

SEMICLASSICAL LIMITS OF QUANTUM AFFINE SPACES

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Abstract Semiclassical limits of generic multi-parameter quantized coordinate rings $A = \mathcal{O}_{\mathbf{q}}(k^n)$ of affine spaces are constructed and related to A , for k an algebraically closed field of characteristic zero and \mathbf{q} a multiplicatively antisymmetric matrix whose entries generate a torsion-free subgroup of k^\times . A semiclassical limit of A is a Poisson algebra structure on the corresponding classical coordinate ring $R = \mathcal{O}(k^n)$, and results of Oh, Park, Shin and the authors are used to construct homeomorphisms from the Poisson-prime and Poisson-primitive spectra of R onto the prime and primitive spectra of A . The Poisson-primitive spectrum of R is then identified with the space of symplectic cores in k^n in the sense of Brown and Gordon, and an example is presented (over \mathbb{C}) for which the Poisson-primitive spectrum of R is not homeomorphic to the space of symplectic leaves in k^n . Finally, these results are extended from quantum affine spaces to quantum affine toric varieties.

Keywords: quantum affine space; prime and primitive spectra; Poisson-prime and Poisson-primitive spectra; symplectic core; semiclassical limit; quantum toric variety

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1. Introduction

This paper is a study of ideal theory in quantum affine n -space and, more generally, in quantum toric varieties. The focus is on the relationship between the prime and primitive spectra of these non-commutative algebras and the Poisson spectra of corresponding commutative semiclassical limits.

1.1. History and context

A basic principle of the *orbit method* is that given a non-commutative algebra A , one should associate to A an algebraic variety V with a Poisson structure and should relate the primitive ideals of A to the symplectic leaves in V . This idea first arose in Lie theory, with the enveloping algebras $A = U(\mathfrak{g})$ of finite-dimensional complex Lie algebras \mathfrak{g} providing fundamental examples. The symmetric algebra $S(\mathfrak{g})$ has a Poisson bracket induced from the Lie bracket on \mathfrak{g} , and the identification of $S(\mathfrak{g})$ with the coordinate ring $\mathcal{O}(\mathfrak{g}^*)$ turns the affine space \mathfrak{g}^* into a Poisson variety, equipped with the *KKS*

(Kirillov–Kostant–Souriau) Poisson structure. In this setting, the Poisson algebra $S(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^*)$ is the *semiclassical limit* of $U(\mathfrak{g})$, because its Poisson bracket can be obtained from the identification of this algebra with the associated graded algebra of $U(\mathfrak{g})$ (with respect to the standard filtration). (See § 1.3 for the semiclassical limit process.) A famous theorem of Kirillov, Kostant and Souriau shows that the symplectic leaves in \mathfrak{g}^* coincide with the coadjoint orbits of the associated simply connected Lie group. If \mathfrak{g} is solvable and algebraic, the Dixmier map gives a homeomorphism from the space of symplectic leaves of \mathfrak{g}^* (equipped with the quotient Zariski topology) onto the primitive ideal space $\text{Prim} U(\mathfrak{g})$.

Analogous patterns are posited for quantum groups (for which a variant of the semiclassical limit is appropriate), particularly for quantized coordinate rings of algebraic varieties (e.g. [10, Introduction]). One quickly sees, via examples, that the best results are to be expected in *generic* cases (meaning that appropriate parameters are not roots of unity). Here a fundamental test case is $\mathcal{O}_q(G)$, the standard single parameter quantized coordinate ring of a complex semisimple algebraic group G . Hodges and Levasseur [10, 11], working with $G = \text{SL}_n(\mathbb{C})$, and Hodges *et al.* [12] extending their results to general G , constructed bijections from the space of symplectic leaves in G (relative to the semiclassical limit Poisson structure) onto the primitive ideal space of $\mathcal{O}_q(G)$. (The structure of the primitive ideals of $\mathcal{O}_q(G)$ underlying these bijections was also developed by Joseph [14, 15].) It is an open problem (solved only in the easy case $G = \text{SL}_2(\mathbb{C})$) whether homeomorphisms can be constructed. The problem may be alternatively expressed in terms of topological quotients. The above results give surjections $G \rightarrow \text{Prim } \mathcal{O}_q(G)$ whose fibres are the symplectic leaves in G , and the question becomes: does there exist such a surjection for which $\text{Prim } \mathcal{O}_q(G)$ has the quotient topology?

We raised the corresponding problem for other quantized coordinate rings in the following form [6, 9]: if A is a generic quantized coordinate ring of an algebraic variety V , is $\text{Prim } A$ a topological quotient of V , and is the prime spectrum $\text{Spec } A$ a topological quotient of $\text{Spec } \mathcal{O}(V)$? We proved that these indeed hold for quantum tori and quantum affine spaces [9]. (In fact, these results hold in non-generic cases as well, modulo a small technical assumption.) Later, Oh *et al.* [17] showed that the maps constructed in [9] induce homeomorphisms from the spaces of Poisson-primitive and Poisson-prime ideals in coordinate rings of tori and affine spaces onto the primitive and prime spectra of corresponding generic quantized coordinate rings. However, the Poisson structures in these results were not exhibited as semiclassical limits.

1.2. Results of this paper

Our purposes here are threefold. First, we construct a semiclassical limit $R = \mathcal{O}(k^n)$ of the quantum coordinate ring $A = \mathcal{O}_q(k^n)$, when the quantizing parameters generate a torsion-free group, such that the above-cited results of Oh *et al.* can be applied to give homeomorphisms, respectively, from the Poisson-prime and Poisson-primitive spectra of R onto the prime and primitive spectra of A . Furthermore, since our homeomorphisms occur in generic settings, the explicit descriptions of the maps involved can be somewhat simplified, and we take the opportunity to do this. Second, we show that the Poisson-

primitive ideals occurring in the coordinate rings $\mathcal{O}(k^n)$ here correspond to the Poisson cores in the affine space k^n , in the sense of Brown and Gordon [5], and do not always correspond to symplectic leaves in general (over \mathbb{C}). Finally, we extend the preceding results to quantum affine toric varieties, as we did for topological quotients in [6, 9].

The paper is organized as follows. In §2, the semiclassical limit is constructed. In §3, the homeomorphisms are presented. Section 3 also contains an example showing that primitive ideals and symplectic leaves do not necessarily correspond bijectively. In §4, the special case in which the q_{ij} are all powers of a single parameter is considered. Finally, §5 contains the generalizations to quantum affine toric varieties and related algebras.

1.3. Recall that a *Poisson algebra* over a field k is a commutative k -algebra R equipped with a *Poisson bracket*, that is, a k -bilinear map $\{\cdot, \cdot\} : R \times R \rightarrow R$ such that $(R, \{\cdot, \cdot\})$ is a Lie algebra and such that $\{\cdot, \cdot\}$ is a derivation in each variable. These can arise as semiclassical limits in the following two ways.

First, suppose that \mathcal{A} is a non-negatively filtered k -algebra whose associated graded ring, $R = \text{gr } \mathcal{A}$, is commutative. Given homogeneous elements $r \in \text{gr}_i \mathcal{A}$ and $s \in \text{gr}_j \mathcal{A}$, choose representatives $\hat{r} \in \mathcal{A}_i$ and $\hat{s} \in \mathcal{A}_j$. The commutativity of R implies that $[\hat{r}, \hat{s}] \in \mathcal{A}_{i+j-1}$, and we set $\{r, s\}$ equal to the coset of $[\hat{r}, \hat{s}]$ in $\text{gr}_{i+j-1} \mathcal{A}$. This provides a well-defined Poisson bracket on R , and the resulting Poisson algebra is called the *semiclassical limit* of \mathcal{A} .

For the second construction, suppose that \mathcal{A} is a k -algebra and that $h \in \mathcal{A}$ is a central non-zero-divisor such that $R = \mathcal{A}/h\mathcal{A}$ is commutative. Given any $r, s \in R$, choose representatives $\hat{r}, \hat{s} \in \mathcal{A}$. Then $[\hat{r}, \hat{s}]$ is uniquely divisible by h in \mathcal{A} , and we set $\{r, s\}$ equal to the coset $(1/h)[\hat{r}, \hat{s}] + h\mathcal{A}$ in R . We again obtain a well-defined Poisson bracket on R , and this Poisson algebra is viewed as the *semiclassical limit* of \mathcal{A} . The algebra \mathcal{A} may be thought of as a family of deformations of R , namely the algebras $\mathcal{A}_q = \mathcal{A}/(h - q)\mathcal{A}$ for $q \in k$. By abuse of terminology, R is also referred to as a semiclassical limit of one of the algebras \mathcal{A}_q , when q is suitably generic.

