Compositio Math. 143 (2007) 82–88
doi:10.1112/S0010437X06002387

On the highest Lyubeznik number of a local ring

Wenliang Zhang

Abstract

Let A be a d-dimensional local ring containing a field. We will prove that the highest Lyubeznik number $\lambda_{d,d}(A)$ is equal to the number of connected components of the Hochster–Huneke graph associated to B, where $B = \widehat{\hat{A}^{\mathrm{sh}}}$ is the completion of the strict Henselization of the completion of A. This was proven by Lyubeznik in characteristic $p > 0$. Our statement and proof are characteristic-free.

1. Introduction

Throughout this paper, all rings are Noetherian and commutative. Let A be a local ring that admits a surjection from an n-dimensional regular local ring $(R, \mathfrak{m})$ containing a field. Let $I \subset R$ be the kernel of the surjection, and let $k = R/\mathfrak{m}$ be the residue field of R. Then the Lyubeznik numbers $\lambda_{i,j}(A)$ (see [Lyu93, Definition 4.1]) are defined to be $\dim_k(\mathrm{Ext}^i_R(k, H^{n-j}_I(R)))$. It was proven in [Lyu93] that they are all finite and depend only on A, i and j, but neither on R nor on the surjection $R \to A$.

The Lyubeznik numbers have been studied by a number of authors, including those of [BB05, Kaw00, Kaw02, Lyu06, GS98, Wal01]. In this paper, we will give an interpretation of $\lambda_{d,d}(A)$ in terms of the topology of $\mathrm{Spec}(A)$.

Firstly, we reproduce the definition of the Hochster–Huneke graph associated to a local ring which was originally given in [HH94] (we recall that the height of an ideal is the minimum of the heights of the minimal primes over that ideal).

Definition 1.1 [HH94, Definition 3.4]. Let B be a local ring. The graph Γ_B associated to B is defined as follows. Its vertices are the top-dimensional minimal prime ideals of B, and two distinct vertices P and Q are joined by an edge if and only if $\mathrm{height}_B(P + Q) = 1$.

In [Lyu02] and [Lyu06], the following question was posed.

Question 1.2 [Lyu06, Question 1.1]. Is $\lambda_{d,d}(A)$ equal to the number of the connected components of the Hochster–Huneke graph Γ_B associated to $B = \widehat{\hat{A}^{\mathrm{sh}}}$, the completion of the strict Henselization of the completion of A?

As is pointed out in [Lyu06], the graph Γ_B can be realized by a much smaller ring than $B = \widehat{\hat{A}^{\mathrm{sh}}}$. Namely, if $\hat{A}$ is the completion of A with respect to the maximal ideal and $k \subset \hat{A}$ is a coefficient field, then there exists a finite separable extension field K of k such that $\Gamma_B = \Gamma_{\hat{A}\otimes_k K}$. In particular, if the residue field of A is separably closed, then $\Gamma_B = \Gamma_{\hat{A}}$.

Received 6 March 2006, accepted in final form 10 May 2006.
2000 Mathematics Subject Classification 13D45, 13H99, 14B15.
Keywords: local cohomology, Lyubeznik numbers.
NSF support through grant DMS-0202176 is gratefully acknowledged.

It is shown in [Lyu06] that the answer to the above question is positive in characteristic $p > 0$. Our main result in this paper is that the answer to the above question is positive in general, without any restriction on the characteristic.

Main Theorem. *Let $A = R/I$ be a local ring, where R is a regular local ring containing a field (of any characteristic), and $\dim(A) = d$. Then $\lambda_{d,d}(A)$ is equal to the number of connected components of the Hochster–Huneke graph Γ_B associated to $B = \widehat{\hat{A}^{\mathrm{sh}}}$.*

Our proof of the Main Theorem is completely characteristic-free. We use the following result from [Lyu06] whose proof in [Lyu06] is completely characteristic-free.

Lemma 1.3 [Lyu06, Corollary 2.4]. *Let A be a local ring of dimension d containing a field and let $B = \widehat{\hat{A}^{\mathrm{sh}}}$ be the completion of the strict Henselization of the completion of A. Let $\Gamma_1, \ldots, \Gamma_r$ be the connected components of Γ_B. Let I_j be the intersection of the minimal primes of B that are the vertices of Γ_j. Let $B_j = B/I_j$. Then $\lambda_{i,d}(A) = \sum_{j=1}^{r} \lambda_{i,d}(B_j)$ for every i.*

Clearly, to prove our Main Theorem it is enough to show that $\lambda_{d,d}(B_j) = 1$ for every j. Since every B_j is complete, local, d-dimensional, reduced, equidimensional, contains a field has a separably closed residue field and the Hochster–Huneke graph associated to B_j is connected, this is proven in the following Theorem 1.4.

