to  $\mathcal{O}(d_{T_j} x_j)$  on  $C_0$ . By the same lemma, all other  $\mathcal{O}(d_T D^{T \cup \{n+1\}})$  restrict trivially to  $C_0$ . Hence,

$$L_1|_{C_0} = \mathcal{O}\left(\sum_{i \in T_0} d_i s_i + \sum_{1 \leq j \leq m} d_{T_j} x_j\right).$$

This line bundle has degree  $\sum_{j=0}^m \sum_{i \in T_j} d_i = 0$ , since  $[n]$  is the disjoint union of all  $T_j$ . □

### 2.5 Base change to the gerbe

DEFINITION 2.5.1. Let  $\overline{M}^{(2)}$  be the  $\mu_r$ -gerbe over  $\overline{M}^{(1)}$  of  $r$ th roots of  $L_2$ . Let  $\pi^{(2)} : \overline{C}^{(2)} \rightarrow \overline{M}^{(2)}$  be the base change of  $\pi^{(1)} : \overline{C}^{(1)} \rightarrow \overline{M}^{(1)}$  via  $\overline{M}^{(2)} \rightarrow \overline{M}^{(1)}$ , and let  $L_2^{1/r}$  be the universal line bundle that is an  $r$ th root of  $L_2$ .

By abuse of notation, we write  $L_1$  also for the pull-back of  $L_1$  to  $\overline{C}^{(2)}$ . The line bundle  $L = L_1 \otimes \pi^{(2)*} L_2^{-1/r}$  together with the obvious isomorphism  $L^r \rightarrow \mathcal{O}_{\overline{C}^{(2)}}$  defines a morphism  $\overline{C}^{(2)} \rightarrow B\mu_r$ .

THEOREM 2.5.2. *The following diagram:*

$$\begin{array}{ccc} \overline{C}^{(2)} & \longrightarrow & B\mu_r \\ \pi^{(2)} \downarrow & & \\ \overline{M}^{(2)} & & \end{array}$$

is a family of twisted stable maps over  $\overline{M}^{(2)}$  which defines an isomorphism

$$m : \overline{M}^{(2)} \rightarrow \overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r).$$

*Proof.* We already showed that  $\pi^{(1)}$  (and thus  $\pi^{(2)}$ ) is a family of balanced twisted curves.

The morphism  $\overline{C}^{(2)} \rightarrow \overline{M}^{(2)} \times B\mu_r$  is representable: away from the sections and nodes, the map  $\pi^{(2)}$  is already representable and since, all nodes and sections do not intersect each other, we can treat them separately. At a section  $s_i$ , the relative inertia group of  $\pi^{(2)}$  is isomorphic to  $B\mu_{r_i}$ ; since that group acts faithfully on  $L_1$  and thus on  $L$ , the map on inertia groups is injective. A similar argument holds for all nodes.

We thus get a morphism  $m$  as claimed in the theorem. By Lemma 2.5.3,  $m$  is an isomorphism if both stacks are smooth and the morphism is birational and a bijection of  $\mathbb{C}$ -valued points which induces isomorphisms of their stabilizer groups. Since  $\overline{M}^{(2)}$  is étale over  $\overline{M}^{(1)}$ , which is a root construction on  $\overline{M}_{0,n}$  at smooth divisors with transversal intersection, it follows that  $\overline{M}^{(2)}$  is smooth. The first-order deformations of an  $n$ -marked, genus-zero twisted stable map to  $B\mu_r$  are the same as those of the marked twisted curve, which has dimension equal to  $n - 3$  (see [ACV03, §3]). As this equals the dimension of  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$ , it is also smooth.

For bijectivity, note that  $\mathbb{C}$ -valued points of  $\overline{M}^{(2)}$  are in bijection with  $\mathbb{C}$ -valued points of  $\overline{M}_{0,n}$ . Hence, it suffices to show for each  $n$ -marked genus-zero curve  $C$  that there is a unique

twisted stable map to  $B\mu_r$  with coarse moduli space  $C$  and contact types  $e_1, \dots, e_n$ . For  $C$  irreducible, this uniqueness is shown in [CC08, 2.1.5]. Otherwise, one can show by induction that the contact types at the nodes are uniquely determined by those at marked points. So, the morphism is unique over each component of  $C$ , and it suffices to show that it glues uniquely over the nodes. Since the morphism to  $B\mu_r$  is equivalent to a line bundle  $L$  and a non-vanishing section of  $L^r$ , the gluing is clearly unique up to isomorphism.

It remains only to check the map on automorphism groups induced by  $m$ . If  $x$  is a closed point of  $\overline{M}^{(2)}$ , the automorphism group  $G_{m(x)}$  of  $m(x)$  in the moduli stack  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$  can be identified with the group of  $\mu_r$ -automorphisms of the  $\mu_r$ -cover  $\tilde{C}_x$  of the fiber  $C_x$  of  $\overline{C}^{(2)}$  over  $x$ . If  $S$  is the set of irreducible components of  $C_x$ , this identifies  $G_{m(x)}$  with the subgroup of  $\mu_r^S$  that acts compatibly over every node; since the preimage of a node  $n_T \in C_x$  in  $\tilde{C}_x$  is isomorphic to  $\mu_r/\mu_{r_T}$ , we can identify  $G_{m(x)}$  with the kernel of the map

$$\Sigma = \prod_T \Sigma_T : \mu_r^S \rightarrow \prod_{T \subset [n-1] | x \in D^T} \mu_r/\mu_{r_T},$$

where  $\Sigma_T$  is given by the quotient of the group elements corresponding to the two irreducible components meeting in  $n_T$ .

The  $\mu_r$ -cover  $\tilde{C}_x$  is given by the  $r$ th roots of unity inside the line bundle  $L|_{C_x}$ ; hence, to understand the map  $G_x \rightarrow G_{m(x)}$ , it is sufficient to look at how the automorphism group  $G_x$  of  $x$  acts on  $L$ . By the construction of  $\overline{M}^{(2)}$ , the automorphism group of  $x$  is

$$G_x = \mu_r \times \prod_{T \subset [n-1] | x \in D^T} \mu_{r_T}.$$

By the definition of  $L$ , the first factor acts via  $L_2$  and thus diagonally, whereas  $\mu_{r_T}$  acts diagonally on the irreducible components of  $D^{T \cup \{n+1\}} \cap C_x$  (and trivially on all others) by its induced action on  $\mathcal{O}(\mathcal{D}^{T \cup \{n+1\}})$ . Let  $C_0$  be the component of  $C_x$  which contains the  $n$ th marking.

Before showing that  $m : G_x \rightarrow G_{m(x)}$  is an isomorphism, we introduce some notation. For  $g \in G_x$ , write  $g_0$  for the projection of  $g$  onto  $\mu_r$ , and write  $g_T$  for its projection onto  $\mu_{r_T}$ . For each irreducible component  $C_i$  of  $C_x$ , let  $C_{j_0}, C_{j_1}, \dots, C_{j_{k_i}}$  be the unique shortest path from  $C_i$  to  $C_0$ . That is to say,  $j_0 = i, j_{k_i} = 0, C_{j_\ell}$  meets  $C_{j_{\ell+1}}$  in a node for each  $0 \leq \ell \leq k_i - 1$ , and there are no repetitions in  $j_0, \dots, j_{k_i}$ . For  $0 \leq \ell \leq k_i - 1$ , let  $T_\ell$  be the subset of  $[n - 1]$  determined by the node joining  $C_{j_\ell}$  to  $C_{j_{\ell+1}}$ . Then  $m(g)$  acts on the restriction of  $L$  to the component  $C_i$  by

$$g_0 \prod_{\ell=0}^{k_i-1} g_{T_\ell}. \tag{2.5.1}$$

Let  $g \in G_x$ , and suppose that  $m(g)$  is trivial. Since  $m(g)$  acts on  $C_0$  by  $g_0$ , it follows that  $g_0$  is trivial. By induction on the number of nodes separating a given node from  $C_0$ , and using (2.5.1), it follows that  $g_T$  is trivial for each  $T$ . Therefore,  $g$  is trivial.

Now suppose that  $h \in G_{m(x)}$ . Let  $g_0$  be the element of  $\mu_r$  by which  $h$  acts on  $C_0$ . By induction over the nodes as in the previous paragraph, and using the fact that the irreducible components of  $C_x$  form a tree, we can now define  $g_T$  for each  $T$  in such a way that  $m(g) = h$ . Therefore,  $m : G_x \rightarrow G_{m(x)}$  is an isomorphism.  $\square$

We remark that [ACV03, §7] contains a careful treatment of automorphism groups for  $G$ -covers.

LEMMA 2.5.3. *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be normal, separated, integral, Deligne–Mumford stacks of finite type over an algebraically closed field  $k$  of characteristic zero. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a birational morphism which induces an equivalence of categories between objects over  $\text{Spec } k$ . Then  $f$  is an isomorphism.*

*Proof.* Let  $V \rightarrow \mathcal{Y}$  be an étale surjective morphism from a scheme  $V$  and let  $U = V \times_{\mathcal{Y}} \mathcal{X}$ . Then  $U \rightarrow V$  is separated and quasi-finite, and hence quasi-affine by [LM00, A.2]. Therefore,  $U$  is a scheme. Let  $U' \subseteq U$  be a connected component and let  $V' \subseteq V$  be its image. The hypotheses of the lemma imply that  $U'$  and  $V'$  are normal varieties and that  $U' \rightarrow V'$  is a birational morphism which is bijective on  $k$ -points. By Zariski’s birational correspondence theorem, it follows that  $U' \rightarrow V'$  is an isomorphism. Applying the argument to each connected component shows that  $U \rightarrow V$  is an isomorphism. It now follows from [LM00, 3.8.1] that  $\mathcal{X} \rightarrow \mathcal{Y}$  is an isomorphism.  $\square$

**2.6 Comments on the construction**

The pull-back of a one-dimensional  $B\mu_r$ -representation is a power of  $L$ . Hence, in order to understand the Chern class of the obstruction bundle  $R^1\pi_*\mathbb{C}^N$ , it is sufficient to understand the Chern classes of the higher direct image  $R^1\pi_*L^w$  of powers of  $L$ , and their products.

It is worth pointing out that while the ghost automorphism groups are isomorphic to  $\mu_{r_T}$ , this isomorphism is not natural; the ghost automorphism group is naturally isomorphic to the relative stabilizer group of the twisted node, and by choosing one of the two components  $D^{T \cup \{n+1\}}$  or  $D^{T^C \cup \{n+1\}}$  (and identifying the stabilizer group by its action on the corresponding tangent bundle) one gets an isomorphism to  $\mu_{r_T}$  whose sign depends on this choice. In our construction, this choice shows up in the definition of  $L_1$ , for which we had to choose one of  $T$  and  $T^C$  for all divisors  $D^T$  of  $\overline{M}_{0,n}$ . As  $L_2$  depends on that choice only up to an  $r$ th power, neither  $\overline{M}^{(2)}$  nor the universal line bundle  $L$  depends on this choice. The map  $m : \overline{M}^{(2)} \rightarrow \overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$  does depend on it, however. Different choices can be related by a composition with a ghost automorphism<sup>3</sup> of the moduli stack.

**3. Weighted stable maps to  $B\mu_r$**

**3.1 Weighted stable maps**

Let  $g$  be a non-negative integer, and  $\mathcal{A} = (a_1, \dots, a_n)$  be *weight data*, which means that  $a_i \in \mathbb{Q} \cap [0, 1]$  satisfy  $2g - 2 + \sum_i a_i > 0$ .

In [Has03], Hassett introduced the notion of weighted stable curves: a weighted stable curve of type  $(g, \mathcal{A})$  over  $S$  is a nodal curve  $\pi : C \rightarrow S$  of genus  $g$  with  $n$  sections  $s_i : S \rightarrow C$  such that:

- (1) every section  $s_i$  with positive weight  $a_i$  is contained in the smooth locus of  $\pi$ ;
- (2) the rational divisor  $K_{C/S} + \sum_i a_i s_i$  is  $\pi$ -relatively ample; and
- (3) for any  $I \subset [n]$  such that the intersection  $\bigcap_i s_i$  is non-empty, we have  $\sum_i a_i \leq 1$ .

We will summarize a few of his results, and refer the reader to [Has03] for details.

If  $a_i = 1$  for all  $i$ , then these are stable curves in the usual sense. The difference is that when points  $s_i, i \in I$ , collide, then only when  $\sum_i a_i > 1$  does a new rational component bubble off. This is enough to make the new rational component stable according to condition (3.1).

---

<sup>3</sup>By which we mean, in this context, an automorphism covering the identity on the coarse moduli space.