2. Construction of the semiclassical limit

2.1. Set-up

Throughout the paper we make the following assumptions.

- (i) k is an algebraically closed field of characteristic zero (with group of units k^\times).
- (ii) $q \neq 0, 1$ is an element of k .
- (iii) n is a positive integer, and $\mathbf{q} = (q_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix over k (i.e. $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1} \in k^\times$ for $1 \leq i, j \leq n$).
- (iv) The multiplicative subgroup $\langle q_{ij} \rangle = \langle q_{ij} \mid 1 \leq i, j \leq n \rangle$ of k^\times is torsion free. (Note that the rank of this free abelian group can be no larger than $n(n-1)/2$.)

- (v) $A = \mathcal{O}_q(k^n)$ is the k -algebra presented by generators x_1, \dots, x_n and relations $x_i x_j = q_{ij} x_j x_i$ for $1 \leq i, j \leq n$. This algebra is commonly referred to as a *multi-parameter quantum affine n -space over k* .
- (vi) $K = k[z]_{\langle (z-1)(z-q) \rangle}$ is the localization of a polynomial ring $k[z]$ at the semi-maximal ideal $\langle (z-1)(z-q) \rangle$. (If desired, K can be replaced by a finitely generated subalgebra, as noted in §2.6.) Write K^\times for the group of units of K . Note that there are well-defined evaluation maps

$$\gamma_1 : K \rightarrow k \quad \text{and} \quad \gamma_q : K \rightarrow k,$$

given by $\gamma_1(f) = f(1)$ and $\gamma_q(f) = f(q)$. Moreover, if $f \in K^\times$, then $\gamma_1(f), \gamma_q(f) \in k^\times$.

Our goal is to realize $\text{Spec } A$ via a suitable semiclassical limit. Our approach depends essentially on [17], which in turn relies on [8, 9, 16].

2.2. We now follow [9, §4] and [8, §1] (cf. [4, §4]).

- (i) Let $\Gamma = \mathbb{Z}^n$, with standard basis elements $\varepsilon_1, \dots, \varepsilon_n$. For $s = (s_1, \dots, s_n)$ and $t = (t_1, \dots, t_n)$ in Γ , set

$$\sigma(s, t) = \prod_{i,j=1}^n q_{ij}^{s_i t_j}.$$

Then $\sigma : \Gamma \times \Gamma \rightarrow k^\times$ is an alternating bicharacter:

$$\sigma(s, s) = 1, \quad \sigma(s, t) = \sigma(t, s)^{-1}, \quad \sigma(s, t + u) = \sigma(s, t)\sigma(s, u)$$

for $s, t, u \in \Gamma$. Moreover, the subgroup $\langle \text{im } \sigma \rangle$ of k^\times is equal to $\langle q_{ij} \rangle$.

- (ii) Let Γ^+ denote the submonoid of Γ of n -tuples without negative entries. For $s = (s_1, \dots, s_n)$ and $t = (t_1, \dots, t_n)$ in Γ^+ , let x^s denote the monomial

$$x_1^{s_1} \cdots x_n^{s_n} \in A.$$

Note, for all $s, t \in \Gamma^+$, that

$$x^s x^t = \sigma(s, t) x^t x^s.$$

Also note that $q_{ij} = \sigma(\varepsilon_i, \varepsilon_j)$ for $1 \leq i, j, \leq n$.

2.3. Since $\langle q_{ij} \rangle$ is torsion free, $-1 \notin \langle q_{ij} \rangle$. Hence, it follows from [9, Lemma 4.2] that there exists an alternating bicharacter $c : \Gamma \times \Gamma \rightarrow k^\times$ such that $\sigma(s, t) = c(s, t)^2$ for all $s, t \in \Gamma$, and such that $\sigma(s, t) = 1$ if and only if $c(s, t) = 1$. In the proof of [9, Lemma 4.2], c is constructed so that the subgroup $\Lambda = \langle \text{im } c \rangle$ of k^\times is contained in a divisible hull of $\langle q_{ij} \rangle$. Since $\langle q_{ij} \rangle$ is torsion free, so is its divisible hull, and therefore Λ is torsion free.

2.4. We now form the twisted monoid algebra $k^c\Gamma^+$, with k -basis $\{x^s \mid s \in \Gamma^+\}$, and with multiplication given via $x^s * x^t = c(s, t)x^{s+t}$, for $s, t \in \Gamma^+$. (The notational overlap with § 2.2 (ii) will be resolved momentarily.) Note that

$$x^s * x^t = c(s, t)^2 x^t * x^s = \sigma(s, t)x^t * x^s$$

for $s, t \in \Gamma^+$. Hence, the assignment $x_i \mapsto x^{\varepsilon_i}$, for $1 \leq i \leq n$, induces an isomorphism from A onto $k^c\Gamma^+$.

Henceforth, we identify A with $k^c\Gamma^+$, via the above isomorphism. In particular, the monomial x^s of § 2.2 (ii), for $s \in \Gamma^+$, is identified with the basis element x^s of $k^c\Gamma^+$.

2.5. Suppose that $\lambda_1, \dots, \lambda_m$ form a basis for A , with

$$c(s, t) = \lambda_1^{\ell_1(s,t)} \dots \lambda_m^{\ell_m(s,t)}$$

for $s, t \in \Gamma$, and for suitable (unique) alternating biadditive maps $\ell_i : \Gamma \times \Gamma \rightarrow \mathbb{Z}$.

2.6. The field k , being algebraically closed, must be infinite dimensional over \mathbb{Q} . Choose \mathbb{Q} -linearly independent elements $\mu_1, \dots, \mu_m \in k$. Observe that the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ q^2 & q & 1 \\ 2 & 1 & 0 \end{bmatrix} \in M_3(k)$$

has determinant $(q - 1)^2 \neq 0$ and so is invertible. Hence, for $1 \leq i \leq m$, there are unique scalars $a_i, b_i, c_i \in k$ such that the quadratic polynomial

$$f_i(z) = a_i z^2 + b_i z + c_i$$

satisfies the following conditions:

$$f_i(1) = 1, \quad f_i(q) = \lambda_i, \quad f'_i(1) = \mu_i,$$

where $f'(z)$ denotes the formal derivative of a rational function $f(z) \in k(z)$. The displayed properties are all that we require of the polynomials f_i . In particular, they need not be quadratic.

Note that $f_1, \dots, f_m \in K^\times$, because neither $z - 1$ nor $z - q$ is a factor of any f_i . Since these are the key properties needed for K , we could replace K by the affine algebra $k[z][f_1^{-1}, \dots, f_m^{-1}]$, if desired.

Further, set

$$\tilde{c}(s, t) = f_1^{\ell_1(s,t)} \dots f_m^{\ell_m(s,t)}$$

for $s, t \in \Gamma$, where ℓ_1, \dots, ℓ_m are as in § 2.5. Then $\tilde{c} : \Gamma \times \Gamma \rightarrow K^\times$ is an alternating bicharacter, such that

$$\tilde{c}(s, t)(q) = \lambda_1^{\ell_1(s,t)} \dots \lambda_m^{\ell_m(s,t)} = c(s, t)$$

for all $s, t \in \Gamma$.

Set $\tilde{A} = \langle \text{im } \tilde{c} \rangle = \langle f_1, \dots, f_m \rangle \subset K^\times$.

Lemma 2.1.