Theorem 1.4. *If A is d-dimensional, complete, reduced, equidimensional, local, contains a field, has a separably closed residue field and the graph associated to A is connected, then $\lambda_{d,d}(A) = 1$.*

Thus, our Main Theorem follows from Lemma 1.3 and Theorem 1.4. To complete the proof of the Main Theorem it remains to prove Theorem 1.4. This is accomplished in the following section.

2. Proof of Theorem 1.4

Throughout this section A is as in Theorem 1.4, i.e. d-dimensional, complete, local, reduced, equidimensional, contains a field, has a separably closed residue field and the Hochster–Huneke graph associated to A is connected. The case that $\dim(A) \leqslant 2$ has been completely settled by Kawasaki [Kaw00] and Walther [Wal01], independently. Thus, it remains to settle the case when $\dim(A) \geqslant 3$. We will do this by induction on $\dim(A)$, the case $\dim(A) \leqslant 2$ being known.

Accordingly, throughout this section we assume that $d \geqslant 3$ and Theorem 1.4 is proven for $d - 1$. By Cohen's structure theorem, A is a homomorphic image of a complete regular local ring $(R, \mathfrak{m})$ containing a field.

Let $\dim(R) = n$. By [Lyu93, Corollary 3.6] the set of the minimal primes of the support of $H_I^{n-d+1}(R)$ is finite. We pick an element $r \in \mathfrak{m}$ as follows. If $\mathrm{Ass}_R(H_I^{n-d+1}(R)) \neq \{\mathfrak{m}\}$, then standard prime avoidance implies that there is an element $r \in \mathfrak{m}$ that does not belong to any minimal prime of I nor to any minimal prime of the support of $H_I^{n-d+1}(R)$ different from $\mathfrak{m}$. If $\mathrm{Ass}_R(H_I^{n-d+1}(R)) = \{\mathfrak{m}\}$, the only condition on r is that $r \in \mathfrak{m}$ and r does not belong to any minimal prime of I. We fix one such element $r \in \mathfrak{m}$ throughout the rest of this section.

Let $\bar{r}$ be the image of r in $A = R/I$. Then $\bar{r}$ is not contained in any minimal prime ideal of A, since r is not contained in any minimal prime of I. Hence, $A/\bar{r}A$ is equidimensional and $\dim(A/\bar{r}A) = d - 1$. We are going to prove the following two propositions.

Proposition 2.1. *We have $\lambda_{d,d}(A) = \lambda_{d-1,d-1}(A/\sqrt{\bar{r}A})$.*

Proposition 2.2. *The Hochster–Huneke graph associated to $A/\sqrt{\bar{r}A}$ is connected.*

https://doi.org/10.1112/S0010437X06002387 Published online by Cambridge University Press

Proof of Theorem 1.4. Proposition 2.2 shows that the ring $A/\sqrt{\bar{r}A}$ satisfies the hypotheses of Theorem 1.4 for dimension $d-1$. The inductive hypotheses (i.e. Theorem 1.4 in dimension $d-1$) implies that $\lambda_{d-1,d-1}(A/\sqrt{\bar{r}A}) = 1$. Proposition 2.1 now shows that $\lambda_{d,d}(A) = 1$. This completes the proof of Theorem 1.4 modulo Propositions 2.1 and 2.2. □

It remains to prove Propositions 2.1 and 2.2. We begin with a proof of Proposition 2.1 which requires Lemmas 2.3 and 2.4 below.

Lemma 2.3. *We have* $\dim(\mathrm{Supp}_R(H_I^{n-d+1}(R))) \leqslant d-2$.