All the moduli spaces  $\overline{M}_{g,\mathcal{A}}$  with  $|\mathcal{A}| = n$  are birational. More precisely, assume that the weight data  $\mathcal{A} = (a_1, \dots, a_n)$  and  $\mathcal{B} = (b_1, \dots, b_n)$  satisfy  $a_i \geq b_i$  for all  $i$ , and  $a_i > b_i$  for at least one  $i$  (we will write  $\mathcal{A} > \mathcal{B}$  from now on). Then there is a birational reduction morphism  $\rho_{\mathcal{B},\mathcal{A}} : \overline{M}_{g,\mathcal{A}} \rightarrow \overline{M}_{g,\mathcal{B}}$ . (It is induced by  $\mathcal{B}$ -stabilizing the family of curves over  $\overline{M}_{g,\mathcal{A}}$ .) There is a chamber decomposition of  $[0, 1]^n$  by a finite number of walls such that the moduli space  $\overline{M}_{g,\mathcal{A}}$  only depends on the chamber in which the weight data  $\mathcal{A}$  lies: the walls are associated to subsets  $T \subset [n]$  and given as

$$w_T = \left\{ a_i \mid \sum_{i \in T} a_i = 1 \right\}. \tag{3.1.1}$$

Further, the contraction morphism for crossing a single wall is given as a smooth blow-up.

It is somewhat convenient to allow at least one weight to be zero, because  $\overline{M}_{g,\mathcal{A} \cup \{0\}}$  is by definition the universal curve over  $\overline{M}_{g,\mathcal{A}}$ .

This notion has been extended to weighted stable maps in [AG08, BM09, MM08]. In particular, in [AG08, BM09] it was shown that Gromov–Witten invariants can be computed for any choice of weights, yielding identical GW invariants, and [AG08] gave wall-crossing formulae for the full Gromov–Witten potential including gravitational descendants.

### 3.2 $\overline{M}_{0,n}$ as a blow-up of $\mathbb{P}^{n-3}$

As an example that will be important later, consider for given  $n$  the weights  $\mathcal{A}_k = (1/k, \dots, 1/k, 1)$  (with  $n - 1$  entries of  $1/k$ ) for  $k = 1, \dots, n - 2$ . The moduli space  $\overline{M}_{0,\mathcal{A}_{n-2}}$  is isomorphic to  $\mathbb{P}^{n-3}$ , and the universal curve is the blow-up  $Bl_x \mathbb{P}^{n-2}$  of  $\mathbb{P}^{n-2}$  at a point  $x$ ; the universal map is the projection of  $\mathbb{P}^{n-2}$  to  $\mathbb{P}^{n-3} \cong \mathbb{P}T_x$  from  $x$ . If we pick  $x$  away from the coordinate hyperplanes, then the image of the special section with weight 1 is the exceptional divisor, while the remaining sections can be given as the coordinate hyperplanes. (The special section cannot intersect with any other, while the only condition on the remaining sections is that they may not all coincide.)

The moduli space  $\overline{M}_{0,\mathcal{A}_{n-3}}$  is the blow-up of  $\overline{M}_{0,\mathcal{A}_{n-2}}$  at the  $n - 1$  points that are the images of the intersections of  $n - 2$  of the  $n - 1$  coordinate hyperplanes. When we successively increase the first  $n - 1$  weights from  $1/(n - 2)$  to 1, one gets a description of  $\overline{M}_{0,n}$  by successive blow-ups from  $\mathbb{P}^{n-3}$ . This is also explained in [Has03, § 6.2]; the description of  $\overline{M}_{0,n}$  as a blow-up of  $\mathbb{P}^{n-3}$  is equivalent to the description by de Concini and Procesi in [dP95].

### 3.3 The moduli spaces of weighted stable maps to $B\mu_r$ via $r$ th roots

We sidestep the question of defining a moduli problem of weighted stable maps to a stack in general. Instead, we give a direct construction of the moduli stacks via  $r$ th root constructions, guided by the construction in the non-weighted case in § 2.

Given  $r$ , weight data  $\mathcal{A} > 0$ , and  $\mathcal{E} = (e_1, \dots, e_n) \in \mu_r^n$ , we want to construct a stack which would resemble  $\overline{M}_{0,\mathcal{A}}(e_1, \dots, e_n; B\mu_r)$  if it were to exist. Choose  $d_i \in (1/r)\mathbb{Z}$  with  $e^{2\pi i d_i} = e_i$  and  $\sum_{i=1}^n d_i = 0$  as before. Boundary divisors on  $\overline{M}_{0,\mathcal{A}}$  are given as  $D_\sigma$  for  $\mathcal{A}$ -stable 2-partitions  $\sigma = (T, T^C)$  of  $[n]$ ;  $\mathcal{A}$ -stable means that the condition  $|T|, |T^C| \geq 2$  is replaced by  $\sum_{i \in T} a_i > 1$  and  $\sum_{i \in T^C} a_i > 1$ . (This of course means that a corresponding rational curve with two components is  $\mathcal{A}$ -stable.)

Let  $\overline{M}_{0,\mathcal{A}}^{(1)}$  be the stack obtained from  $\overline{M}_{0,\mathcal{A}}$  by taking the  $r_T$ th root at every divisor  $D^T$  such that  $(T, T^C)$  is  $\mathcal{A}$ -stable (where  $r_T$  is defined as before as the order of  $\prod_{i \in T} e_i$ ). To obtain  $\overline{C}_{0,\mathcal{A}}^{(1)}$

from  $\overline{C}_{0,\mathcal{A}} = \overline{M}_{0,\mathcal{A}\cup\{0\}}$ , we start with the same construction, but additionally construct the  $r_i$ th root at every section  $s_i$ .<sup>4</sup> The same proof as in Lemma 2.3.2 shows the following.

LEMMA 3.3.1. *There is a canonical map  $\overline{C}_{0,\mathcal{A}}^{(1)} \rightarrow \overline{M}_{0,\mathcal{A}}^{(1)}$ .*

This is a stacky curve with balanced nodes, but it can have points (in the relative smooth locus of the coarse moduli space) with automorphism group  $\mu_r^{|I|}$  for curves where  $s_i, i \in I$ , are identical; so, this is not a twisted stable curve in the sense of [AV02].

PROPOSITION 3.3.2. *Let  $L_{1,\mathcal{A}}$  be the line bundle on  $\overline{C}^{(1)}$  defined by*

$$L_{1,\mathcal{A}} := \bigotimes_{i=1}^n \mathcal{O}(s_i)^{d_i} \otimes \bigotimes_T \mathcal{O}(D^{T \cup \{n+1\}})^{d_T}, \tag{3.3.1}$$

where the second tensor product goes over all subsets  $T \subset [n]$  with  $n \notin T$  such that  $(T, T^c)$  is  $\mathcal{A}$ -stable. Then  $L_{1,\mathcal{A}}^r = \pi^* L_{2,\mathcal{A}}$  for some line bundle  $L_{2,\mathcal{A}}$  on  $\overline{M}^{(1)}$ .

Again, this has the same proof as before.

Let  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$  be the  $\mu_r$ -gerbe over  $\overline{M}_{0,\mathcal{A}}^{(1)}$  of  $r$ th roots of  $L_{2,\mathcal{A}}$ , and let  $\overline{C}_{0,\mathcal{A}}^{\mu_r}$  be the base change of  $\overline{C}_{0,\mathcal{A}}^{(1)}$  to  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$ . The line bundle  $L_{\mathcal{A}} := L_{1,\mathcal{A}} \otimes \pi^* L_{2,\mathcal{A}}^{-1/r}$  on  $\overline{C}_{\mathcal{A}}$  has trivial  $r$ th power and thus defines a map  $f : \overline{C}_{0,\mathcal{A}}^{\mu_r} \rightarrow B\mu_r$ .

However, again  $f : \overline{C}_{0,\mathcal{A}}^{\mu_r} \rightarrow B\mu_r$  is not a twisted stable map in the sense of [AV02]; most importantly,  $f$  is not representable (not representable even after a base change to a scheme  $S \rightarrow \overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$ ).

### 3.4 $\mathbb{P}^{n-3}$ -weight data

For any weight  $w$  of a one-dimensional  $\mu_r$ -representation, we call

$$H_{\mathcal{A}}^w = R^1 \pi_* L_{\mathcal{A}}^w$$

the generalized dual Hodge bundle on  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$  for the weight data  $\mathcal{A}$ .

Our computation of the Chern class builds up from a direct computation for the weight data  $\mathcal{A} = (1/(n-2), \dots, 1/(n-2), 1)$ , which yields  $\overline{M}_{0,\mathcal{A}} \cong \mathbb{P}^{n-3}$  (see § 3.1). Given  $w$ , let  $\delta_i^w \in [0, 1)$  be the age of the line bundle  $L^w$  at the  $i$ th section; it is determined by  $e^{2\pi i \delta_i^w} = e_i^w$ . For any subset  $T \subset [n]$ , we let  $\delta_T^w = \sum_{i \in T} \delta_i^w$ .

PROPOSITION 3.4.1. *The generalized dual Hodge bundle  $H_{\mathcal{A}}^w$  has the following class in the  $K$ -group.<sup>5</sup>*

$$[H_{\mathcal{A}}^w] = \sum_{p=\langle \delta_{[n-1]}^w \rangle}^{\delta_{[n-1]}^w - 1} [\mathcal{O}(-pH)].$$

*Proof.* The moduli space is a  $\mu_r$ -gerbe over  $\mathbb{P}^{n-3}$ , and the universal curve is constructed from  $Bl_x \mathbb{P}^{n-2}$  by the  $r_i$ th root construction at the section  $s_i$  for all  $i$ , and the base change to the  $\mu_r$ -gerbe.

<sup>4</sup> In the case of  $\overline{M}_{0,n}$ , the section  $s_i$  is equivalent to the boundary divisor given by  $T = \{i, n+1\}$ ; however, this does not yield an  $\mathcal{A}$ -stable 2-partition and hence we need to list them separately.

<sup>5</sup> See 1.4 for other notation conventions used in this formula.

We choose  $d_i$  such that  $d_i \in [0, 1)$  for  $i = 1, \dots, n - 1$  and  $d_n = -\sum_{i=1}^{n-1} d_i$ . Then  $L_{2,\mathcal{A}}$  can be computed by  $L_2 = s_n^*(L_{1,\mathcal{A}})^r = \mathcal{O}(-rd_n)$ , and so by the projection formula  $H_{\mathcal{A}}^w = \mathcal{O}(wd_n) \otimes R^1\pi_*L_{1,\mathcal{A}}^w$ . To compute the higher direct image of  $L_{1,\mathcal{A}}^w$ , we break up  $\pi$  into the composition  $\pi = \pi_2 \circ \pi_1$  of the map  $\pi_1$ , forgetting the roots along the sections, with the map  $\pi_2$  that is the base change of the natural projection  $\text{Bl}_x \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-3}$  to the  $\mu_r$ -gerbe.

The push-forward along  $\pi_1$  follows easily from (2.1.1). Using  $-wd_n = \delta_{[n-1]}^w + \sum_{i=1}^{n-1} [wd_i]$ , we get (where we write  $E$  for the exceptional divisor of  $\text{Bl}_x \mathbb{P}^{n-2}$ )

$$(\pi_1)_*L_{1,\mathcal{A}}^w = \mathcal{O}\left(\sum_{i=1}^n [wd_i]s_i\right) = \mathcal{O}((-wd_n - \delta_{[n-1]}^w)H + [wd_n]E).$$

The relative canonical bundle of  $\pi_2$  is  $-H - E$ , and the pull-back of the hyperplane class on  $\mathbb{P}^{n-3}$  is  $H - E$ . Hence,

$$\begin{aligned} R^1(\pi_2)_*(\pi_1)_*L_{1,\mathcal{A}}^w &= ((\pi_2)_*\mathcal{O}((wd_n + \delta_{[n-1]}^w - 1)H + (-[wd_n] - 1)E))^\vee \\ &= \mathcal{O}(-\langle \delta_{[n-1]}^w \rangle - wd_n) \otimes ((\pi_2)_*\mathcal{O}([\delta_{[n-1]}^w] - 2)H)^\vee, \end{aligned}$$

where we used

$$wd_n - [wd_n] = \{wd_n\} = 1 - \langle -wd_n \rangle = 1 - \langle \delta_{[n-1]}^w \rangle$$

and

$$\delta_{[n-1]}^w - \langle \delta_{[n-1]}^w \rangle = [\delta_{[n-1]}^w] - 1.$$

Applying  $(\pi_2)_*$  to the short exact sequences  $\mathcal{O}((a - 1)H) \rightarrow \mathcal{O}(aH) \rightarrow \mathcal{O}(aH)|_H$ , and using  $(\pi_2)_*\mathcal{O} = \mathcal{O}$ , implies that in the  $K$ -group,

$$[R^1(\pi_2)_*(\pi_1)_*L_{1,\mathcal{A}}^w] = \sum_{p=0}^{[\delta_{[n-1]}^w]-2} [\mathcal{O}(-\langle \delta_{[n-1]}^w \rangle - wd_n - p)].$$

Tensoring this with  $\mathcal{O}(wd_n)$  and re-indexing yields the statement of the proposition. □

Since the hyperplane class of  $\mathbb{P}^{n-3}$  agrees with the  $\psi$ -class of the  $n$ th marking (which is special by having weight 1), this implies the following corollary.