(i) *The specialization $z \mapsto q$ induces a group isomorphism*

$$\gamma_q : \tilde{\Lambda} \xrightarrow{f(z) \mapsto f(q)} \Lambda.$$

(ii) *The elements f_1, \dots, f_m form a basis for $\tilde{\Lambda}$, and so $\tilde{\Lambda}$ is free abelian of rank m .*

Proof. Consider the group homomorphism

$$\gamma_q : K^\times \xrightarrow{f(z) \mapsto f(q)} k^\times.$$

Note, for $1 \leq i \leq m$, that $\gamma_q(f_i) = \lambda_i$. Since f_1, \dots, f_m generate $\tilde{\Lambda}$ and $\lambda_1, \dots, \lambda_m$ form a basis for Λ , we see both that the f_i form a basis for $\tilde{\Lambda}$ and that γ_q maps $\tilde{\Lambda}$ isomorphically onto Λ . □

Remark 2.2. By the construction, $\gamma_q(\tilde{c}(s, t)) = c(s, t)$, for all $s, t \in \Gamma$.

Let k^+ denote the additive group underlying the field k .

Lemma 2.3. *The rule $\psi(f) = f'(1)$ gives a well-defined injective group homomorphism*

$$\psi : \tilde{\Lambda} \xrightarrow{f(z) \mapsto f'(1)} k^+.$$

Proof. Observe that K is closed under formal differentiation, and so $f'(1)$ is defined for $f \in K$. Thus, ψ is a well-defined map from $\tilde{\Lambda}$ to k . Now $f_i(1) = 1$ for $1 \leq i \leq m$, and so $f(1) = 1$ for all $f \in \tilde{\Lambda}$. Therefore, for all $f, g \in \tilde{\Lambda}$,

$$\psi(fg) = f'(1)g(1) + f(1)g'(1) = f'(1) + g'(1) = \psi(f) + \psi(g),$$

proving that ψ is a group homomorphism. Second, for $1 \leq i \leq m$,

$$\psi(f_i) = f'_i(1) = \mu_i.$$

Since μ_1, \dots, μ_m are \mathbb{Z} -linearly independent, we conclude that ψ is injective. □

2.7. Now set $\tilde{R} = K^{\tilde{c}}\Gamma^+$, the twisted monoid K -algebra with K -basis $\{x^s \mid s \in \Gamma^+\}$ and multiplication given via $x^s * x^t = \tilde{c}(s, t)x^{s+t}$, for $s, t \in \Gamma^+$. We again use x_i to denote x^{ε_i} , for $1 \leq i \leq n$.

Recall that $q_{ij} = \sigma(\varepsilon_i, \varepsilon_j) = c(\varepsilon_i, \varepsilon_j)^2$, for $1 \leq i, j \leq n$, and set

$$\tilde{q}_{ij} = \tilde{q}_{ij}(z) = \tilde{c}(\varepsilon_i, \varepsilon_j)^2 \in K^\times.$$

Then

$$\tilde{R} = K\langle x_1, \dots, x_n \mid x_i * x_j = \tilde{q}_{ij}x_j * x_i \text{ for } 1 \leq i, j \leq n \rangle.$$

For $\mu \in k^\times$, set

$$R_\mu = \tilde{R}/\langle z - \mu \rangle.$$

We see that $R_q \cong A$, since

$$\tilde{q}_{ij}(q) = \gamma_q(\tilde{q}_{ij}) = \gamma_q\tilde{c}(\varepsilon_i, \varepsilon_j) = c(\varepsilon_i, \varepsilon_j) = q_{ij}$$

for all i, j , and we use this isomorphism to identify A with R_q . Under this identification, the cosets $x_i + \langle z - q \rangle \in R_q$ correspond to the elements $x_i \in A$.

2.8. Next, consider the group homomorphism

$$\gamma_1 : K^\times \xrightarrow{f(z) \mapsto f(1)} k^\times.$$

Note that $\gamma_1(\tilde{A}) = 1$ and, in particular, that $\gamma_1(\tilde{c}(s, t)) = 1$ for all $s, t \in \Gamma$. We therefore have an isomorphism from R_1 onto $R := k[x_1, \dots, x_n]$, sending

$$x_i + \langle z - 1 \rangle \mapsto x_i,$$

and we identify R_1 with R via this map. We also identify R and R_1 with the commutative monoid algebra $k\Gamma^+$, with k -basis $\{x^s \mid s \in \Gamma^+\}$ and multiplication given by $x^s x^t = x^{s+t}$ for $s, t \in \Gamma^+$.

Since $\tilde{R}/\langle z - 1 \rangle = R$ is commutative and $z - 1$ is a central non-zero-divisor in \tilde{R} , there is a Poisson bracket on R as in § 1.3, and R becomes the semiclassical limit of $A = R_q$ (or, more accurately, the semiclassical limit of the family of algebras R_μ). The Poisson bracket on R is given by

$$\{\bar{a}, \bar{b}\} = \left. \frac{ab - ba}{z - 1} \right|_{z=1}$$

for all $a, b \in \tilde{R}$, where \bar{a} and \bar{b} denote the cosets of a and b in R . In particular,

$$\{x_i, x_j\} = \left(\left. \frac{\tilde{q}_{ij}(z) - 1}{z - 1} \right|_{z=1} \right) x_i x_j = \tilde{q}'_{ij}(1) x_i x_j$$

for $1 \leq i, j \leq n$. (The last equality holds because $\tilde{q}_{ij}(1) = \gamma_1(\tilde{q}_{ij}) = 1$.)

We treat R as a Poisson algebra in this way, for the remainder of the paper. In the notation of [17, 3.1], $R = k_u \Gamma^+$, where $u : \Gamma \times \Gamma \rightarrow k$ is the alternating biadditive map such that $u(\varepsilon_i, \varepsilon_j) = \tilde{q}'_{ij}(1)$, for $1 \leq i, j \leq n$. In this notation,

$$\{x^s, x^t\} = u(s, t) x^s x^t$$

for $s, t \in \Gamma^+$.

3. The homeomorphisms

Retain the notation of the previous section. In particular,

$$A = R_q = k\langle x_1, \dots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for } 1 \leq i, j \leq n \rangle$$

is as in § 2.7, with $q_{ij} = \tilde{q}_{ij}(q)$, and

$$R = R_1 = k[x_1, \dots, x_n]$$

is the Poisson algebra with bracket

$$\{x_i, x_j\} = \tilde{q}'_{ij}(1) x_i x_j,$$

following § 2.8. In particular, R is a semiclassical limit of A .

3.1. (We now follow [5, 7, 16, 17].) An ideal I of R is a *Poisson ideal* if $\{R, I\} \subseteq I$. We let $\text{PSpec } R$ denote the (Zariski) subspace of $\text{Spec } R$ comprised of the prime Poisson ideals. Each maximal ideal \mathfrak{m} of R contains a unique largest Poisson ideal $\mathcal{P}(\mathfrak{m})$, called the *Poisson core* of \mathfrak{m} . The Poisson cores of the maximal ideals of R are termed *Poisson-primitive ideals* [5, 3.2] (or *symplectic ideals* [16, Definition 1.2]) and are prime (see [5, 3.2] or [16, Lemma 1.3]). The subspace of $\text{PSpec } R$ consisting of the Poisson-primitive ideals will be denoted $\text{PPrim } R$. The *Poisson centre* $Z_p(R)$ is the set of $z \in R$ such that $\{R, z\} = 0$.

In our main result, Theorem 3.2, we will describe a homeomorphism from $\text{PSpec } R$ onto $\text{Spec } A$.