Proof. Let P be an arbitrary element of $\mathrm{Supp}_R(H_I^{n-d+1}(R))$. Assume $\mathrm{height}_R(P) \leqslant n-d+1$. If $\mathrm{height}_R(P) \leqslant n-d$, then $(H_I^{n-d+1}(R))_P \cong H_{IR_P}^{n-d+1}(R_P) = 0$ since the dimension of R_P is less than $n-d+1$. If $\mathrm{height}_R(P) = n-d+1$, then the dimension of R_P is $n-d+1$ and by the Hartshorne–Lichtenbaum vanishing theorem (see [BS98, Theorem 8.2.1])

$$(H_I^{n-d+1}(R))_P = H_{IR_P}^{n-d+1}(R_P) = 0$$

(since $I\widehat{R_P}$ is not $P\widehat{R_P}$-primary as R is regular and every minimal prime of IR_P has height $n-d$). So, $\mathrm{height}_R(P) \geqslant n-d+2$ for every $P \in \mathrm{Supp}_R(H_I^{n-d+1}(R))$, hence $\dim(\mathrm{Supp}_R(H_I^{n-d+1}(R))) \leqslant d-2$. □

Lemma 2.4. *We have* $\dim(\mathrm{Supp}_R(H_{(r)}^0(H_I^{n-d+1}(R)))) \leqslant d-3$.

Proof. If $\mathrm{Supp}_R(H_I^{n-d+1}(R)) = \{\mathfrak{m}\}$, then $\dim(\mathrm{Supp}_R(H_{(r)}^0(H_I^{n-d+1}(R)))) = 0 \leqslant d-3$, since $d \geqslant 3$ by our assumption and $H_{(r)}^0(H_I^{n-d+1}(R))$ is a submodule of $H_I^{n-d+1}(R)$. If $\mathrm{Supp}_R(H_I^{n-d+1}(R)) \neq \{\mathfrak{m}\}$, let P be an arbitrary element of $\mathrm{Supp}_R(H_{(r)}^0(H_I^{n-d+1}(R)))$. Then P has to contain r and a minimal element P' in $\mathrm{Supp}_R(H_I^{n-d+1}(R))$. Since $r \notin P'$ by Lemma 2.1, it follows from Krull's principal ideal theorem that $\mathrm{height}_R(P'+(r)) = \mathrm{height}_R(P')+1$. Hence, $\mathrm{height}_R(P) \geqslant n-d+2+1 = n-d+3$. Therefore, $\dim(\mathrm{Supp}_R(H_{(r)}^0(H_I^{n-d+1}(R)))) \leqslant d-3$. □

Proof of Proposition 2.1. We always have the following exact sequence as part of the long exact sequence [BS98, Proposition 8.1.2]

$$\cdots \to H_{I+(r)}^{n-d}(R) \to H_I^{n-d}(R) \xrightarrow{\alpha} H_I^{n-d}(R_r) \xrightarrow{\beta} H_{I+(r)}^{n-d+1}(R) \to H_I^{n-d+1}(R) \to \cdots.$$

From our choice of r, the height of $I+(r)$ is $n-d+1$, hence $H_{I+(r)}^{n-d}(R) = 0$. Let L be the cokernel of α, then we have a short exact sequence

$$0 \to H_I^{n-d}(R) \xrightarrow{\alpha} H_I^{n-d}(R_r) \to L \to 0. \tag{1}$$

Since $r \in \mathfrak{m}$ and the multiplication by r is injective on $H_I^{n-d}(R_r)$, $H_{\mathfrak{m}}^i(H_I^{n-d}(R_r)) = 0$ for all integers $i \geqslant 0$. Therefore, applying the functor $R\Gamma_{\mathfrak{m}}$ to (1), we will have

$$H_{\mathfrak{m}}^{d-1}(L) \cong H_{\mathfrak{m}}^d(H_I^{n-d}(R)).$$

Let M be the cokernel of β. Then we have a short exact sequence

$$0 \to L \to H_{I+(r)}^{n-d+1}(R) \to M \to 0. \tag{2}$$

Applying the functor $R\Gamma_{\mathfrak{m}}$ to (2), we have

$$\cdots \to H_{\mathfrak{m}}^{d-2}(M) \to H_{\mathfrak{m}}^{d-1}(L) \to H_{\mathfrak{m}}^{d-1}(H_{I+(r)}^{n-d+1}(R)) \to H_{\mathfrak{m}}^{d-1}(M) \to \cdots. \tag{3}$$

Since M is a quotient module of $H_{I+(r)}^{n-d+1}(R)$, every element of M is annihilated by some power of $I+(r)$. On the other hand, M is isomorphic to a submodule of $H_I^{n-d+1}(R)$, so M is isomorphic to a submodule of $H_{(r)}^0(H_I^{n-d+1}(R))$. Therefore, $\dim(\mathrm{Supp}_R(M)) \leqslant d-3$ by Lemma 2.4.