**COROLLARY 3.4.2.** *In the situation of the previous proposition, the Chern class of  $H_{\mathcal{A}}^2$  is given as*

$$c(H_{\mathcal{A}}^w) = \prod_{p=\langle \delta_{[n-1]}^w \rangle}^{\delta_{[n-1]}^w-1} (1 - p\psi_n).$$

(Note that  $\psi_n$  denotes the pull-back of the corresponding class in  $\overline{M}_{0,\mathcal{A}}$  by our convention for the rational Chow groups of the moduli stacks.)

### 3.5 Relation to the twisted $I$ -function

In Givental’s formalism for Gromov–Witten theory [CCIT09, CG07, Giv01, Giv04], the so-called  $J$ -function plays an essential role. Let  $X = [\mathbb{C}^N/\mu_r]$ , where  $\mu_r$  acts diagonally with weights  $w_1, \dots, w_N$ . The Chen–Ruan orbifold cohomology of  $X$  is  $H = H_{CR}^*(X) = \bigoplus_{e \in \mu_r} \mathbb{C} \cdot h_e$ , and

the  $J$  function is a map  $H \rightarrow H[z][[z^{-1}]]$  defined by the following formula:

$$\begin{aligned} J^X(z, t) &= z + t + \sum_{n \geq 0} \sum_{e \in \mu_r} \frac{1}{n!} \left\langle t, \dots, t, \frac{h_e}{z - \psi} \right\rangle_{0, n+1}^X \cdot r h_{e^{-1}} \\ &= z + t + \sum_{n \geq 0} \sum_{e \in \mu_r} \sum_{k \geq 0} \frac{1}{n! z^{k+1}} \langle t, \dots, t, \psi_n^k h_e \rangle_{0, n+1}^X \cdot r h_{e^{-1}}. \end{aligned} \tag{3.5.1}$$

For example, by the results of [JK02], the  $J$ -function of  $B\mu_r$  is given as

$$J^{B\mu_r}(z, t) = z + t + z \sum_{\underline{k}=(k_0, k_1, \dots, k_{r-1})} \prod_{j=0}^{r-1} \frac{t_j^{k_j}}{k_j! z^{|k_j|}} \cdot h_{\prod_j (\omega^j)^{k_j}},$$

where we wrote  $t = \sum_{j=0}^{r-1} t_j h_{\omega^j}$ . The idea of [CCIT09] (and Givental’s formalism in general), specialized to our setting, is to determine the  $J$ -function  $X$  from the  $J$ -function of  $B\mu_r$ ; the former is called the *twisted  $J$ -function* in [CCIT09].

As an approximation to the twisted  $J$ -function, Coates *et al.* defined a *twisted  $I$ -function* in [CCIT09, § 4]; specialized to our case, and translated into our notation, it is given by<sup>6</sup>

$$I^{\text{tw}}(z, t) = z + t + \sum_{(e_1, \dots, e_n)} J^{(e_1, \dots, e_n)} \cdot M_{(e_1, \dots, e_n)}(z).$$

Here

$$J^{(e_1, \dots, e_n)} = z^{-n+2} \delta_{1, \prod_i e_i} \cdot h_{e_n^{-1}}$$

is the part of the  $J$ -function of  $B\mu_r$  that comes from invariants computed on  $\overline{M}_{0, n}(e_1, \dots, e_n; B\mu_r)$ , and  $M_{(e_1, \dots, e_n)}(z)$  is defined by

$$M_{(e_1, \dots, e_n)}(z) = \prod_{a=1}^N \prod_{p=\langle \delta_{[n-1]}^{w_a} \rangle} (\delta_{[n-1]}^{w_a})^{-1} (1 - pz).$$

To show that  $I^{\text{tw}}$  has the desired properties,<sup>7</sup> Tseng’s Grothendieck–Riemann–Roch computation of the Chern character of the obstruction bundle was used in [Tse10].

We can define a *weighted  $J$ -function* of  $X$  by

$$J^{X; \text{weighted}}(z, t) = z + t + \sum_{n \geq 0} \sum_{e \in \mu_r} \frac{1}{n!} \left\langle t, \dots, t, \frac{h_e}{z - \psi} \right\rangle_{0, n+1}^{X; \text{weighted}} \cdot r h_{e^{-1}},$$

where the invariant with superscript ‘weighted’ denotes the invariant computed by the moduli of weighted stable maps  $\overline{M}_{0, \mathcal{A}, \mathcal{E}}^{\mu_r}$  considered in the previous section, i.e. for  $\mathcal{A} = (1/(n - 2), \dots, 1/(n - 2), 1)$ . Then the result of the previous section can be formulated as

$$J^{X; \text{weighted}}(z, t) = z + t + \sum_{(e_1, \dots, e_n)} J^{(e_1, \dots, e_n)} \cdot \widetilde{M}_{(e_1, \dots, e_n)}(z),$$

where  $\widetilde{M}_{(e_1, \dots, e_n)}(z)$  is the truncation of  $M_{(e_1, \dots, e_n)}(z)$  by  $z^{n-2} = 0$ : when the Euler class of the obstruction bundle for  $(e_1, \dots, e_n)$  is given as a polynomial  $P(\psi_n)$  in  $\psi_n$ , then the contribution

<sup>6</sup> The formula on p. 9 of this reference defining the ‘modification factor’  $M_\theta(z)$  has to be applied with  $s_i$  specialized such that  $e^{s(ch)}$  for the Chern character  $ch$  of some bundle  $E$  gives the Euler class of  $-[E]$ .

<sup>7</sup> It has the same image as the twisted  $J$ -function, a subset of Givental’s Lagrangian cone  $\mathcal{L}_X$ .



to the  $J$ -function is the  $\psi_n^{n-3}$ -coefficient of  $P(\psi_n)(1/(z - \psi_n))$ . This coefficient is given by  $z^{-n+2}$  times the truncation of  $P(z)$ .

Independently of  $n$ , this shows that  $J^{X;\text{weighted}}$  is obtained from  $I^{\text{tw}}$  by removing all terms of non-negative degree in  $z$  except the first two; such terms would correspond geometrically to a negative number of  $\psi_n$ -insertions.

### 4. Weight change

The goals of this section are the wall-crossing Theorems 4.3.1 and 4.3.2.

#### 4.1 Preparations

We begin with several lemmas we will need in the proof.

LEMMA 4.1.1. *Let  $D_1, D_2$  be two smooth divisors with transversal intersection on a smooth Deligne–Mumford stack  $X$ . Let  $\tilde{X}$  be the blow-up of  $X$  at their intersection, with exceptional divisor  $E$  and proper transforms  $\tilde{D}_1$  and  $\tilde{D}_2$ . On the other hand, consider the  $r$ th root construction  $X_{(D_1,r),(D_2,r)}$  and its blow-up  $Z$  at the intersection of the two gerbes  $\mathcal{D}_1$  and  $\mathcal{D}_2$  lying over  $D_1$  and  $D_2$ , respectively. Then  $Z$  is isomorphic to the  $r$ th root construction  $\tilde{X}_{(\tilde{D}_1,r),(\tilde{D}_2,r),(E,r)}$ .*

*Proof.* Let  $\tilde{\mathcal{D}}_i$  and  $\mathcal{E}$  be the gerbes in  $\tilde{X}_{(\tilde{D}_1,r),(\tilde{D}_2,r),(E,r)}$  lying over  $\tilde{D}_i$  and  $E$ , respectively. The line bundles  $\mathcal{O}(\tilde{\mathcal{D}}_i \otimes \mathcal{E})$  with their canonical sections are  $r$ th roots of the pull-backs of  $D_i \subset X$ , determining a morphism

$$f : \tilde{X}_{(\tilde{D}_1,r),(\tilde{D}_2,r),(E,r)} \rightarrow X_{(D_1,r),(D_2,r)}.$$

The pull-back of the ideal sheaf of the intersection  $\mathcal{D}_1 \cap \mathcal{D}_2$  along  $f$  is the ideal sheaf of  $\mathcal{E}$ : this is easy to see locally, where we can assume that  $\mathcal{D}_i$  is cut out by an equation ( $\mathfrak{s}_i = 0$ ); its pull-back  $f^*\mathfrak{s}_i$  cuts out  $\mathcal{E} \cup \tilde{\mathcal{D}}_i$ . Since  $\mathcal{E}$  is Cartier, the universal property of blow-ups yields a map

$$g : \tilde{X}_{(\tilde{D}_1,r),(\tilde{D}_2,r),(E,r)} \rightarrow Z.$$

To go the other way, we first show that  $Z \rightarrow X$  lifts to  $\tilde{X}$ . The preimage of  $D_1 \cap D_2$  in  $X_{(D_1,r),(D_2,r)}$  is  $\mathcal{D}_1^r \cap \mathcal{D}_2^r$ . One can check after étale base change to a scheme that the preimage of this in  $Z$  is  $r$  times the exceptional divisor and hence is Cartier. So, the universal property of blow-ups gives us a morphism  $Z \rightarrow \tilde{X}$ . The preimage of  $E$  under this morphism is  $r$  times the exceptional divisor of  $Z$ , and it follows that the preimage of  $\tilde{D}_i$  is  $r$  times the proper transform of  $\mathcal{D}_i$ . This gives us a lifting to

$$h : Z \rightarrow \tilde{X}_{(\tilde{D}_1,r),(\tilde{D}_2,r),(E,r)}.$$

As neither  $Z$  nor  $\tilde{X}_{(\tilde{D}_1,r),(\tilde{D}_2,r),(E,r)}$  has non-trivial automorphisms over the identity of  $X$ , both  $gh$  and  $hg$  must be (2-isomorphic to) the identity. □

LEMMA 4.1.2. *Let  $X, Y$  be Deligne–Mumford stacks, and let  $f : X \rightarrow Y$  be a composition of  $r$ th root constructions and blow-ups at regularly embedded centers (i.e. the normal sheaf of the center is a vector bundle). Then  $\mathbb{R}f_*\mathbb{L}f^*\mathcal{F} = \mathcal{F}$  for any quasi-coherent sheaf  $\mathcal{F}$  on  $Y$ .*

*Proof.* By the projection formula, it is enough to prove that  $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$ . For  $r$ th root constructions, it is obvious that the higher direct images vanish and, by [Cad07],  $f_*\mathcal{O}_X = \mathcal{O}_Y$ .

For blow-ups, this is well known in the case of schemes. Since blow-ups are representable, one can reduce to the case of schemes by taking an étale base change to a scheme covering  $Y$ .  $\square$

LEMMA 4.1.3 (Base change). *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{h} & Y \\ f' \downarrow & \square & \downarrow f \\ X' & \xrightarrow{g} & X \end{array}$$

be a 2-cartesian square of Deligne–Mumford stacks, where  $X$  is quasi-compact,  $f$  is quasi-compact and quasi-separated, and  $Y$  and  $X'$  are tor-independent over  $X$ . Let  $\mathcal{E} \in D^b(Y)$  be a complex having quasi-coherent cohomology. Suppose either that  $g$  has finite tor-dimension or that  $\mathcal{E}$  has flat, finite amplitude relative to  $f$ . Then there is a natural isomorphism

$$\mathbb{L}g^*\mathbb{R}f_*\mathcal{E} \rightarrow \mathbb{R}f'_*\mathbb{L}h^*\mathcal{E}.$$

We will apply this in the case where  $f$  is flat and  $\mathcal{E}$  is a line bundle on  $Y$ .

*Proof.* The existence of a natural morphism  $\mathbb{L}g^*\mathbb{R}f_*\mathcal{E} \rightarrow \mathbb{R}f'_*\mathbb{L}h^*\mathcal{E}$  follows from the adjointness of pull-back and push-forward. Indeed, the natural morphism  $\mathcal{E} \rightarrow \mathbb{R}h_*\mathbb{L}h^*\mathcal{E}$  determines a morphism

$$\mathbb{R}f_*\mathcal{E} \rightarrow \mathbb{R}(fh)_*\mathbb{L}h^*\mathcal{E} = \mathbb{R}(gf')_*\mathbb{L}h^*\mathcal{E},$$

which is equivalent to the morphism above by adjointness of  $\mathbb{R}g_*$  and  $\mathbb{L}g^*$ .