3.2. To proceed further, we need more of the notation of [17].

- (i) Let W denote the set of subsets of $\{1, \dots, n\}$, and let $w \in W$.
- (ii) Let I_w denote the ideal of R generated by the x_i for $i \in w$, and let Y_w denote the multiplicatively closed subset of R/I_w generated by 1 and the cosets of the x_j for $j \notin w$. Let R_w denote the localization of R/I_w at Y_w . We let each x_i also denote its image in R_w .
- (iii) Let Γ_w denote the subgroup of Γ generated by the basis elements ε_j for $j \notin w$, and let c_w denote the restriction of c , defined in § 2.3, to $\Gamma_w \times \Gamma_w$.
- (iv) Identify R_w with the group algebra $k\Gamma_w$, via $x_j \leftrightarrow x^{\varepsilon_j}$, for $j \notin w$.
- (v) Set $H = \text{Hom}(\Gamma, k^\times)$, which is a group under pointwise multiplication, isomorphic to the algebraic torus $(k^\times)^n$. This group acts on $R = k\Gamma^+$ and on $R_w = k\Gamma_w$ by k -algebra automorphisms such that

$$h \cdot x^s = \langle h, s \rangle h_s,$$

for $h \in H$ and $s \in \Gamma^+$ or $s \in \Gamma_w$. Further, set

$$S_w = \text{rad}(c_w) = \{s \in \Gamma_w \mid c_w(s, t) = 1 \text{ for all } t \in \Gamma_w\},$$

$$S_w^\perp = \{h \in H \mid \langle h, s \rangle = 1 \text{ for all } s \in S_w\}.$$

Then S_w^\perp is a subgroup of H , and it acts on both R and R_w through the H -action. To match the notation of [6, 3.4], let σ_w denote the restriction of $\sigma = c^2$ to Γ_w . The conditions on c in § 2.3 then imply that

$$S_w = \{s \in \Gamma_w \mid \sigma_w(s, t) = 1 \text{ for all } t \in \Gamma_w\} = \text{rad}(\sigma_w).$$

- (vi) Let $\text{Spec}_w R$ denote the set of prime ideals of R that contain x_i for $i \in w$ but do not contain x_j for $j \notin w$. Localization provides a natural homeomorphism between $\text{Spec}_w R$ and $\text{Spec } R_w$, and this homeomorphism is H -equivariant.
- (vii) For $P \in \text{Spec}_w R$, let $(P : S_w^\perp)$ denote the intersection of the prime ideals in the S_w^\perp -orbit of P .

3.3. Both R and A have k -basis $\{x^s \mid s \in \Gamma^+\}$. We will use $\Phi : A \rightarrow R$ to denote the k -linear isomorphism such that $\Phi(x^s) = x^s$ for all $s \in \Gamma^+$.

3.4.

(i) For $w \in W$, set

$$\begin{aligned} \text{PSpec}_w R &= \text{PSpec } R \cap \text{Spec}_w R, \\ \text{PPrim}_w R &= \text{PPrim } R \cap \text{Spec}_w R. \end{aligned}$$

(ii) The maps $\text{PSpec } R \rightarrow \text{Spec } A$ and $\text{PPrim } R \rightarrow \text{Prim } A$ used in [17, Theorem 3.5] are defined by the formula

$$P \mapsto \Phi^{-1}(P : S_w^\perp)$$

for $w \in W$ and $P \in \text{PSpec}_w R$. Moreover, as mentioned in the proof of [17, Proposition 3.4], $(P : S_w^\perp) = P$ for all $P \in \text{PSpec}_w R$ under our present hypotheses, and so the formula reduces to

$$P \mapsto \Phi^{-1}(P).$$

Since this key point is somewhat hidden in [17], we excerpt the result and its proof in the next lemma. Before doing so, however, we need the following ingredient.

(iii) Recall the isomorphism γ_q of Lemma 2.1 and the homomorphism ψ of Lemma 2.3. Let φ denote the composite homomorphism

$$A \xrightarrow{\gamma_q^{-1}} \tilde{A} \xrightarrow{f \mapsto f^2} \tilde{A} \xrightarrow{\psi} k^+,$$

which is injective because $\varphi(\lambda_i) = \psi(f_i^2) = 2\mu_i$ for $1 \leq i \leq n$. Recalling u from § 2.8, observe that

$$\varphi(c(\varepsilon_i, \varepsilon_j)) = \psi(\gamma_q^{-1}(c(\varepsilon_i, \varepsilon_j))^2) = \psi\gamma_q^{-1}(q_{ij}) = \psi(\tilde{q}_{ij}) = \tilde{q}'_{ij}(1) = u(\varepsilon_i, \varepsilon_j)$$

for $1 \leq i, j \leq n$, and so $u = \varphi c$.

Lemma 3.1 (Oh et al. [17]). *Let $w \in W$ and let P be a Poisson-prime ideal in $\text{Spec}_w R$. Then P is stable under the action of S_w^\perp . Consequently, $P = (P : S_w^\perp)$.*

Proof. Let u_w denote the restriction of u to Γ_w , and note that $u_w = \varphi c_w$. The induced Poisson structure on the localization $R_w = k\Gamma_w$ satisfies

$$\{x^s, x^t\} = u_w(s, t)x^s x^t$$

for $s, t \in \Gamma_w^+$, and so we have $R_w = k_{u_w} \Gamma$.

The Poisson-prime ideal P/I_w in R/I_w induces a Poisson-prime ideal Q in R_w , which contracts to a prime ideal Q' in the Poisson centre $Z_p(R_w)$. By [18, Lemma 1.2] (which is valid over any base field of characteristic zero), Q is generated by Q' , and $Z_p(R_w)$ equals the group algebra of the radical of u_w . However,

$$\text{rad}(u_w) = \{s \in \Gamma_w \mid u_w(s, t) = 0 \text{ for all } t \in \Gamma_w\} = \text{rad}(c_w) = S_w,$$

because $u_w = \varphi c_w$ and φ is injective. Thus, $Z_p(R_w) = kS_w$. By definition of S_w^\perp , this group acts trivially on kS_w , and so it fixes Q' (pointwise). Therefore, S_w^\perp stabilizes Q , and hence also P/I_w and finally P . \square

We now apply [17, Theorem 3.5]. Note that the following theorem asserts, in particular, that the linear map $\Phi^{-1} : R \rightarrow A$ sends Poisson-prime ideals of R to prime ideals of A .

Theorem 3.2. *Let $A = \mathcal{O}_{\mathbf{q}}(k^n)$, where k is an algebraically closed field of characteristic zero, and where $\mathbf{q} = (q_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix over k such that the group $\langle q_{ij} \rangle \subseteq k^\times$ is torsion free. Let $R = k[x_1, \dots, x_n]$, equipped with the Poisson structure described in § 2.8, and let $\Phi : A \rightarrow R$ be the k -linear isomorphism of § 3.3. Then the rule $P \mapsto \Phi^{-1}(P)$ determines a homeomorphism*

$$\text{PSpec } R \rightarrow \text{Spec } A,$$

which restricts to a homeomorphism

$$\text{PPrim } R \rightarrow \text{Prim } A.$$

Proof. We have a homomorphism φ , from § 3.4 (iii), and an alternating biadditive map u , from § 2.8, exactly as described in [17, 2.3]. We can identify R as a Poisson k -algebra with $k_u \Gamma^+$, following the notation of [17, 3.1] (see § 2.8). Similarly, we have identified A as a k -algebra with $k^c \Gamma^+$, also following the notation of [17, 3.1] (see § 2.4). The theorem now follows directly from Lemma 3.1 and [17, Theorem 3.5]. \square

3.5.

- (i) Identify the affine space k^n with the maximal ideal space of R . The rule $\mathfrak{m} \mapsto \mathcal{P}(\mathfrak{m})$ gives a surjective map $k^n \rightarrow \text{PPrim } R$, the fibres of which are called *symplectic cores* [5, 3.3], and are algebraic analogs of symplectic leaves (cf. [5, 3.3, 3.5, 3.7]). Specifically, the symplectic core containing a point \mathfrak{m} is the set

$$\mathcal{C}(\mathfrak{m}) = \{\mathfrak{m}' \in k^n \mid \mathcal{P}(\mathfrak{m}') = \mathcal{P}(\mathfrak{m})\}.$$

- (ii) Let $\text{SymCr } k^n$ denote the set of symplectic cores in k^n . The rule $\mathfrak{m} \mapsto \mathcal{C}(\mathfrak{m})$ gives a surjective map $k^n \rightarrow \text{SymCr } k^n$, and we give $\text{SymCr } k^n$ the quotient (Zariski) topology via this map. By the definition of symplectic cores, there is a bijection $\text{SymCr } k^n \rightarrow \text{PPrim } R$ such that $\mathcal{C}(\mathfrak{m}) \mapsto \mathcal{P}(\mathfrak{m})$ for $\mathfrak{m} \in k^n$.
- (iii) It follows from [7, Theorem 4.1 (b)] that the Zariski topology on $\text{PPrim } R$ is the quotient topology from the canonical map $k^n \rightarrow \text{PPrim } R$ in (i). (To verify the hypotheses of [7, Theorem 4.1 (b)], observe that the action of the torus H on R is a rational action by Poisson automorphisms, and observe that R has only finitely many H -stable prime Poisson ideals, namely the ideals I_w of § 3.2 (ii).) Hence, the bijection $\text{SymCr } k^n \rightarrow \text{PPrim } R$ of (ii) is a homeomorphism.