https://doi.org/10.1112/S0010437X06002387 Published online by Cambridge University Press

By Grothendieck's vanishing theorem, we have

$$H^{d-1}_{\mathfrak{m}}(M) = H^{d-2}_{\mathfrak{m}}(M) = 0.$$

Combining this with (3), we have

$$H^{d-1}_{\mathfrak{m}}(L) \cong H^{d-1}_{\mathfrak{m}}(H^{n-d+1}_{I+(r)}(R)).$$

Therefore,

$$H^{d}_{\mathfrak{m}}(H^{n-d}_{I}(R)) \cong H^{d-1}_{\mathfrak{m}}(H^{n-d+1}_{I+(r)}(R)).$$

Since $H^{n-d+1}_{I+(r)}(R) = H^{n-d+1}_{\sqrt{I+(r)}}(R)$, we have

$$\dim_{R/\mathfrak{m}} \mathrm{Hom}_R(R/\mathfrak{m}, H^{d}_{\mathfrak{m}}(H^{n-d}_{I}(R))) = \dim_{R/\mathfrak{m}} \mathrm{Hom}_R(R/\mathfrak{m}, H^{d-1}_{\mathfrak{m}}(H^{n-(d-1)}_{\sqrt{I+(r)}}(R))),$$

which implies that

$$\lambda_{d,d}(A) = \lambda_{d,d}(R/I) = \lambda_{d-1,d-1}(R/\sqrt{I+(r)}) = \lambda_{d-1,d-1}(A/\sqrt{\bar{r}A}).$$

This completes the proof of Proposition 2.1. □

To complete the proof of Theorem 1.4, it remains to prove Proposition 2.2. Let $\Theta = \{P_1, \ldots, P_s\}$ be the set of the minimal prime ideals of A. Let $\Sigma_i = \{Q \in \mathrm{Spec}(A) \mid Q \text{ is minimal over } P_i + \sqrt{\bar{r}A}\}$, and let $\Sigma = \bigcup_i \Sigma_i$. There is one-to-one correspondence between Σ and the set of the minimal prime ideals in $A/\sqrt{\bar{r}A}$.

Lemma 2.5. *Let P_1 and P_2 be two arbitrary elements in Θ. If for any $Q_\alpha \in \Sigma_1$ and any $Q_\beta \in \Sigma_2$, $\mathrm{height}_{A/\sqrt{\bar{r}A}}((Q_\alpha + Q_\beta)/\sqrt{\bar{r}A}) \geqslant 2$, then $\mathrm{height}_A(P_1 + P_2) \geqslant 2$.*

Proof. Otherwise, $\mathrm{height}_A(P_1 + P_2) = 1$ (obviously, $\mathrm{height}_A(P_1 + P_2) \geqslant 1$). By the principal ideal theorem and considering that A is catenary because it is complete, we have

$$\mathrm{height}_A(P_1 + P_2 + \sqrt{\bar{r}A}) \leqslant 2. \tag{4}$$

Let $\mathcal{Q}$ be an arbitrary prime ideal of A minimal over $P_1 + P_2 + \sqrt{\bar{r}A}$. Then $\mathcal{Q}$ must contain some $Q_1 \in \Sigma_1$ and some $Q_2 \in \Sigma_2$. Therefore,

$$\mathrm{height}_{A/\sqrt{\bar{r}A}}(\mathcal{Q}/\sqrt{\bar{r}A}) \geqslant \mathrm{height}_{A/\sqrt{\bar{r}A}}((Q_1 + Q_2)/\sqrt{\bar{r}A}) \geqslant 2.$$

Thus,

$$\mathrm{height}_{A/\sqrt{\bar{r}A}}((P_1 + P_2 + \sqrt{\bar{r}A})/\sqrt{\bar{r}A}) \geqslant 2.$$

Hence, for any prime ideal $\tilde{Q}$ minimal over $P_1 + P_2 + \sqrt{\bar{r}A}$, there exist prime ideals Q and $\bar{Q}$ so that we have a chain of ideals

$$\sqrt{\bar{r}A} \subset Q \subsetneq \bar{Q} \subsetneq \tilde{Q}.$$

Q contains $\sqrt{\bar{r}A}$, thus Q properly contains some element in Θ, say, P_3. Then we have a chain of ideals

$$P_3 \subsetneq Q \subsetneq \bar{Q} \subsetneq \tilde{Q}.$$

Therefore, for every prime ideal $\tilde{Q}$ minimal over $P_1 + P_2 + \sqrt{\bar{r}A}$,

$$\mathrm{height}_A(\tilde{Q}) \geqslant 3,$$

hence,

$$\mathrm{height}_A(P_1 + P_2 + \sqrt{\bar{r}A}) \geqslant 3,$$

contrary to (4). □

https://doi.org/10.1112/S0010437X06002387 Published online by Cambridge University Press