For schemes, the proposition is the same as [BGI71, IV, 3.1.0], and the reduction to the case of schemes is identical to the proof of [LM00, 13.1.9].  $\square$

LEMMA 4.1.4. *Let  $\tau : \tilde{X} \rightarrow X$  be the blow-up of a smooth Deligne–Mumford stack  $X$  at a smooth center  $Z \subset X$  of codimension two, with normal bundle  $N$  and exceptional divisor  $E$ . Then, for  $n \geq 0$ ,*

$$[R^1\tau_*\mathcal{O}(n \cdot E)] = \sum_{k=0}^{n-2} [\Lambda^2 N \otimes \text{Sym}^k N].$$

*Proof.* As a blow-up is representable, it is sufficient to check this for schemes. Due to the short exact sequences  $\mathcal{O}((n-1) \cdot E) \rightarrow \mathcal{O}(nE) \rightarrow \mathcal{O}_E(-n)$  on  $\tilde{X}$ , it follows from induction if we show that  $R^1\pi_*\mathcal{O}_E(-n) = [\Lambda^2 N \otimes \text{Sym}^{n-2} N]$ . This is easily checked by Serre duality, since

$$R^0\pi_*\mathcal{O}_E(n) = \text{Sym}^n(N^\vee)$$

and the relative dualizing sheaf of  $\tau|_E$  is  $\mathcal{O}_E(-2) \otimes \tau^*(\Lambda^2 N)^{-1}$ .  $\square$

### 4.2 Constructing the reduction map

Consider two weight data  $\mathcal{A} > \mathcal{B}$ ; then there is a reduction morphism  $\rho_{\mathcal{B},\mathcal{A}} : \overline{M}_{0,\mathcal{A}} \rightarrow \overline{M}_{0,\mathcal{B}}$ . Further, we assume the following property.

There is exactly one 2-partition  $\sigma = (T_0, T_0^C)$  of  $[n]$  such that  $\sigma$  is  $\mathcal{A}$ -stable but  $\mathcal{B}$ -unstable. (\*)

In other words, there is just one wall between  $\mathcal{A}$  and  $\mathcal{B}$  in the chamber decomposition of the set of weight data discussed in § 3.1. Specifically, this means that  $\sum_{i \in T_0} a_i > 1$  but  $\sum_{i \in T_0} b_i \leq 1$ , and that  $T_0$  is the only such subset of  $[n]$ .

Then  $\rho_{\mathcal{B},\mathcal{A}} : \overline{M}_{0,\mathcal{A}} \rightarrow \overline{M}_{0,\mathcal{B}}$  is the blow-up of  $\overline{M}_{0,\mathcal{B}}$  at the locus  $Z_{T_0}$  of curves where all  $s_i, i \in T_0$  agree; the exceptional divisor is  $D^{T_0}$ . The universal curve  $\overline{C}_{\mathcal{A}}$  is obtained from  $\overline{C}_{\mathcal{B}}$  in two steps.

- (1) Blowing up at the preimage  $\pi^{-1}Z_{T_0}$  of  $Z_{T_0}$ , i.e. taking the base change for  $\rho_{\mathcal{B},\mathcal{A}}$ ; we denote the resulting family over  $\overline{M}_{0,\mathcal{A}}$  by  $\overline{C}_{\mathcal{B}/\mathcal{A}}$ .
- (2) Blowing up at the preimage (with respect to the previous blow-up) of the common image  $s_i(Z_{T_0})$  for any  $i \in T_0$ . (See [BM09, Remark 3.1.2].)

There is a canonical map

$$\rho_{\mathcal{B},\mathcal{A}}^{(1)} : \overline{M}_{0,\mathcal{A}}^{(1)} \rightarrow \overline{M}_{0,\mathcal{B}}^{(1)}.$$

This follows from the fact that the pull-back of any boundary divisor  $D^S \subset \overline{M}_{0,\mathcal{B}}$  is just the corresponding divisor  $D^S \subset \overline{M}_{0,\mathcal{A}}$ , and that the  $r$ th root construction commutes with such base change.

A matching map  $\overline{C}_{\mathcal{A}}^{(1)} \rightarrow \overline{C}_{\mathcal{B}}^{(1)}$  does not exist in general, for the following reason: the preimage of a section  $s_i \subset \overline{C}_{\mathcal{B}}$  with  $i \in T_0$  is the union of the corresponding section  $s_i \in \overline{C}_{\mathcal{A}}$  with the exceptional divisor  $D^{T_0 \cup \{n+1\}}$  of the second blow-up step in the construction of  $\overline{C}_{0,\mathcal{A}}$  above. However, for example when  $\prod_{i \in T_0} e_i = 1$ , there is no root construction along the divisor  $D^{T_0 \cup \{n+1\}}$  in the construction of  $\overline{C}_{\mathcal{A}}^{(1)}$  at all, and so an  $r_i$ th root of the pull-back of  $s_i \subset \overline{C}_{0,\mathcal{B}}$  does not exist.

In order to compare the Hodge bundles, we will later construct some auxiliary spaces to overcome this problem.

For simplicity, we make the following additional assumption.

We assume that for  $T_0$  as in (\*), we have  $n \notin T_0$ . (\*\*)

(This simplifies the computation with respect to our choice of  $L_{1,\mathcal{A}}$  in the definition (3.3.1) and, on the other hand, always holds when we start with the  $\mathbb{P}^{n-3}$ -weight data used in § 3.4.)

LEMMA 4.2.1. *Assuming (\*) and (\*\*), we have  $\rho_{\mathcal{B},\mathcal{A}}^* L_{2,\mathcal{B}} = L_{2,\mathcal{A}}$ .*

This is immediate from  $L_{2,\mathcal{A}} = s_n^*(L_{1,\mathcal{A}})^r$ , as  $s_n$  does not meet any of the divisors appearing in the definition (3.3.1) of  $L_{1,\mathcal{A}}$ , except itself. As a consequence, we have the following corollary.

COROLLARY 4.2.2. *There is a well-defined reduction map*

$$\rho_{\mathcal{B},\mathcal{A}} : \overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r} \rightarrow \overline{M}_{0,\mathcal{B},\mathcal{E}}^{\mu_r}.$$

### 4.3 Weight change and Hodge bundles

We continue with the assumptions (\*) and (\*\*) from the previous section. We want to compare  $\rho_{\mathcal{B},\mathcal{A}}^* H_{\mathcal{B}}^w$  and  $H_{\mathcal{A}}^w$  in the  $K$ -group of  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$  where, due to the projection formula,  $H_{\mathcal{A}}^w$  can be computed by

$$H_{\mathcal{A}}^w = R^1 \pi_* L_{1,\mathcal{A}}^w \otimes (L_{2,\mathcal{A}})^{-w/r}.$$

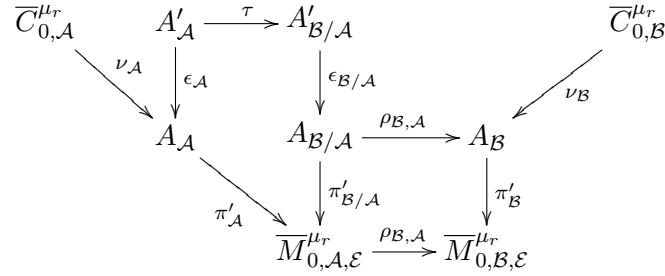
We will introduce several auxiliary spaces; the goal is to have a reduction map as a smooth blow-up between spaces that are very close to the universal curves. This is achieved in the map  $\tau$  below.

Let  $A_{\mathcal{A}}$  be the intermediate space obtained from  $\overline{C}_{0,\mathcal{A}}^{\mu_r}$  by forgetting the root construction along all sections. In other words, it is constructed from the universal curve  $\overline{C}_{\mathcal{A}}$  by the  $r_T$ th root construction for every divisor  $D^T$ , where  $T \subset [n + 1]$  and  $T$  is  $\mathcal{A} \cup \{0\}$ -stable, followed by the base change along  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r} \rightarrow \overline{M}_{0,\mathcal{A}}^{(1)}$ . Let  $\nu_{\mathcal{A}} : \overline{C}_{0,\mathcal{A}}^{\mu_r} \rightarrow A_{\mathcal{A}}$  be the induced map, and  $\pi'_{\mathcal{A}} : A_{\mathcal{A}} \rightarrow \overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$  the projection to the moduli space.

Then

$$\nu_{\mathcal{A}*}(L_{1,\mathcal{A}}^w) = \mathcal{O}\left(\sum_{i=1}^n [wd_i] s_i + \sum_{\substack{T \subset [n-1] \\ (T, T^c) \text{ is } \mathcal{A}\text{-stable}}} wd_T \cdot D^{T \cup \{n+1\}}\right). \tag{4.3.1}$$

Now let  $A_{\mathcal{B}/\mathcal{A}}$  be the base change of  $A_{\mathcal{B}} \rightarrow \overline{M}_{0,\mathcal{B},\mathcal{E}}^{\mu_r}$  along  $\rho_{\mathcal{B},\mathcal{A}}$ . While a map  $A_{\mathcal{A}} \rightarrow A_{\mathcal{B}/\mathcal{A}}$  exists, we prefer not to use it and instead consider two more additional spaces: pick any section  $s_{j_0}$  with  $j_0 \in T_0$  and let  $A'_{\mathcal{A}} = (A_{\mathcal{A}})_{s_{j_0}, r_T}$  be the stack obtained from  $A_{\mathcal{A}}$  by adding the  $r_T$ th root construction at the  $j_0$ th section. We define  $A'_{\mathcal{B}/\mathcal{A}}$  analogously.



Applying Lemma 4.1.1 to the divisors  $s_{j_0}$  and  $\pi^{-1}D^T$  on the coarse moduli space of  $A_{\mathcal{B}/\mathcal{A}}$ , we see that the map  $\tau$  is the blow-up at the intersection of the tautological gerbes over  $s_{j_0}$  and  $(\pi'_{\mathcal{B}/\mathcal{A}})^{-1}D^{T_0}$  in  $A'_{\mathcal{B}/\mathcal{A}}$ .

Setting  $L_{1,\mathcal{B}/\mathcal{A}}^w = \rho_{\mathcal{B},\mathcal{A}}^* \nu_{\mathcal{B}*} L_{1,\mathcal{B}}^w$ , we have

$$\begin{aligned} (\rho_{\mathcal{B},\mathcal{A}})^* H_{\mathcal{B}}^w &= (\rho_{\mathcal{B},\mathcal{A}})^* R^1(\pi'_{\mathcal{B}})_* \nu_{\mathcal{B}*} L_{1,\mathcal{B}}^w \otimes L_{2,\mathcal{B}}^{-w/r} \\ &= R^1(\pi'_{\mathcal{B}/\mathcal{A}})_* \rho_{\mathcal{B},\mathcal{A}}^* \nu_{\mathcal{B}*} L_{1,\mathcal{B}}^w \otimes (L_{2,\mathcal{A}})^{-w/r} \tag{Lemma 4.1.3} \\ &= R^1(\pi'_{\mathcal{B}/\mathcal{A}})_* L_{1,\mathcal{B}/\mathcal{A}}^w \otimes (L_{2,\mathcal{A}})^{-w/r} \\ &= R^1(\pi'_{\mathcal{B}/\mathcal{A}} \epsilon_{\mathcal{B}/\mathcal{A}} \tau)_* (\epsilon_{\mathcal{B}/\mathcal{A}} \tau)^* L_{1,\mathcal{B}/\mathcal{A}}^w \otimes (L_{2,\mathcal{A}})^{-w/r} \tag{Lemma 4.1.2} \end{aligned} \tag{4.3.2}$$

on the other hand,

$$\begin{aligned} H_{\mathcal{A}}^w &= R^1(\pi'_{\mathcal{A}})_* \nu_{\mathcal{A}*} L_{1,\mathcal{A}}^w \otimes (L_{2,\mathcal{A}})^{-w/r} \\ &= R^1(\pi'_{\mathcal{A}} \epsilon_{\mathcal{A}})_* \epsilon_{\mathcal{A}}^* \nu_{\mathcal{A}*} L_{1,\mathcal{A}}^w \otimes (L_{2,\mathcal{A}})^{-w/r} \tag{Lemma 4.1.2} \\ &= R^1(\pi'_{\mathcal{B}/\mathcal{A}} \epsilon_{\mathcal{B}/\mathcal{A}} \tau)_* \epsilon_{\mathcal{A}}^* \nu_{\mathcal{A}*} L_{1,\mathcal{A}}^w \otimes (L_{2,\mathcal{A}})^{-w/r}. \end{aligned} \tag{4.3.3}$$

Now, for any  $T \subset [n - 1]$  such that  $(T, T^c)$  is  $\mathcal{B}$ -stable, it is also  $\mathcal{A}$ -stable, and

$$(\rho_{\mathcal{B},\mathcal{A}} \epsilon_{\mathcal{B}/\mathcal{A}} \tau)^*(D^{T \cup \{n+1\}}) = \epsilon_{\mathcal{A}}^*(D^{T \cup \{n+1\}}).$$