Combining these observations with Theorem 3.2, we obtain the following.

Corollary 3.3. *Under the hypotheses of Theorem 3.2, there is a homeomorphism*

$$\text{SympCr } k^n \rightarrow \text{Prim } A$$

given by the rule $\mathcal{C}(\mathfrak{m}) \mapsto \Phi^{-1}(\mathcal{P}(\mathfrak{m}))$.

The symplectic cores $\mathcal{C}(\mathfrak{m})$, which make up the points of the space $\text{SympCr } k^n$ in Corollary 3.3, have good geometric structure themselves: they are homogeneous smooth irreducible quasi-affine varieties, as the following corollary shows. In keeping with the notation of §3.4 (i), set

$$\text{Max}_w R = \text{Max } R \cap \text{Spec}_w R$$

for $w \in W$. Since we have identified $\text{Max } R$ with k^n , the sets $\text{Max}_w R$ partition k^n .

Corollary 3.4. *Let $w \in W$ and $\mathfrak{m} \in \text{Max}_w R$. Then the symplectic core $\mathcal{C}(\mathfrak{m})$ is a smooth irreducible locally closed subset of k^n , and it equals the S_w^\perp -orbit of \mathfrak{m} in k^n .*

Proof. Irreducibility and local closedness will follow from [5, Lemma 3.3] once we know that all Poisson-primitive ideals of R are locally closed points in $\text{PSpec } R$. The latter fact will follow from the Poisson Dixmier–Moeglin equivalence of [7, Theorem 4.3]. As we have already noted in §3.5 (iii), the torus H acts rationally on R by Poisson automorphisms, and there are only finitely many H -stable Poisson-prime ideals in R . Hence, the hypotheses of [7, Theorem 4.3] are satisfied. Investing that theorem into [5, Lemma 3.3], we find that $\mathcal{C}(\mathfrak{m})$ is locally closed and that its closure is the set

$$\{\mathfrak{m}' \in \text{Max } R \mid \mathfrak{m}' \supseteq \mathcal{P}(\mathfrak{m})\},$$

which is irreducible because $\mathcal{P}(\mathfrak{m})$ is a prime ideal. Therefore, $\mathcal{C}(\mathfrak{m})$ is irreducible. Lemma 3.3 of [5] also shows that $\mathcal{C}(\mathfrak{m})$ is smooth in its closure. Smoothness in k^n will follow from the general theory of algebraic group actions (e.g. [2, Proposition 1.8]), once we exhibit $\mathcal{C}(\mathfrak{m})$ as an orbit of an algebraic group.

The S_w^\perp -orbit of \mathfrak{m} appears as the fibre of the quotient map $\text{Max } R \rightarrow \text{Prim } A$ in [9, Theorem 4.11]. Hence, we need to show that this quotient map, call it μ , agrees with the one obtained in our present setting, namely the map

$$\tau : k^n \rightarrow \text{Prim } A, \quad \mathfrak{m} \mapsto \Phi^{-1}(\mathcal{P}(\mathfrak{m})),$$

which is the composition of the homeomorphism in Corollary 3.3 with the quotient map $k^n \rightarrow \text{SympCr } k^n$. By construction, the fibres of τ are the symplectic cores in k^n .

Since $\mathfrak{m} \in \text{Max}_w R$, [17, Proposition 3.4] shows that $(\mathfrak{m} : S_w^\perp) = (P : S_w^\perp)$ for some $P \in \text{PSpec}_w R$. But $(P : S_w^\perp) = P$ by Lemma 3.1, and so $(\mathfrak{m} : S_w^\perp) = P$ is a Poisson-prime ideal. In particular, $(\mathfrak{m} : S_w^\perp)$ is a Poisson ideal contained in \mathfrak{m} , whence $(\mathfrak{m} : S_w^\perp) \subseteq \mathcal{P}(\mathfrak{m})$. On the other hand, the Poisson-primitive ideal $\mathcal{P}(\mathfrak{m})$ is S_w^\perp -stable by Lemma 3.1. Since it is contained in \mathfrak{m} , it must be contained in $(\mathfrak{m} : S_w^\perp)$. Therefore, $\mathcal{P}(\mathfrak{m}) = (\mathfrak{m} : S_w^\perp)$, and we conclude that

$$\tau(\mathfrak{m}) = \Phi^{-1}(\mathfrak{m} : S_w^\perp).$$

This shows that τ agrees with μ , as desired. By [9, Theorem 4.11], the fibres of μ over points in $\text{Prim}_w A$ consist precisely of the S_w^\perp -orbits within k^n . Therefore, $\mathcal{C}(\mathfrak{m})$ equals the S_w^\perp -orbit of \mathfrak{m} .

Note from its definition that S_w^\perp is a closed subgroup of the torus H , so that it is an affine algebraic group. Since H acts rationally on R , its induced action on $\text{Max } R = k^n$ is morphic, as is the corresponding action of S_w^\perp . Standard results (e.g. [2, Proposition 1.8]) thus imply that the S_w^\perp -orbit $\mathcal{C}(\mathfrak{m})$ is smooth (and locally closed). \square

When $k = \mathbb{C}$, an affine variety equipped with a Poisson structure can also be partitioned into symplectic leaves (e.g. see [5, 3.5]), and it has been a goal of research in quantum groups to represent primitive spectra of quantized algebras as spaces of symplectic leaves. This correspondence between primitive ideals and symplectic leaves, however, can break down when the symplectic leaves are not algebraic (i.e. not locally closed in the Zariski topology), as noted by Hodges *et al.* [12, p. 52], Vancliff [18, Theorem 3.8] and Brandl [3, Example 6.4]. The following provides an explicit example of this phenomenon, in the form of a quantum affine 3-space.

Example 3.5. (i) Let $k = \mathbb{C}$, choose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and choose $q \in \mathbb{C}$ transcendental over the field $\mathbb{Q}(\alpha)$. We would like to construct an example of the form $\mathcal{O}_q(\mathbb{C}^3)$, with some $q_{ij} = q^\alpha$. However, that would require working with z^α in our semiclassical limit construction § 2.8, and we cannot form z^α in K . To replace z^α , we use the first-order Taylor approximation $1 + \alpha(z - 1)$, and consequently we use $1 + \alpha(q - 1)$ in place of q^α in the defining relations for this example. Because q is transcendental over $\mathbb{Q}(\alpha)$, the elements $\lambda_1 = q$ and $\lambda_2 = 1 + \alpha(q - 1)$ generate a free abelian subgroup of \mathbb{C}^\times of rank 2.

We now take

$$\mathbf{q} = \begin{bmatrix} 1 & \lambda_1^2 & \lambda_2^2 \\ \lambda_1^{-2} & 1 & 1 \\ \lambda_2^{-2} & 1 & 1 \end{bmatrix}$$

and form $A = \mathcal{O}_{\mathbf{q}}(\mathbb{C}^3)$.