The pull-back of the divisor class of sections is given by

$$(\rho_{\mathcal{B},\mathcal{A}} \epsilon_{\mathcal{B}/\mathcal{A}} \tau)^*(s_i) = \begin{cases} s_i & \text{if } i \notin T_0, \\ s_i + D^{T_0 \cup \{n+1\}} & \text{if } i \in T_0. \end{cases}$$

Using these formulae and (4.3.1) for  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, yields

$$(\epsilon_{\mathcal{A}})^* \nu_{\mathcal{A}*} L_{1,\mathcal{A}}^w = (\epsilon_{\mathcal{B}/\mathcal{A}} \tau)^* L_{1,\mathcal{B}/\mathcal{A}}^w \otimes \mathcal{O}\left(\sum_{i \in T_0} (wd_i - [wd_i]) \cdot D^{T_0 \cup \{n+1\}}\right).$$

Note that  $\delta_{T_0}^w = \sum_{i \in T_0} (wd_i - \lfloor wd_i \rfloor)$ . The projection formula yields

$$\mathbb{R}(\epsilon_{\mathcal{B}/\mathcal{A}}\tau)_*(\epsilon_{\mathcal{A}})^*\nu_{\mathcal{A}*}L_{1,\mathcal{A}}^w = L_{1,\mathcal{B}/\mathcal{A}}^w \otimes \mathbb{R}(\epsilon_{\mathcal{B}/\mathcal{A}}\tau)_*\mathcal{O}(\delta_{T_0}^w \cdot D^{T_0 \cup \{n+1\}}). \tag{4.3.4}$$

Combining (4.3.2), (4.3.3), and (4.3.4), and using  $R^0(\epsilon_{\mathcal{B}/\mathcal{A}}\tau)_*\mathcal{O}(\delta_{T_0}^w \cdot D^{T_0 \cup \{n+1\}}) = \mathcal{O}$ , it follows that

$$[H_{\mathcal{A}}^w] = [\rho_{\mathcal{B},\mathcal{A}}^*H_{\mathcal{B}}^w] + [(\pi'_{\mathcal{B}/\mathcal{A}})_*(R^1(\epsilon_{\mathcal{B}/\mathcal{A}}\tau)_*\mathcal{O}(\delta_{T_0}^w D^{T_0 \cup \{n+1\}}) \otimes L_{1,\mathcal{B}/\mathcal{A}}^w) \otimes L_{2,\mathcal{A}}^{-w/r}]. \tag{4.3.5}$$

Write  $\delta_{T_0}^w$  as the fraction  $p_0/r_0$ , where  $r_0 = r_{T_0}$ . The exceptional divisor of  $\tau$  is  $\mathcal{D}^{T_0 \cup \{n+1\}} = (1/r_0)D^{T_0 \cup \{n+1\}}$ ; so, by Lemma 4.1.4,

$$[R^1\tau_*\mathcal{O}(\delta_{T_0}^w \cdot D^{T_0 \cup \{n+1\}})] = \sum_{k=0}^{p_0-2} [\Lambda^2 N \otimes \text{Sym}^k N], \tag{4.3.6}$$

where  $N$  is the normal bundle to the center of the blow-up  $\tau$ . To compute the right-hand side, we introduce additional normal bundles. Let  $N_{\mathfrak{s}_{j_0}}$  be the normal bundle to the gerbe  $\mathfrak{s}_{j_0}$  over  $s_{j_0}$  in  $A'_{\mathcal{B}/\mathcal{A}}$ , let  $N_{\mathcal{D}^{T_0}}$  be the normal bundle to the gerbe  $\mathcal{D}^{T_0}$  over  $D^{T_0}$  in  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$ , and let  $N_{s_{j_0}}$  be the normal bundle to  $s_{j_0}$  in  $A_{\mathcal{B}/\mathcal{A}}$  (equivalently,  $N_{s_{j_0}}$  is the relative tangent bundle of  $\pi'_{\mathcal{B}/\mathcal{A}}$  restricted to  $s_{j_0}$ ). Then  $N_{\mathfrak{s}_{j_0}}$  is the restriction of the tautological bundle of the  $r_0$ th root construction  $\epsilon_{\mathcal{B}/\mathcal{A}}$  at the section  $s_{j_0}$  to the gerbe, and

$$(\epsilon_{\mathcal{B}/\mathcal{A}})_*N_{\mathfrak{s}_{j_0}}^k = \begin{cases} N_{s_{j_0}}^{k/r_0} & \text{if } r_0 \text{ divides } k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.7}$$

The section  $s_{j_0}$  induces a splitting of the tangent bundle of  $A_{\mathcal{B}/\mathcal{A}}$  along  $s_{j_0}$  into the relative tangent bundle and the push-forward of the tangent bundle of  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$  along  $s_{j_0}$ . This induces a splitting of  $N$  as

$$N = ((\epsilon_{\mathcal{B}/\mathcal{A}}\pi'_{\mathcal{B}/\mathcal{A}})^*N_{\mathcal{D}^{T_0}} \oplus N_{\mathfrak{s}_{j_0}})|_{Z_{T_0}}.$$

Applying this splitting to (4.3.6), we obtain

$$[R^1\tau_*\mathcal{O}(\delta_{T_0}^w \cdot D^{T_0 \cup \{n+1\}})] = \sum_{k=0}^{p_0-2} \sum_{a+b=k} [N_{\mathfrak{s}_{j_0}}^{b+1} \otimes (\epsilon_{\mathcal{B}/\mathcal{A}}\pi'_{\mathcal{B}/\mathcal{A}})^*N_{\mathcal{D}^{T_0}}^{a+1}].$$

We can rewrite the summation as  $\sum_{b=0}^{p_0-2} \sum_{a=0}^{p_0-b-2}$  or, equivalently,  $\sum_{b=0}^{p_0-1} \sum_{a=0}^{p_0-b-2}$ . By (4.3.7), only the terms with  $mr_0 = b + 1$  for some  $m \in \mathbb{Z}$  are surviving the push-forward along  $\epsilon_{\mathcal{B}/\mathcal{A}}$ , which yields

$$[R^1(\epsilon_{\mathcal{B}/\mathcal{A}}\tau)_*\mathcal{O}(\delta_{T_0}^w \cdot D^{T_0 \cup \{n+1\}})] = \sum_{m=1}^{\lceil \delta_{T_0}^w \rceil - 1} \sum_{a=0}^{p_0-r_0m-1} [N_{s_{j_0}}^m \otimes (\pi'_{\mathcal{B}/\mathcal{A}})^*N_{\mathcal{D}^{T_0}}^{a+1}].$$

Combining this with (4.3.5) yields the wall-crossing theorem in the  $K$ -group.

**THEOREM 4.3.1.** *The generalized dual Hodge bundles  $H_{\mathcal{A}}^w$  and  $H_{\mathcal{B}}^w$  can be related in the  $K$ -group of  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$  as follows:*

$$[H_{\mathcal{A}}^w] = [\rho_{\mathcal{B},\mathcal{A}}^*H_{\mathcal{B}}^w] + \sum_{m=1}^{\lceil \delta_{T_0}^w \rceil - 1} \sum_{a=1}^{p_0-r_0m} [N_{\mathcal{D}^{T_0}}^a \otimes s_{j_0}^*N_{s_{j_0}}^m \otimes s_{j_0}^*L_{1,\mathcal{B}/\mathcal{A}}^w \otimes L_{2,\mathcal{A}}^{-w/r}], \tag{4.3.8}$$

where  $s_{j_0}$  is the section  $s_{j_0} : \overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r} \rightarrow A_{\mathcal{B}/\mathcal{A}}$ .

Let  $\alpha$  be the first Chern class of the line bundle  $s_{j_0}^* \nu_{\mathcal{B}}^* L_{1,\mathcal{B}}^w \otimes L_{2,\mathcal{B}}^{-w/r}$  on  $\overline{M}_{0,\mathcal{B},\mathcal{E}}^{\mu_r}$ . The first Chern class of  $s_{j_0}^* N_{s_{j_0}}$  on  $\overline{M}_{0,\mathcal{B},\mathcal{E}}^{\mu_r}$  is  $-\psi_{j_0}$ , and so the first Chern class of  $\mathcal{K} := s_{j_0}^* N_{s_{j_0}}^m \otimes s_{j_0}^* L_{1,\mathcal{B}/\mathcal{A}}^w \otimes L_{2,\mathcal{A}}^{-w/r}$  is  $\rho_{\mathcal{B},\mathcal{A}}^*(\alpha - m\psi_{j_0})$ .

Tensoring the short exact sequences

$$0 \rightarrow \mathcal{O}\left(\frac{1}{r_0}D^{T_0}\right)^{a-1} \rightarrow \mathcal{O}\left(\frac{1}{r_0}D^{T_0}\right)^a \rightarrow N_{\mathcal{D}^{T_0}}^a \rightarrow 0$$

with  $\mathcal{K}$  yields Chern classes for all summands of the right-hand side of (4.3.8), and thus the following formula relating the Chern classes of the dual Hodge bundles:

$$\begin{aligned} c(H_{\mathcal{A}}^w) &= \rho_{\mathcal{B},\mathcal{A}}^*(c(H_{\mathcal{B}}^w)) \cdot \prod_{m=1}^{[\delta_{T_0}^w]-1} \prod_{a=1}^{p_0-r_0m} \frac{1 + (a/r_0)D^{T_0} + \rho_{\mathcal{B},\mathcal{A}}^*(\alpha - m\psi_{j_0})}{1 + ((a-1)/r_0)D^{T_0} + \rho_{\mathcal{B},\mathcal{A}}^*(\alpha - m\psi_{j_0})} \\ &= \rho_{\mathcal{B},\mathcal{A}}^*(c(H_{\mathcal{B}}^w)) \cdot \prod_{m=1}^{[\delta_{T_0}^w]-1} \frac{1 + (\delta_{T_0}^w - m)D^{T_0} + \rho_{\mathcal{B},\mathcal{A}}^*(\alpha - m\psi_{j_0})}{1 + \rho_{\mathcal{B},\mathcal{A}}^*(\alpha - m\psi_{j_0})} \\ &= \rho_{\mathcal{B},\mathcal{A}}^*(c(H_{\mathcal{B}}^w)) \cdot \prod_{m=1}^{[\delta_{T_0}^w]-1} \left(1 + \frac{(\delta_{T_0}^w - m)D^{T_0}}{1 + \rho_{\mathcal{B},\mathcal{A}}^*(\alpha - m\psi_{j_0})}\right). \end{aligned}$$

Equation (4.3.8) implies that this formula does not depend on  $\alpha$  itself but only on its restriction  $\alpha \cdot Z_{T_0}$  to the center of the blow-up part of  $\rho_{\mathcal{B},\mathcal{A}}$ . The Chern class of  $L_{1,\mathcal{B}}$  on  $\overline{C}_{0,\mathcal{B}}^{\mu_r}$  is by construction equal to the Chern class of  $\pi_{\mathcal{B}}^* L_{2,\mathcal{B}}^{1/r}$ . Hence,  $\alpha$  can be computed as the difference of the Chern classes of  $\nu_{\mathcal{B}}^* L_{1,\mathcal{B}}^w$  and  $L_{1,\mathcal{B}}^w$ , which is given by  $\sum_{i=1}^n ([wd_i] - wd_i)s_i = -\sum_{i=1}^n \delta_i^w s_i$ . Pulling this back via  $s_{j_0}$ , and ignoring everything that restricts as zero to  $Z_{T_0}$ , gives  $\alpha = -\sum_{i \in T_0} \delta_i^w (-\psi_{j_0}) = \delta_{T_0}^w \psi_{j_0}$ .

We claim that the restriction of  $\rho_{\mathcal{B},\mathcal{A}}^* \psi_{j_0}$  to  $D^{T_0}$  is the  $\psi$ -class of the node on the component corresponding to the complement of  $T_0$ . To see this, note that the irreducible component over  $D^{T_0}$  corresponding to  $T_0^C$  is the pull-back of the family over  $Z_{T_0}$ , with  $s_{j_0}$  being pulled back to the node.

Since in the final formula, after expanding the fraction using a geometric series,  $\rho_{\mathcal{B},\mathcal{A}}^* \psi_{j_0}$  only appears in monomials that also have a factor of  $D^{T_0}$ , we can replace  $\rho_{\mathcal{B},\mathcal{A}}^* \psi_{j_0}$  by  $\psi_{T_0}$ ; here  $\psi_{T_0}$  is for now any divisor on that restricts as the  $\psi$ -class of the node to  $D^{T_0}$  (but see (A.1.4) for a somewhat canonical global definition). So the formula simplifies further.

**THEOREM 4.3.2.** *Assume that there is a single wall  $w_{T_0}$  as defined in (3.1.1) between the two weight data  $\mathcal{A} \geq \mathcal{B}$ . The Chern classes of the generalized dual Hodge bundles can be related on  $\overline{M}_{0,\mathcal{A},\mathcal{E}}^{\mu_r}$  as follows:*

$$c(H_{\mathcal{A}}^w) = \rho_{\mathcal{B},\mathcal{A}}^*(c(H_{\mathcal{B}}^w)) \cdot \prod_{p=\langle \delta_{T_0}^w \rangle}^{\delta_{T_0}^w - 1} \left(1 + \frac{pD^{T_0}}{1 + p\psi_{T_0}}\right).$$

5. Chern class formula

5.1 Main theorem

Theorem 4.3.2 and Corollary 3.4.2 immediately give a closed formula for the equivariant Euler class of the generalized dual Hodge bundle.

Assume that  $\mu_r$  is acting diagonally on  $\mathbb{C}^N$  with weights  $w_1, \dots, w_N$ . Given  $e_1, \dots, e_n \in \mu_r$ , let  $\delta_i^{(a)} \in [0, 1)$  be the age of  $e_i$  acting on the  $a$ th coordinate direction, i.e.  $e^{2\pi i \delta_i^{(a)}} = e_i^{w_a}$ . For all subsets  $T \subset [n]$ , let  $\delta_T^{(a)} = \sum_{i \in T} \delta_i^{(a)}$ .

THEOREM 5.1.1. *On the connected component  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$  of the moduli space of twisted stable maps  $\overline{M}_{0,n}(B\mu_r)$ , the equivariant Euler class of the obstruction bundle is given as*

$$e_{\mathcal{T}}([R^1\pi_*f^*\mathbb{C}^N]) = \prod_{a=1}^N \prod_{p=\langle \delta_{[n-1]}^{(a)} \rangle}^{\delta_{[n-1]}^{(a)}-1} (t_a - p\psi_n) \cdot \prod_{\substack{T \subsetneq [n-1] \\ 2 \leq |T|}} \prod_{p=\langle \delta_T^{(a)} \rangle}^{\delta_T^{(a)}-1} \left( 1 + \frac{pD^T}{t_a + p\psi_T} \right).$$

*Proof.* If we start with weight data  $\mathcal{A} = (1/(n-2), \dots, 1/(n-2), 1)$  as in §3.4, we can choose a path in  $[0, 1]^n$  leading to  $\mathcal{A} = (1, 1, \dots, 1)$  such that we pass every wall  $w_T = \{a_i \mid \sum_{i \in T} a_i = 1\}$  for  $T \subsetneq [n-1], |T| \geq 2$  exactly once, and only one wall at a time. By Theorem 4.3.2, we pick up exactly the factor in the above product corresponding to  $D^T$  when we cross the wall  $w_T$ , after we set  $t_a = 1$ . To get the equivariant Euler class from the total Chern class, we just have to multiply the  $i$ th Chern class of the higher direct image of the  $a$ th coordinate direction  $\mathbb{C} \subset \mathbb{C}^N$  with  $t_a^{k-i}$ , as the torus is acting trivially on the moduli space, and linearly with multiplication by  $t_a$  on the fibers of the vector bundle. □

5.2 Remarks on the formula

Using the notation  $D^{[n-1]} = -\psi_n$ ,  $D^i = -\psi_i$  and  $\psi_{[n-1]} = 0$  as explained in the appendix, the formula can be written in more compact form:

$$e_{\mathcal{T}}([R^1\pi_*f^*\mathbb{C}^N] - [R^0\pi_*f^*\mathbb{C}^N]) = \prod_{a=1}^N t_a^{\delta_{[n]}^{(a)}-1} \prod_{\emptyset \neq T \subsetneq [n-1]} \prod_{p=\langle \delta_T^{(a)} \rangle}^{\delta_T^{(a)}-1} \frac{t_a + p\psi_T + pD^T}{t_a + p\psi_T}. \tag{5.2.1}$$

Here  $\delta_{[n]}^{(a)} - 1$  is the virtual dimension of the contribution from the  $a$ th coordinate direction  $\mathbb{C} \subset \mathbb{C}^N$  to obstruction bundle.

The convenience of this formulation is that it remains correct (up to an overall power of  $t_a$ ) as long as  $\delta_{\{i\}}^{(a)}$  is any real number such that  $e_i^{w_a} = e^{2\pi i \delta_{\{i\}}^{(a)}}$  (if we still define  $\delta_T^{(a)} = \sum_{i \in T} \delta_{\{i\}}^{(a)}$ ). This is shown in the appendix; see Lemma A.1.1.

This version of the formula also gives the correct answer for the necessary localization computation in the case where all  $e_i$  act trivially on one of the coordinate directions, that is if  $e_i^{w_a} = 1$  for some  $a$  and all  $i$ . In that case,  $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$  is the fixed point locus of  $\overline{M}_{0,n}(e_1, \dots, e_n; [\mathbb{C}^N/\mu_r])$  (instead of being isomorphic to it). The factor of  $1/t_a$  we get in the above formula is the contribution of the  $a$ th coordinate direction to the inverse of the equivariant Euler class of the normal bundle of the fixed point locus.

### 6. Recursions for Gromov–Witten invariants

#### 6.1 Inclusion–exclusion principle

The formula gives particularly nice recursions when the invariants are (almost) non-equivariant. To expand the formula, we use the following fact, which we think of as a generalized inclusion–exclusion principle.

LEMMA 6.1.1. *Let  $S$  be a partially ordered set. Let  $U(S)$  be the set of non-empty subsets  $I \subset S$  such that no two elements of  $I$  are comparable. For every subset  $I \subset S$ , let  $C(I) \subset S$  be the ‘ordered complement’ of  $I$ : the set of elements of  $S$  that are not less than or equal to any element of  $I$ . Then*

$$\prod_{T \in S} (1 + x_T) = 1 + \sum_{I \in U(S)} (-1)^{|I|+1} \prod_{T \in I} x_T \prod_{T \in C(I)} (1 + x_T). \tag{6.1.1}$$

*Proof.* For any subset  $J \subset S$ , the monomial  $\prod_{T \in J} x_T$  appears on the right-hand side of the above product whenever  $I$  is a subset of the set of minimal elements of  $J$ . It is easily checked that it overall has coefficient 1.  $\square$

We use this for  $S$  being the set of subsets  $T \subset [n - 1]$  with  $2 \leq |T| \leq n - 2$ , ordered by inclusion, and

$$\begin{aligned} x_T &= -1 + \prod_{a=1}^N \prod_{p=\langle \delta_T^{(a)} \rangle}^{\delta_T^{(a)}-1} \left( 1 + \frac{pD^T}{t_a + p\psi_T} \right) \\ &= \frac{\prod_{a,p}(1 - t_a^{-1}p\psi_{TC}) - \prod_{a,p}(1 + t_a^{-1}p\psi_T)}{\prod_{a,p}(1 + t_a^{-1}p\psi_T)}, \end{aligned}$$

where  $\psi_{TC} = -D^T - \psi_T$  (see Appendix A). Then  $x_T$  is a class with support on  $D^T$ . Thus, if  $T, T'$  are not comparable, then  $x_T x_{T'}$  can only be non-zero if  $T$  and  $T'$  are disjoint; hence, the expansion of Lemma 6.1.1 reduces to a sum over a combination of pairwise disjoint subsets  $T_1, \dots, T_k \subset [n - 1]$ . Since every  $T_i$  has size at least two, the datum of  $\{T_1, \dots, T_k\}$  can be identified with a partition  $\mathcal{P}$  of  $[n - 1]$ . Given  $\mathcal{P}$ , we write  $\mathcal{P}_{\geq 2}$  for the sets in  $\mathcal{P}$  that have size at least two, recovering the list of the  $T_i$ , and we write  $|\mathcal{P}_{\geq 2}| = \sum_i |T_i|$  for the total size of their union. We can similarly simplify the second product of (6.1.1) to a product over  $T$  which are either disjoint from or fully contain  $T_i$ , for all  $i$ ; in other words, we can identify  $T$  with a subset of the quotient set  $[n - 1]/(\mathcal{P})$  having at least two elements. Thus,

$$\begin{aligned} &e_{\mathcal{T}}(R^1\pi_*f^*\mathbb{C}^N) \\ &= \left( 1 + \sum_{\mathcal{P}} (-1)^{|\mathcal{P}_{\geq 2}|+1} \prod_{T \in \mathcal{P}_{\geq 2}} x_T \prod_{\substack{T \subset [n-1]/(\mathcal{P}) \\ 2 \leq |T|}} (1 + x_T) \right) \prod_{a=1}^N \prod_{p=\langle \delta_{[n-1]}^{(a)} \rangle}^{\delta_{[n-1]}^{(a)}-1} (t_a - p\psi_n), \tag{6.1.2} \end{aligned}$$

where the sum goes over all non-trivial partitions  $\mathcal{P}$  of  $[n - 1]$  (excluding the partitions of sizes one and  $n - 1$ ), and we identify a subset  $T \subset [n - 1]/(\mathcal{P})$  with its preimage in  $[n - 1]$ .

The class associated to the partition  $\mathcal{P}$  in the above expansion has support on  $\bigcap_{T \in \mathcal{P}_{\geq 2}} D^T$ , which explains why we call it an inclusion–exclusion principle. This intersection is a moduli space of comb curves as in Figure 1.



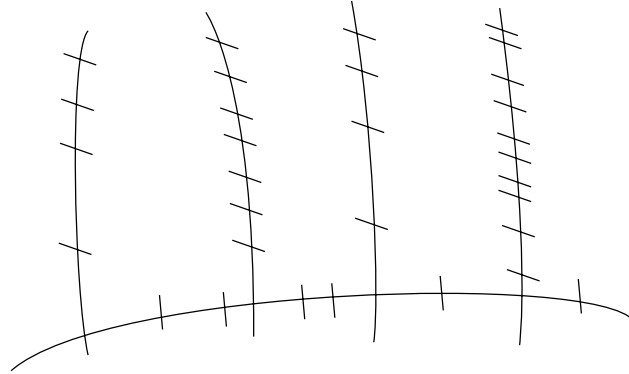


FIGURE 1. Comb for  $n = 30$ ,  $p = 7$ , and  $\underline{m} = (1, 1, 2, 3)$ .

**6.2 Non-equivariant recursions for  $[\mathbb{C}^3/\mu_r]$**

Let  $\mu_r$  act non-trivially on  $\mathbb{C}^3$  so that it leaves the volume form of  $\mathbb{C}^3$  invariant. Up to isomorphism of  $\mu_r$ , we may assume that the generator is acting with age 1; then the weights  $w_1, w_2, w_3$  of the one-dimensional representations satisfy  $w_1 + w_2 + w_3 = r$ . The age of the action of a non-trivial group element  $e_i$  is given by  $\text{age}(e_i, \mathbb{C}^3) = \text{age}(e_i^{w_1}) + \text{age}(e_i^{w_2}) + \text{age}(e_i^{w_3})$ . In this section, we will develop recursions for invariants of the form

$$\langle h_{e_1} \otimes \cdots \otimes h_{e_{n-1}} \otimes \psi_n^\nu h_{e_n} \rangle_{0,n}^{[\mathbb{C}^3/\mu_r]}, \tag{6.2.1}$$

where  $e_1, \dots, e_{n-1} \in \mu_r$  are group elements of age 1, and  $e_n$  is arbitrary, and also the only element for which we allow insertion of a  $\psi$ -class. (This implies that  $\delta_T^{(1)} + \delta_T^{(2)} + \delta_T^{(3)} = |T|$  for all  $T \subset [n - 1]$ .)

We want to determine the integral of a summand on the right-hand side of (6.1.2) related to a partition  $\mathcal{P}$  of  $[n - 1]$ , after inserting an additional  $\psi$ -class at the  $n$ th marked point. Let  $T_1, \dots, T_k$  be the elements of  $\mathcal{P}_{\geq 2}$ .

$$\psi_n^\nu \cdot c(\mathcal{P}) = \psi_n^\nu \cdot \prod_{i=1}^k x_{T_i} \prod_{\substack{T \subsetneq [n-1]/(\mathcal{P}) \\ 2 \leq |T|}} (1 + x_T) \prod_{a=1}^N \prod_{p=\langle \delta_{[n-1]}^{(a)} \rangle}^{\delta_{[n-1]}^{(a)} - 1} (t_a - p\psi_n).$$

This term is supported on the intersection  $D^{T_1} \cap \cdots \cap D^{T_k}$ , isomorphic to  $\overline{M}_{0,T_1 \cup \{*\}} \times \cdots \times \overline{M}_{0,T_k \cup \{*\}} \times \overline{M}_{0,[n]/(\mathcal{P})}$ ; so, in order to determine the integral of  $c(\mathcal{P}) \cdot \psi_n^\nu$ , we will write it as a product of  $D^{T_1} \cdots D^{T_k}$  with factors that are pulled back from one of the components above.