(ii) The primitive spectrum of A is easily calculated by the methods of [8], as follows. Let W denote the set of subsets of $\{1, 2, 3\}$, and for $w \in W$ set

$$\begin{aligned} J_w &= \langle x_i \mid i \in w \rangle \in \text{Spec } A, \\ A_w &= (A/J_w)[x_j^{-1} \mid j \notin w], \\ \text{Prim}_w A &= \{P \in \text{Prim } A \mid P \cap \{x_1, x_2, x_3\} = \{x_i \mid i \in w\}\}, \\ S_w &= \left\{ a \in \mathbb{Z}^3 \mid a_i = 0 \text{ for } i \in w \text{ and } \prod_{i \notin w} q_{ij}^{a_i} = 1 \text{ for all } j \notin w \right\}. \end{aligned}$$

Then $\text{Prim } A$ is the disjoint union of the sets $\text{Prim}_w A$, and each $\text{Prim}_w A$ is homeomorphic to $\text{Prim } A_w$ via localization and contraction [8, Theorem 2.3]. Moreover, since A_w is a quantum torus, $\text{Prim } A_w$ consists precisely of the ideals induced from maximal ideals of the centre $Z(A_w)$ [8, Corollary 1.5], and $Z(A_w)$ is spanned by the (cosets of the) monomials x^a for $a \in S_w$ (e.g. [8, Lemma 1.2]). In particular, when $Z(A_w) = \mathbb{C}$

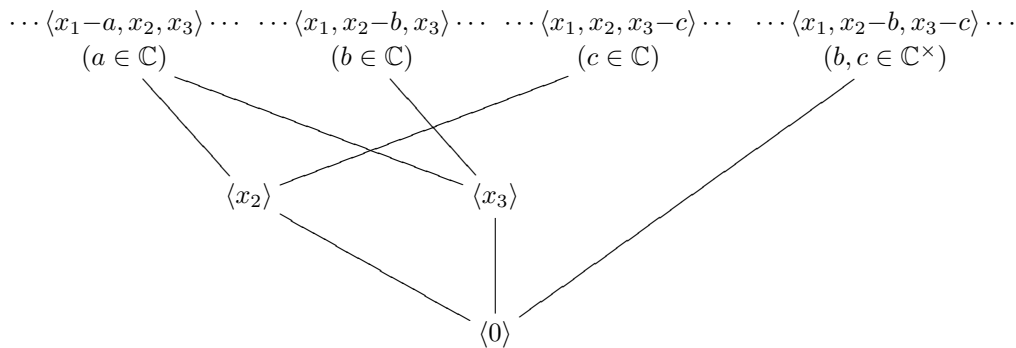
(equivalently, when A_w is simple), $\text{Prim}_w A$ consists of just the ideal J_w . In the present example, this occurs in the cases $w = \emptyset, \{2\}, \{3\}, \{1, 2, 3\}$. Hence,

$$\begin{aligned} \text{Prim}_\emptyset A &= \{\langle 0 \rangle\}, & \text{Prim}_{\{2\}} A &= \{\langle x_2 \rangle\}, \\ \text{Prim}_{\{3\}} A &= \{\langle x_3 \rangle\}, & \text{Prim}_{\{1,2,3\}} A &= \{\langle x_1, x_2, x_3 \rangle\}. \end{aligned}$$

In the remaining four cases, A_w is a commutative Laurent polynomial ring over \mathbb{C} , and so we obtain

$$\begin{aligned} \text{Prim}_{\{1\}} A &= \{\langle x_1, x_2 - b, x_3 - c \mid b, c \in \mathbb{C}^\times \rangle\}, \\ \text{Prim}_{\{1,2\}} A &= \{\langle x_1, x_2, x_3 - c \mid c \in \mathbb{C}^\times \rangle\}, \\ \text{Prim}_{\{1,3\}} A &= \{\langle x_1, x_2 - b, x_3 \mid b \in \mathbb{C}^\times \rangle\}, \\ \text{Prim}_{\{2,3\}} A &= \{\langle x_1 - a, x_2, x_3 \mid a \in \mathbb{C}^\times \rangle\}. \end{aligned}$$

Hence, $\text{Prim} A$ may be pictured as follows:



(iii) In setting up the semiclassical limit, we may choose c as in § 2.3 so that

$$c(\varepsilon_1, \varepsilon_2) = \lambda_1, \quad c(\varepsilon_1, \varepsilon_3) = \lambda_2, \quad c(\varepsilon_2, \varepsilon_3) = 1.$$

In view of (i), the group $\Lambda = \langle \text{im } c \rangle$ is free abelian with a basis λ_1, λ_2 . Since α is irrational, we may (and do) choose $\mu_1 = 1$ and $\mu_2 = \alpha$. The polynomials f_i of § 2.6 are then given by

$$f_1(z) = z, \quad f_2(z) = 1 + \alpha(z - 1),$$

whence $\tilde{q}_{12}(z) = z^2$ and $\tilde{q}_{13}(z) = (1 + \alpha(z - 1))^2$, while $\tilde{q}_{23}(z) = 1$. Consequently,

$$\tilde{q}'_{12}(1) = 2, \quad \tilde{q}'_{13}(1) = 2\alpha, \quad \tilde{q}'_{23}(1) = 0.$$

(iv) The semiclassical limit of A in this example is thus $R = \mathbb{C}[x_1, x_2, x_3]$, equipped with the Poisson structure such that

$$\{x_1, x_2\} = 2x_1x_2, \quad \{x_1, x_3\} = 2\alpha x_1x_3, \quad \{x_2, x_3\} = 0.$$

This is a quadratic analog of the KKS Poisson structure on the dual of the standard example of a non-algebraic solvable Lie algebra (see, for example, [19, Example 2.43]).

The Poisson-primitive ideals of R can be computed via the Poisson analogue of the methods of [8] (see [7, Theorems 4.2, 4.3]). These ideals are given by the same sets of generators as the primitive ideals of A described in (ii); in particular, the picture of $\text{Prim } A$ obtained there serves also as a picture of $\text{PPrim } R$. From this picture, we see that the symplectic cores in \mathbb{C}^3 for the Poisson structure under consideration are the following sets:

- (a) the individual points on the x_1 -axis;
- (b) the individual points in the x_2x_3 -plane;
- (c) the x_1x_2 -plane with the x_1 - and x_2 -axes removed;
- (d) the x_1x_3 -plane with the x_1 - and x_3 -axes removed;
- (e) the space \mathbb{C}^3 with the three coordinate planes removed.

(v) Finally, we indicate how to find the symplectic leaves for this Poisson structure on \mathbb{C}^3 . These are not all algebraic, just as in the case of the KKS Poisson structure on the dual of a non-algebraic solvable Lie algebra (cf. [19, Example 2.43 and discussion on p. 67] and [5, Remark 1, p. 203]). We first recall from [5, Proposition 3.6 (1)] that each symplectic core is a union of symplectic leaves. This immediately implies that those individual points which are symplectic cores are also symplectic leaves. We determine the other leaves by considering Hamiltonian paths, as follows.

In full, the Poisson bracket on R is given by the formula

$$\{f, g\} = 2x_1x_2 \left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1} \right) + 2\alpha x_1x_3 \left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_3} - \frac{\partial f}{\partial x_3} \frac{\partial g}{\partial x_1} \right),$$

which also defines the unique extension of the bracket to the algebra S of smooth complex functions on \mathbb{C}^3 . For $f \in S$, the derivation $H_f = \{f, \cdot\}$ on S gives a smooth vector field on \mathbb{C}^3 , and the flows (integral curves) of such vector fields H_f are the *Hamiltonian paths* for the given Poisson structure.* By definition [20, p. 529], the symplectic leaves of \mathbb{C}^3 are the equivalence classes for the relation ‘connected by piecewise Hamiltonian paths’.

Paths of the form $\mathbf{c}(t) = (a, be^{at}, ce^{\alpha at})$ with $a, b, c \in \mathbb{C}$ are flows of $H_{x_1/2}$, and paths $\mathbf{c}(t) = (ae^{bt}, b, c)$ with $a, b, c \in \mathbb{C}$ are flows of $H_{-x_2/2}$. It follows that the third and fourth symplectic cores listed in (iv) are connected with respect to piecewise Hamiltonian paths, and hence these cores are also symplectic leaves. Moreover, each of the surfaces

$$\Sigma_d = \{(a_1, a_2, a_3) \in (\mathbb{C}^\times)^3 \mid a_3 = da_2^\alpha\},$$

for $d \in \mathbb{C}^\times$, is connected with respect to piecewise Hamiltonian paths. On the other hand, any Hamiltonian path within $(\mathbb{C}^\times)^3$ must satisfy $\dot{x}_3(t)/x_3(t) = \alpha \dot{x}_2(t)/x_2(t)$ and so must

* Specifically, a flow of H_f is a path $\mathbf{c}(t) = (x_1(t), x_2(t), x_3(t))$ such that

$$\dot{\mathbf{c}}(t) = H_f \mathbf{c}(t) = \left(-2x_1x_2 \frac{\partial f}{\partial x_2} - 2\alpha x_1x_3 \frac{\partial f}{\partial x_3}, 2x_1x_2 \frac{\partial f}{\partial x_1}, 2\alpha x_1x_3 \frac{\partial f}{\partial x_1} \right),$$

where the dot denotes d/dt .

be contained within one of the Σ_d . Therefore, the Σ_d are the remaining symplectic leaves of \mathbb{C}^3 . These form a one-parameter family of non-algebraic surfaces whose union is a single symplectic core.