The numerator of  $x_{T_j}$  is the only factor that has terms coming from  $\overline{M}_{0,T_j \cup \{*\}}$ , while its denominator involves  $\psi_{T_j}$ , which is the  $\psi$ -class of the node corresponding to the marking  $T_j$  on  $\overline{M}_{0,[n]/(\mathcal{P})}$ . To examine the numerator more closely, we first factor out  $D^{T_j}$ :

$$\begin{aligned} \prod_{a,p} (1 - t_a^{-1} p\psi_{T_j^C}) - \prod_{a,p} (1 + t_a^{-1} p\psi_{T_j}) &= \sum_{k>0} \beta_k ((-1)^k \psi_{T_j^C}^k - \psi_{T_j}^k) \\ &= D^{T_j} \cdot \sum_{k>0} \beta_k \sum_{\ell=0}^{k-1} (-\psi_{T_j^C})^\ell \psi_{T_j}^{k-1-\ell}. \end{aligned}$$

The largest power of  $\psi_{T_j^{\mathbb{C}}}$  which appears in the last expression is

$$\sum_{a=1}^3 (\delta_{T_j}^{(a)} - \langle \delta_{T_j}^{(a)} \rangle) - 1 = |T_j| - 1 - \sum_{a=1}^3 \langle \delta_{T_j}^{(a)} \rangle.$$

As the dimension of  $\overline{M}_{0,T_j \cup \{*\}}$  is  $|T_j| - 2$ , the expression only has a term in the top degree if  $\sum_{a=1}^3 \langle \delta_{T_j}^{(a)} \rangle = 1$ , which means that  $e_{T_j} = \prod_{i \in T_j} e_i$  acts with age 1 on  $\mathbb{C}^3$  and acts non-trivially in each coordinate direction. By the balancing condition,  $e_{T_j}$  is prescribing the monodromy of the node as seen from the component corresponding to  $[n]/(\mathcal{P})$ . As the integral of  $\psi_{T_j^{\mathbb{C}}}^{|T_j|-2}$  is 1, the integral of the above product on  $\overline{M}_{0,T_j \cup \{*\}}$  is  $\prod_{a=1}^3 t_a^{-[\delta_{T_j}^{(a)}]} (\delta_{T_j}^{(a)} - 1)!$  if the condition on  $e_{T_j}$  is satisfied, and 0 otherwise.

On  $\overline{M}_{0,[n]/(\mathcal{P})}$ , we are left with the following product:

$$\psi_n^\nu \cdot \prod_{a=1}^3 t_a^{-1+\delta_{[n]}^{(a)}} \prod_{\emptyset \neq T \subseteq [n-1]/(\mathcal{P})} \prod_{p=\langle \delta_T^{(a)} \rangle}^{\delta_T^{(a)}-1} \frac{t_a + p\psi_T + pD^T}{t_a + p\psi_T}.$$

Here we used the same conventions as for formula (5.2.1), applied to the set  $[n-1]/(\mathcal{P})$  (so for example  $D^{\{T_j\}}$  is identified with  $-\psi_{\{T_j\}}$ , which is the  $\psi$ -class of a single marking; the term related to  $T_j$  in the above product is the denominator of  $x_{T_j}$ ); and we extended  $\delta_T^{(a)}$  in the obvious way from subsets of  $[n-1]$  to subsets of the quotient set  $[n-1]/(\mathcal{P})$ . Now, by the remarks in § 5.2, based on Lemma A.1.1, this product computes the Chern class of the obstruction bundle on  $\overline{M}_{0,[n]/(\mathcal{P})}((e_i)_{i \in [n]/(\mathcal{P})}; B\mu_r)$ . Its integral is thus given by the equivariant Gromov–Witten invariant

$$\left\langle \bigotimes_{i \in [n-1]/(\mathcal{P})} h_{e_i} \otimes \psi_n^\nu h_{e_n} \right\rangle_{0,[n]/(\mathcal{P})}^{[\mathbb{C}^3/\mu_r]}.$$

This proves the following recursion.

PROPOSITION 6.2.1. *For an equivariant Gromov–Witten invariant of  $[\mathbb{C}^3/\mu_r]$  as in (6.2.1) with the assumptions above, let  $S$  be the set of non-trivial partitions  $\mathcal{P}$  of  $[n-1]$  such that for every  $T \in \mathcal{P}_{\geq 2}$ , the group element  $e_{T_j} = \prod_{i \in T_j} e_i$  acts with age 1 on  $\mathbb{C}^3$ , and is non-trivial in every coordinate direction. Then the following recursive formula holds:*

$$\begin{aligned} \langle h_{e_1} \otimes \cdots \otimes h_{e_{n-1}} \otimes \psi_n^\nu h_{e_n} \rangle_{0,n}^{[\mathbb{C}^3/\mu_r]} &= \langle h_{e_1} \otimes \cdots \otimes h_{e_{n-1}} \otimes \psi_n^\nu h_{e_n} \rangle_{0,n}^{[\mathbb{C}^3/\mu_r]; \text{weighted}} \\ &+ \sum_{\mathcal{P} \in S} (-1)^{|\mathcal{P}_{\geq 2}|+1} \prod_{T \in \mathcal{P}_{\geq 2}} \prod_{a=1}^3 (\delta_T^{(a)} - 1)! \left\langle \bigotimes_{i \in [n-1]/(\mathcal{P})} h_{e_i} \otimes \psi_n^\nu h_{e_n} \right\rangle_{0,[n]/(\mathcal{P})}^{[\mathbb{C}^3/\mu_r]}. \end{aligned}$$

Here the invariant with superscript ‘weighted’ means the invariant as computed by using the moduli space of weighted stable maps instead of the ordinary moduli space, with weight data chosen as in § 3.4. These invariants are given, up to a multiplication with a monomial in the  $t_a$ , by the  $(n-3-\nu)$ th elementary symmetric function of the variables

$$t_a^{-1}(\delta_{[n-1]}^{(a)} - 1), t_a^{-1}(\delta_{[n-1]}^{(a)} - 2), \dots, t_a^{-1} \langle \delta_{[n-1]}^{(a)} \rangle \quad \text{for } a = 1, 2, 3.$$

**6.3 Recursions for  $[\mathbb{C}^3/\mu_3]$**

The recursion of Proposition 6.2.1 simplifies further in the case  $[\mathbb{C}^3/\mu_3]$  for the diagonal representation of  $\mu_3$ . The only group element of age 1 is  $\omega = e^{2\pi i/3}$ . We have  $\delta_T^{(a)} = |T|/3$  for all  $T \subset [n - 1]$  and  $a = 1, 2, 3$ . The set  $S$  contains the partitions  $\mathcal{P}$  of  $[n - 1]$ , so that every  $T \in \mathcal{P}$  has size  $3m_T + 1$  for some  $m_T \in \mathbb{Z}_{\geq 0}$ . The summand for  $\mathcal{P}$  in the formula of Proposition 6.2.1 depends only on the sizes of the subsets, not on the actual subsets; if we set  $p = \sum_T m_T$ , we can thus reduce the above sum to a sum over partitions  $\underline{m} = (m_1, \dots, m_k)$  of  $p$ , for all  $p \geq 1$  with  $n - 3p \geq 3$ . For any such partition, let  $M(n - 1, \underline{m})$  be the multinomial coefficient

$$M(n - 1, \underline{m}) = \binom{n - 1}{3m_1 + 1, \dots, 3m_k + 1, n - 1 - \sum_j (3m_j + 1)}$$

counting the ways to distribute  $n - 1$  markings on the different components.

PROPOSITION 6.3.1.

$$\begin{aligned} \langle h_\omega^{\otimes n-1} \otimes \psi_n^\nu h_{e_n} \rangle_{0,n}^{[\mathbb{C}^3/\mu_3]} &= \langle h_\omega^{\otimes n-1} \otimes \psi_n^\nu h_{e_n} \rangle_{0,n}^{[\mathbb{C}^3/\mu_3]; \text{weighted}} \\ &+ \sum_{p=1}^{\lfloor (n-3)/3 \rfloor} \sum_{\underline{m}} \frac{(-1)^{p+1}}{|\text{Aut } \underline{m}|} \prod_{j=1}^k \left( \binom{m_j - 2}{3} \right)^3 M(n - 1, \underline{m}) \langle h_\omega^{\otimes n-1-3p} \otimes \psi_n^\nu h_{e_n} \rangle_{0,n}^{[\mathbb{C}^3/\mu_3]}. \end{aligned}$$

The superscript ‘weighted’ means the same as before; if  $e_n = \omega$  and  $\nu = 0$ , this weighted Gromov–Witten invariant is just given as  $(-1)^{n+1} ((\frac{1}{3}(n - 4))!)^3 \frac{1}{3}$ ; otherwise, it is an elementary symmetric function.

A Maple program implementing some of these recursions is available from the authors upon request. The numbers match the calculations of [ABK08, CC09, CCIT09].

As the recursions are linear, it is not hard to invert the matrix and obtain a direct formula. Let  $I_\ell = \langle h_\omega^{\otimes 3\ell+3} \rangle_{0,3\ell+3}^{[\mathbb{C}^3/\mu_3]}$ . Proposition 6.3.1 implies that

$$\sum_{p=0}^{\ell} (-1)^p C_{p,\ell} I_{\ell-p} = (-1)^\ell \left( \left( \ell - \frac{1}{3} \right) ! \right)^3 \frac{1}{3},$$

where  $C_{0,\ell} = 1$  and, for  $0 < p \leq \ell$ ,  $C_{p,\ell}$  is the sum over partitions  $\underline{m} = (m_1, \dots, m_k)$  of  $p$ , with  $k \leq 3(\ell - p) + 2$ , of the quantity

$$\frac{1}{|\text{Aut } \underline{m}|} \prod_{j=1}^k \left( \binom{m_j - 2}{3} \right)^3 \cdot \binom{3\ell + 2}{3m_1 + 1, \dots, 3m_k + 1, 3(\ell - p) + 2 - k}.$$

Let  $D_{p,\ell} = C_{\ell-p,\ell}$ . By inverting the matrix, we obtain the formula

$$3(-1)^\ell I_\ell = \sum_{\substack{S \subseteq [0, \ell-1] \\ S = \{x_0, \dots, x_q\}}} (-1)^{|S|} \left( \left( x_0 - \frac{1}{3} \right) ! \right)^3 D_{x_0, x_1} \cdots D_{x_{q-1}, x_q} D_{x_q, \ell},$$

where  $[0, \ell - 1] = \{0, 1, \dots, \ell - 1\}$  and it is assumed that  $x_0 < x_1 < \dots < x_q$ . For  $S = \emptyset$ , the summand is taken to be  $((\ell - 1/3)!)^3$ .

**6.4 Inversion of the ‘mirror map’**

The recursion in this case can also be derived from the results in [CCIT09]. We explained above in §3.5 that the  $I$ -function  $I^{\text{tw}}(t)$  is almost identical (up to high powers of  $z$ ) to the ‘weighted  $J$ -function’  $J^{X; \text{weighted}}$ .

From general principles of Givental’s formalism, it was deduced that for the coordinate change  $\tau(t) : H \rightarrow H$ , called the ‘mirror map’, given by

$$\tau(t_0h_1 + t_1h_\omega) = t_0h_1 + \sum_{k \geq 0} \frac{(-1)^{3k}(t_1)^{3k+1}}{(3k + 1)!} \left( \left( k - \frac{2}{3} \right) ! \right)^3 h_\omega,$$

the twisted  $I$ -function and the twisted  $J$ -function can be related after setting the dual coordinate of  $h_{\omega^2}$  equal to zero:

$$I^{\text{tw}}(t_0h_1 + t_1h_\omega, z) = J^X(\tau(t_0h_1 + t_1h_\omega), z).$$

Our recursive formula can be recovered by comparing coefficients of these two power series; in other words, the sum over partitions  $\mathcal{P}$  in Proposition 6.3.1 is a combinatorial inversion of the mirror map  $\tau(t)$ .

More precisely, given any two power series  $A(t_0h_1 + t_1h_\omega, z)$ ,  $B(t_0h_1 + t_1h_\omega, z)$  related by

$$A(t_0h_1 + t_1h_\omega, z) = B(\tau(t_0h_1 + t_1h_\omega), z),$$

we can recursively recover the coefficients of  $B$  by comparing coefficients of powers of  $t_1$ . If we write  $A(t_1h_1) = \sum_k (a_k/k!)t_1^k$  and  $B(t_1h_1) = \sum_k (b_k/k!)t_1^k$  with  $a_k, b_k \in H[[z^{-1}]]$ , the recursion will look exactly as Proposition 6.3.1 with the ‘weighted invariants’ replaced by  $a_k$  and the actual invariants replaced by  $b_k$ . In particular, setting  $A$  to the identity power series  $A(t_0h_1 + t_1h_\omega) = t_0h_1 + t_1h_\omega$  yields an inversion of the mirror map that can also be interpreted as a sum over comb curves.