We conclude that, while $\text{Prim } A$ is homeomorphic to $\text{SympCr } \mathbb{C}^3$ (Corollary 3.3), it is not homeomorphic to the space of symplectic leaves in \mathbb{C}^3 . For instance, $\text{Prim } A$ has just one dense point, while the space of symplectic leaves in \mathbb{C}^3 (equipped with the quotient Zariski topology) has uncountably many.

4. Uniparameter quantum affine spaces

The Poisson structure on k^n in Theorem 3.2 can be given more explicitly in the uniparameter case, namely when the scalars q_{ij} are powers of a single scalar which is not a root of unity. We treat this scalar as a square (as we may, since k is algebraically closed), and write it in the form q^2 to match our existing notation.

4.1.

- (i) Assume that the scalar $q \in k^\times$ is not a root of unity. Let $\mathbf{r} = (r_{ij})$ be an additively antisymmetric $n \times n$ matrix over \mathbb{Z} , and take $q_{ij} = q^{2r_{ij}}$ for all i, j . Assume that $\mathbf{r} \neq \mathbf{0}$, so that at least one $q_{ij} \neq 1$.
- (ii) Under the present assumptions, the natural choice for the multiplicatively antisymmetric bicharacter c , as in § 2.3, is to define it so that

$$c(\varepsilon_i, \varepsilon_j) = q^{r_{ij}}$$

for all i, j . The group $\Lambda = \langle \text{im } c \rangle$ is then cyclic, of the form $\langle q^r \rangle$, where r is the greatest common divisor of the integers r_{ij} . Note that $r \neq 0$ because $\mathbf{r} \neq \mathbf{0}$.

- (iii) Take $\lambda_1 = q^r$ as the basis element for Λ and let ℓ_1 be the corresponding alternating biadditive map on Γ as in § 2.5, so that $c(s, t) = q^{r\ell_1(s, t)}$ for $s, t \in \Gamma$. Thus, $\ell_1(\varepsilon_i, \varepsilon_j) = r_{ij}/r$ for $1 \leq i, j \leq n$.
- (iv) Since $r \neq 0$, we may take $\mu_1 = r$. It is most convenient to take a possibly non-quadratic choice for the polynomial f_1 in § 2.6, namely $f_1(z) = z^r$. It is easily seen that this satisfies the desired conditions:

$$f_1(1) = 1, \quad f_1(q) = \lambda_1, \quad f'_1(1) = \mu_1.$$

Define \tilde{c} as in § 2.6, and observe that

$$\tilde{c}(\varepsilon_i, \varepsilon_j) = z^{r\ell_1(\varepsilon_i, \varepsilon_j)} = z^{r_{ij}}$$

for $1 \leq i, j \leq n$.

- (v) Defining \tilde{q}_{ij} as in § 2.7, we have $\tilde{q}_{ij}(z) = \tilde{c}(\varepsilon_i, \varepsilon_j)^2 = z^{2r_{ij}}$ for $1 \leq i, j \leq n$. Consequently,

$$\tilde{q}'_{ij}(1) = 2r_{ij}$$

for $1 \leq i, j \leq n$.

In view of § 4.1, the uniparameter cases of Theorem 3.2 and Corollary 3.3 can be stated as follows.

Theorem 4.1. *Let $A = \mathcal{O}_{\mathbf{q}}(k^n)$, assuming that k is an algebraically closed field of characteristic zero and $\mathbf{q} = (q^{2r_{ij}})$, where $q \in k^\times$ is not a root of unity and (r_{ij}) is a non-zero additively antisymmetric $n \times n$ matrix over \mathbb{Z} . Let $R = k[x_1, \dots, x_n]$, equipped with the Poisson structure such that*

$$\{x_i, x_j\} = 2r_{ij}x_ix_j$$

for $1 \leq i, j \leq n$, and let $\Phi : A \rightarrow R$ be the k -linear isomorphism of § 3.3. Then the rule $P \mapsto \Phi^{-1}(P)$ determines a homeomorphism

$$\text{PSpec } R \rightarrow \text{Spec } A$$

that restricts to a homeomorphism

$$\text{PPrim } R \rightarrow \text{Prim } A.$$

Moreover, the rule $\mathcal{C}(\mathfrak{m}) \mapsto \Phi^{-1}(\mathcal{P}(\mathfrak{m}))$ determines a homeomorphism

$$\text{SymCr } k^n \rightarrow \text{Prim } A.$$

5. Quantum affine toric varieties

We extend our main results to quantizations of affine toric varieties and, somewhat more generally, to certain cocycle twists of affine commutative algebras. By a *quantum affine toric variety* over k we mean, as in [13], an affine domain over k equipped with a rational action of an algebraic torus H by k -algebra automorphisms, such that the H -eigenspaces are one-dimensional. Since a rational action of H is equivalent to a grading by the character group of H (e.g. [4, Lemma II.2.11]), the quantum affine toric varieties over k are also the affine domains over k , graded by free abelian groups of finite rank, with one-dimensional homogeneous components. The latter description is convenient for our present purposes, as it allows us to define quantizations via cocycle twists. As in [9, § 6] and [6, § 4], neither one-dimensionality of homogeneous components nor absence of zero-divisors is needed in our proofs, and so we can work with a more general class of twists of graded algebras.

5.1.

- (i) Let R be a commutative affine k -algebra graded by an abelian group G , and let $c : G \times G \rightarrow k^\times$ be a 2-cocycle. The *twist of R by c* [1, § 3] is a G -graded k -algebra R' , with a G -graded vector space isomorphism $r \mapsto r'$ from $R \rightarrow R'$ (the *twist map*), and multiplication given by $r's' = c(\alpha, \beta)(rs)'$ for $\alpha, \beta \in G$ and $r \in R_\alpha$, $s \in R_\beta$.

In [9, Theorem 6.3], topological quotient maps $\text{Spec } R \rightarrow \text{Spec } R'$ and $\max R \rightarrow \text{Prim } R'$ are constructed, under the assumptions that G is torsion free and $-1 \notin$

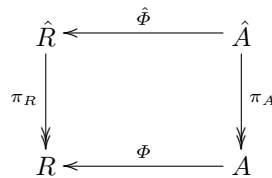
$\langle \text{im } c \rangle$ (or $\text{char } k = 2$). As discussed in [6, 4.3], the proof of [9, Theorem 6.3] provides an alternating bicharacter d on G such that R' is isomorphic to the twist of R by d and, after replacing c by d , the torsion-freeness hypothesis on G is no longer needed.

Thus, we now assume that c is an alternating bicharacter on G . Moreover, we assume that the subgroup $\langle \text{im } c \rangle \subseteq k^\times$ is torsion free. Set $A = R'$, and let $\Phi : A \rightarrow R$ be the inverse of the twist map.

- (ii) Since R is affine, we can choose a finite set of homogeneous k -algebra generators for R , say r_1, \dots, r_n . Set $\delta_i = \text{deg } r_i$ for $1 \leq i \leq n$. Define Γ and Γ^+ as in § 2.2, let $\rho : \Gamma \rightarrow G$ be the group homomorphism such that $\rho(\varepsilon_i) = \delta_i$ for $1 \leq i \leq n$, and set $\hat{c} = c \circ (\rho \times \rho)$, which is an alternating bicharacter on Γ . Also, set $\hat{q}_{ij} = \hat{c}(\varepsilon_i, \varepsilon_j) = c(\delta_i, \delta_j)$ for $1 \leq i, j \leq n$ and $\hat{q} = (\hat{q}_{ij})$.

Now set $\hat{A} = \mathcal{O}_{\hat{q}}(k^n)$ and identify \hat{A} with $k_{\hat{c}}\Gamma^+$, with k -basis $\{x^s \mid s \in \Gamma^+\}$. The corresponding semiclassical limit, as in § 2.8, is the Poisson algebra $\hat{R} = k_{\hat{u}}\Gamma^+$, for a suitable alternating biadditive map $\hat{u} = \varphi\hat{c} : \Gamma \times \Gamma \rightarrow k$, where φ is an injective group homomorphism from $\langle \hat{q}_{ij} \rangle = \langle \text{im } \hat{c} \rangle$ to k^+ , as in § 3.4 (iii). We also write \hat{R} with k -basis $\{x^s \mid s \in \Gamma^+\}$. Hence, there is a k -linear isomorphism $\hat{\Phi} : \hat{A} \rightarrow \hat{R}$ such that $\hat{\Phi}(x^s) = x^s$ for all $s \in \Gamma^+$, as in § 3.3.