### 6.5 Equivariant recursions

The methods of this section are sufficient to produce a linear recursion for the equivariant descendant Gromov–Witten invariants of  $[\mathbb{C}^N/\mu_r]$ . However, this requires one to allow  $\psi$ -classes at every marked point. As in § 6.2, one can use Lemma 6.1.1 to expand the equivariant Euler class of the obstruction bundle, and to each partition of  $[n - 1]$  one should associate a comb as before, where the  $n$ th marked point is on the head of the comb. For each tooth of the comb, we can write the numerator of  $x_T$  as

$$D^T \cdot \sum_{k > 0} \beta_k \sum_{\ell = 0}^{k-1} (-\psi_{TC})^\ell \psi_T^{k-1-\ell},$$

just as in § 6.2. The exponent of  $\psi_{TC}$  which leads to a non-zero integral is determined by the descendant exponents chosen for the marked points in  $T$ . The integral over the tooth can then be computed using the well-known formula

$$\int_{\overline{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \binom{n-3}{a_1, \dots, a_n}.$$

We are left with a polynomial in  $\psi_T$ , the  $\psi$ -class of the node on the main component (and hence our method does not yield a recursion for non-descendant invariants only).

In summary, for each partition of  $[n - 1]$ , one gets a linear combination of equivariant Gromov–Witten invariants with fewer marked points, each with a combinatorial factor. These must be summed together and added to the weighted invariant, just as in § 6.2.

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Appendix A. Combinatorics of divisors on  $\overline{M}_{0,n}$

A.1 Notation for divisors

This section reviews notation for divisors on  $\overline{M}_{0,n}$ , some of which is introduced in this paper. The standard relations are reviewed, and a combinatorial proof of a key simplifying relation, used in § 6, is worked out.

First recall the vital divisor  $D^T$ , introduced by Keel [Kee92], where  $T$  is any subset of  $[n]$  having at least two and at most  $n - 2$  elements. This divisor is the locus of curves having a node which separates the markings into  $T$  and  $T^C$ . Here complements are always taken within  $[n]$ . To make this into a correspondence, assume throughout this appendix that  $S$  and  $T$  are subsets of  $[n - 1]$ . It is natural to define  $D^{[n-1]}$  to be  $-\psi_n$ , which comes from the work of de Concini and Procesi [dP95]. From their point of view,  $D^{[n-1]}$  is the pull-back of minus the hyperplane class under a sequence of blow-ups producing  $\overline{M}_{0,n}$  from  $\mathbb{P}^{n-3}$ . This sequence of blow-ups is the same one discussed in § 3.2 in the context of weighted stable maps. Under this blow-up description of  $\overline{M}_{0,n}$ ,  $D^T$  is the exceptional divisor of the blow-up in the proper transform of the linear space generated by the points labeled by  $[n - 1] \setminus T$ .

The ring  $H^*(\overline{M}_{0,n})$  is generated by the divisors  $D^T$  for  $T \subseteq [n - 1]$ ,  $|T| \geq 2$ , with relations given by

$$D^S D^T \quad \text{if } S \text{ and } T \text{ are incomparable and } S \cap T \neq \emptyset, \tag{A.1.1}$$

$$\sum_{i,j \in T} D^T \quad \text{for every } i \neq j \in [n - 1] \text{ [dP95]}. \tag{A.1.2}$$

Geometrically, the first relation is due to the fact that the exceptional divisors  $D^S$  and  $D^T$  are disjoint, and the second is due to the fact that the preimage of the hyperplane in  $\mathbb{P}^{n-3}$  generated by all points except  $i$  and  $j$  consists of the proper transform  $D^{i,j}$  together with all the exceptional divisors  $D^T$  for  $i, j \in T$ . Fix  $T \subsetneq [n - 1]$  and choose  $i \notin T$  and  $j \in T$ . Then the relations (A.1.1) and (A.1.2) imply that

$$D^T \sum_{\substack{S:i \in S \\ S \supset T}} D^S = 0. \tag{A.1.3}$$

For any non-empty subset  $T \subseteq [n - 1]$ , we introduce the notation

$$\psi_T := \sum_{S \supseteq T} D^S. \tag{A.1.4}$$

If  $T = \{i\}$ , then this recovers the  $\psi$ -class at the  $i$ th marked point. Indeed, we have for any distinct  $j, k$  different from  $i$  (recalling the convention  $n \notin T$ ),

$$\psi_i = \sum_{\substack{j,k \in T \\ i \notin T}} D^T + \sum_{\substack{i \in T \\ j,k \notin T}} D^T = - \sum_{i,j,k \in T} D^T + \sum_{\substack{i \in T \\ j,k \notin T}} D^T + \sum_{i,j \in T} D^T + \sum_{i,k \in T} D^T = \psi_{\{i\}},$$

the last equality following from an inclusion–exclusion argument. By definition,  $\psi_{[n-1]} = 0$ , and we also set  $D^i = -\psi_i$  for  $1 \leq i \leq n - 1$ . Finally, we define

$$\psi_{T^c} := -D^T - \psi_T$$

for  $T \subseteq [n - 1]$ , which we found useful in § 6.

The Chern class formula of Theorem 5.1.1 can now be expressed as

$$c = \prod_{\emptyset \neq T \subseteq [n-1]} \prod_{p=\langle \delta_T \rangle}^{\delta_T-1} \left( \frac{1 + p(D^T + \psi_T)}{1 + p\psi_T} \right), \tag{A.1.5}$$

where  $\delta_T = \sum_{i \in T} \delta_i$  and  $\delta_i$  are chosen so that  $0 \leq \delta_i < 1$ . The following lemma shows that if every  $\delta_i \geq 0$ , this expression is periodic in each  $\delta_i$  with period one; so, it defines a continuous, piecewise-analytic function from an  $(n - 1)$ -dimensional real torus into the cohomology of  $\overline{M}_{0,n}$  with real coefficients. This is used in § 6 to produce recursions for the Gromov–Witten invariants.

LEMMA A.1.1. *Let  $\delta_1, \dots, \delta_{n-1}$  be real numbers and, for any subset  $T \subseteq [n - 1]$ , define  $\delta_T = \sum_{i \in T} \delta_i$ . Then*

$$\prod_{1 \in T} \frac{1 + \delta_T(D^T + \psi_T)}{1 + \delta_T\psi_T} = 1. \tag{A.1.6}$$

*Proof.* For  $1 \leq k \leq n - 1$ , let

$$E_k = \sum_{i=2}^{n-1} \delta_i \sum_{\substack{1, i \in T \\ |T| > k}} D^T + \delta_1 \sum_{\substack{1 \in T \\ |T| > k}} D^T,$$

and let

$$A_k = (1 + E_k) \prod_{\substack{1 \in T \subseteq [n-1] \\ |T| \leq k}} \frac{1 + \delta_T(D^T + \psi_T)}{1 + \delta_T\psi_T}.$$

Then it must be shown that  $A_{n-1} = 1$ . As  $E_{n-2} = \delta_{[n-1]}D^{[n-1]}$ , it follows that  $A_{n-2} = A_{n-1}$ . Moreover,  $E_1 = \delta_1\psi_1$ , so that  $A_1 = 1$ . It remains to show that  $A_k = A_{k-1}$  for  $1 < k < n - 1$ .

Note that for any divisors  $x, y, z$ ,

$$\frac{(1+x)(1+y)}{1+z} = 1 + x + y - z \quad \text{if } (x-z)(y-z) = 0. \tag{A.1.7}$$

Fix  $T$  with  $1 \in T$  and  $|T| = k$ , and let  $x$  be any expression of the form  $E_k + \sum_{S \in \sigma} \alpha_S D^S$ , where  $\sigma$  is a collection of subsets  $S \subseteq [n - 1]$  with  $|S| = k$  and  $1 \in S \neq T$ . Then  $D^T(x - \delta_T\psi_T) = 0$  by the following argument. For any  $S \in \sigma$ ,  $1 \in S \cap T$  and  $|S| = |T|$ , so  $D^S D^T = 0$ . Moreover,

$$E_k - \delta_T\psi_T = \sum_{i:i \notin T} \delta_i \sum_{\substack{S:i \in S \\ T \subseteq S}} D^S + \sum_{\substack{|S| > k \\ S \not\supseteq T \\ 1 \in S}} \delta_S D^S.$$

Since  $D^T$  annihilates the first term on the right-hand side by (A.1.3) and annihilates the second by (A.1.1), it follows that  $D^T(E_k - \delta_T\psi_T) = 0$ , completing the verification of the claim.

It follows that one can iterate through all sets  $T$  with  $|T| = k$  to eliminate those factors from the expression for  $A_k$  and apply the relation (A.1.7) at each step. Since

$$E_k + \sum_{\substack{1 \in T \\ |T|=k}} \delta_T D^T = E_{k-1},$$

it follows that  $A_k = A_{k-1}$ , completing the proof. □

If, in (A.1.5), one were to add 1 to  $\delta_1$  (and thus add 1 to each  $\delta_T$  for  $1 \in T$ ), then one would be multiplying  $c$  by the left-hand side of (A.1.6). Therefore,  $c$  does not change after the translation  $\delta_i \mapsto \delta_i + 1$ . Using the notational conventions of § 1.4, the formula for  $c$  makes sense for negative values of  $\delta_i$ , and the same argument shows that it remains invariant under integer translations. It seems natural to regard  $\delta_1, \dots, \delta_n$  as coordinates on  $(\mathbb{R}/\mathbb{Z})^n$  satisfying  $\sum_{i=1}^n \delta_i = 0$ .

### A.2 Restricting to $D^T$

It is a standard fact that for any proper subset  $T \subset [n - 1]$  containing at least two elements,  $D^T \cong \overline{M}_{0,|T|+1} \times \overline{M}_{0,n-|T|+1}$ , with the node counting as an extra marked point on each factor. The restrictions of divisors  $D^S$  and  $\psi_S$  to  $D^T$  are easily computed if one uses subsets of  $T$  for the divisors on the first factor and subsets of the quotient set  $[n - 1]/T$  for divisors on the second factor. So, on the first factor, the node counts as the extra marked point that all sets must avoid. For sets  $S \subset T$ , we use  $D_*^S$  and  $\psi_S^*$  to denote divisors on  $\overline{M}_{0,|T|+1}$  and we likewise use  $D_{\bullet}^S$  and  $\psi_S^{\bullet}$  for divisors on  $\overline{M}_{0,n-|T|+1}$ . Now the following formulae hold for any non-empty set  $S \subseteq [n - 1]$ .

$$D^S|_{D^T} = \begin{cases} D_*^S \otimes 1 & \text{if } S \subsetneq T \\ 1 \otimes D_{\bullet}^S & \text{if } S \cap T = \emptyset \\ 1 \otimes D_{\bullet}^{S/T} & \text{if } S \supsetneq T \\ D_*^T \otimes 1 + 1 \otimes D_{\bullet}^{T/T} & \text{if } S = T \\ 0 & \text{otherwise} \end{cases} \tag{A.2.1}$$

$$\psi_S|_{D^T} = \begin{cases} \psi_S^* \otimes 1 & \text{if } S \subsetneq T \\ 1 \otimes \psi_S^{\bullet} & \text{if } S \cap T = \emptyset \\ 1 \otimes \psi_{S/T}^{\bullet} & \text{if } S \supsetneq T \end{cases} \tag{A.2.2}$$

Note that the fourth line of (A.2.1) is another way of writing the standard fact that the restriction of  $-D^T$  to  $D^T$  is the sum of the  $\psi$ -classes at the node on the two components. Moreover, the third line of (A.2.2) shows that  $\psi_T$  restricts to  $D^T$  as the  $\psi$ -class of the node on the component corresponding to  $T^C$ . So, our definition of  $\psi_T^C$  as  $-D^T - \psi_T$  ensures that  $\psi_T^C$  restricts to the  $\psi$ -class of the node on the other component.

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Arend Bayer [bayer@math.uconn.edu](mailto:bayer@math.uconn.edu)

Department of Mathematics, University of Utah, 155 South 1400 East,  
Salt Lake City, UT 84112, USA

Current address: Department of Mathematics, University of Connecticut,  
196 Auditorium Road, Storrs, CT 06269-3009, USA

Charles Cadman [math@charlescadman.com](mailto:math@charlescadman.com)

Department of Mathematics, University of British Columbia, 1984 Mathematics Rd, Vancouver,  
BC, Canada V6P 4M8

Current address: 14401 SE 55th St, Bellevue, WA 98006, USA