- (iii) Let $\pi_A : \hat{A} \rightarrow A$ and $\pi_R : \hat{R} \rightarrow R$ be the natural k -algebra quotient maps, such that $\pi_A(x_i) = r'_i$ and $\pi_R(x_i) = r_i$ for $1 \leq i \leq n$. Then we obtain a diagram of k -linear maps as follows:



This diagram commutes because

$$\begin{aligned}
 \Phi\pi_A(x^s) &= \left(\prod_{1 \leq i < j \leq n} \tilde{c}(s_i\varepsilon_i, s_j\varepsilon_j) \right)^{-1} \Phi\pi_A(x_1^{s_1} \cdots x_n^{s_n}) \\
 &= \left(\prod_{1 \leq i < j \leq n} c(s_i\delta_i, s_j\delta_j) \right)^{-1} \Phi((r_1^{s_1})' \cdots (r_n^{s_n})') \\
 &= \Phi((r_1^{s_1} \cdots r_n^{s_n})') \\
 &= r_1^{s_1} \cdots r_n^{s_n} \\
 &= \pi_R\hat{\Phi}(x^s)
 \end{aligned}$$

for $s \in \Gamma^+$.

The Γ -grading on \hat{R} induces a G -grading via the homomorphism ρ , which we write in the form $\hat{R} = \bigoplus_{\alpha \in G} \hat{R}[\alpha]$, where

$$\hat{R}[\alpha] = \bigoplus_{s \in \rho^{-1}(\alpha) \cap \Gamma^+} kx^s$$

for $\alpha \in G$. With respect to this G -grading, π_R is G -homogeneous, in the sense that $\pi_R(\hat{R}[\alpha]) \subseteq R_\alpha$ for all $\alpha \in G$. Hence, $\ker \pi_R$ is a G -homogeneous ideal of \hat{R} .

- (iv) We next show that $\ker \pi_R$ is a Poisson ideal of \hat{R} . To see this, let $\alpha, \beta \in G$ and note that, whenever $s \in \rho^{-1}(\alpha) \cap \Gamma^+$ and $t \in \rho^{-1}(\beta) \cap \Gamma^+$, we have

$$\{x^s, x^t\} = \varphi \hat{c}(s, t)x^s x^t = \varphi c(\alpha, \beta)x^s x^t.$$

It follows that $\{a, b\} = \varphi c(\alpha, \beta)ab$ for all $a \in \hat{R}[\alpha]$ and $b \in \hat{R}[\beta]$. Consequently, any G -homogeneous ideal of \hat{R} , and in particular $\ker \pi_R$, is a Poisson ideal.

Now R becomes a Poisson algebra quotient of \hat{R} , such that

$$\{a, b\} = \varphi c(\alpha, \beta)ab$$

whenever $\alpha, \beta \in G$ and $a \in R_\alpha, b \in R_\beta$. (In particular, $\{r_i, r_j\} = \varphi c(\delta_i, \delta_j)r_i r_j$ for $1 \leq i, j \leq n$.) We view R , equipped with this Poisson structure, as a semiclassical limit of A .

Theorem 5.1. *Let k be an algebraically closed field of characteristic zero and R a commutative affine k -algebra, graded by an abelian group G . Let $c : G \times G \rightarrow k^\times$ be an alternating bicharacter such that the group $\langle \text{im } c \rangle \subseteq k^\times$ is torsion free, let A be the twist of R by c and let $\Phi : A \rightarrow R$ be the inverse of the twist map. Equip R with the Poisson structure described in § 5.1 (iv). Then the rule $P \mapsto \Phi^{-1}(P)$ determines a homeomorphism*

$$\text{PSpec } R \rightarrow \text{Spec } A,$$

which restricts to a homeomorphism

$$\text{PPrim } R \rightarrow \text{Prim } A.$$

Moreover, the rule $\mathcal{C}(\mathfrak{m}) \mapsto \Phi^{-1}(\mathcal{P}(\mathfrak{m}))$ determines a homeomorphism

$$\text{SymCr Max } R \rightarrow \text{Prim } A.$$

Proof. By Theorem 3.2, the rule $P \mapsto \hat{\Phi}^{-1}(P)$ determines homeomorphisms

$$\text{PSpec } \hat{R} \rightarrow \text{Spec } \hat{A} \quad \text{and} \quad \text{PPrim } \hat{R} \rightarrow \text{Prim } \hat{A}.$$

Observe that the first homeomorphism restricts to a homeomorphism $\eta : V \rightarrow W$, where

$$V = \{P \in \text{PSpec } \hat{R} \mid P \supseteq \ker \pi_R\} \quad \text{and} \quad W = \{P \in \text{Spec } \hat{A} \mid P \supseteq \ker \pi_A\}.$$

The quotient maps π_R and π_A induce homeomorphisms $\pi_R^* : \text{PSpec } R \rightarrow V$ and $\pi_A^* : \text{Spec } A \rightarrow W$, which fit into the following commutative diagram:

$$\begin{array}{ccc}
 \text{PSpec } \hat{R} & \xrightarrow{P \mapsto \hat{\Phi}^{-1}(P)} & \text{Spec } \hat{A} \\
 \uparrow \subseteq & & \uparrow \subseteq \\
 V & \xrightarrow{\eta} & W \\
 \uparrow \pi_R^* & & \uparrow \pi_A^* \\
 \text{PSpec } R & \xrightarrow{P \mapsto \Phi^{-1}(P)} & \text{Spec } A
 \end{array}$$

Thus, we have the desired homeomorphism $\text{PSpec } R \rightarrow \text{Spec } A$.

The fact that $P \mapsto \Phi^{-1}(P)$ also determines a homeomorphism $\text{PPrim } R \rightarrow \text{Prim } A$ follows in the same manner, once one observes that π_R^* and π_A^* map $\text{PPrim } R$ and $\text{Prim } A$ homeomorphically onto $V \cap \text{PPrim } \hat{R}$ and $W \cap \text{Prim } \hat{A}$, respectively.

The final homeomorphism will follow from the results above in the same manner as Corollary 3.3 once we show that the Zariski topology on $\text{PPrim } R$ is the quotient topology induced by the Poisson core map $\mathcal{P}(\cdot) : \text{Max } R \rightarrow \text{PPrim } R$. Set

$$X = \{\mathfrak{m} \in \text{Max } \hat{R} \mid \mathfrak{m} \supseteq \ker \pi_R\},$$

and observe that we have a commutative diagram

$$\begin{array}{ccc}
 \text{Max } \hat{R} & \xrightarrow{\mathcal{P}(\cdot)} & \text{PPrim } \hat{R} \\
 \uparrow \subseteq & & \uparrow \subseteq \\
 X & \xrightarrow{\theta} & V \cap \text{PPrim } \hat{R} \\
 \uparrow \pi_R^* & & \uparrow \pi_R^* \\
 \text{Max } R & \xrightarrow{\mathcal{P}(\cdot)} & \text{PPrim } R
 \end{array}$$

with surjective horizontal maps. As in §3.5 (iii), it follows from [7, Theorem 4.1(b)] that the topology on $\text{PPrim } \hat{R}$ is the quotient topology from the top map in the diagram. It follows that θ is a topological quotient map, and therefore so is the bottom map, as desired. □

5.2. The uniparameter case of Theorem 5.1 is the case in which $c = q^d$ where $q \in k^\times$ is not a root of unity and $d : G \times G \rightarrow k$ is an antisymmetric biadditive map. We can then take $\varphi : (\text{im } c) \rightarrow k$ to be the q -logarithm, so that $\varphi c(\alpha, \beta) = d(\alpha, \beta)$ for $\alpha, \beta \in G$.

The Poisson structure on R is then given by

$$\{a, b\} = d(\alpha, \beta)ab$$

for $\alpha, \beta \in G$ and $a \in R_\alpha, b \in R_\beta$.

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