Lemma 2.6. *Assume that the Hochster–Huneke graph associated to $A/\sqrt{P_i + \bar{r}A}$ is connected for all $P_i \in \Theta$. Then so is the Hochster–Huneke graph associated to $A/\sqrt{\bar{r}A}$.*

Proof. Indeed, assume that the graph associated to $A/\sqrt{\bar{r}A}$ is not connected. Then Σ can be divided into 2 non-empty disjoint subsets: Σ^* and Σ^{**}, such that if $Q_1 \in \Sigma^*$ and $Q_2 \in \Sigma^{**}$ then $\text{height}_{A/\sqrt{\bar{r}A}}((Q_1+Q_2)/\sqrt{\bar{r}A}) \geqslant 2$. Since the graph associated to $A/\sqrt{P_i + \bar{r}A}$ is connected, Σ_i has to be completely contained in Σ^* or completely contained in Σ^{**} for every i. Therefore, we can divide Θ into 2 non-empty disjoint subsets:

$$\Theta^* = \{P \in \Theta \mid \textit{the prime ideals minimal over } P + \sqrt{\bar{r}A} \textit{ are contained in } \Sigma^*\}$$

and

$$\Theta^{**} = \{P \in \Theta \mid \textit{the prime ideals minimal over } P + \sqrt{\bar{r}A} \textit{ are contained in } \Sigma^{**}\}.$$

For an arbitrary element $P_1 \in \Theta^*$ and an arbitrary element $P_2 \in \Theta^{**}$,

$$\text{height}_{A/\sqrt{\bar{r}A}}((Q_1+Q_2)/\sqrt{\bar{r}A}) \geqslant 2, \quad \forall Q_1 \in \Sigma_1, \forall Q_2 \in \Sigma_2.$$

Lemma 2.5 implies $\text{height}_A(P_1+P_2) \geqslant 2$. Hence, the graph associated to A is not connected either, which is a contradiction. □

Proof of Proposition 2.2. According to Lemma 2.6, it is enough to prove that the graph associated to $A/\sqrt{P_i + \bar{r}A}$ is connected for all $P_i \in \Theta$. Denoting A/P_i by A and the image of $\bar{r}$ in A/P_i by $\bar{r}$ again, we are reduced to proving that if A is a domain and $\bar{r} \in \mathfrak{m}$ is nonzero, then the graph associated to $A/\sqrt{\bar{r}A}$ is connected.

The following result is not explicitly stated in [HH94], but is a straightforward consequence of [HH94, 3.6c,e] and [HH94, 3.9b,c]: *Let S be a complete local equidimensional ring. If S satisfies Serre's condition S_2 and $x_1, \ldots, x_k$ is a part of a system of parameters of S, then the graph associated to $S/\sqrt{(x_1, \ldots, x_k)}$ is connected.*

Let S be the normalization of A. Since A is a complete local domain, so is S. Serre's criterion of normality shows that S is S_2. Since $\bar{r} \in S$ is S-regular, the graph associated to $S/\sqrt{\bar{r}S}$ is connected by the above-quoted result. As S is module-finite over A, the going-up theorem implies that $\sqrt{\bar{r}S} \cap A = \sqrt{\bar{r}A}$, hence the natural map $\phi : A/\sqrt{\bar{r}A} \to S/\sqrt{\bar{r}S}$ is injective and $S/\sqrt{\bar{r}S}$ is a finite $A/\sqrt{\bar{r}A}$ module via ϕ. The ring $S/\sqrt{\bar{r}S}$ is catenary since S is complete.

Setting $B = A/\sqrt{\bar{r}A}$ and $C = S/\sqrt{\bar{r}S}$, we have that $B \subset C$ is an injective finite extension of equidimensional local rings and C is catenary. The graph associated to C is connected, and we need to show that the graph associated to B is connected. This is shown below.

The graph associated to an equidimensional local ring R is connected if and only if for every pair of minimal primes P_α and P_β of R there is a sequence of prime ideals $P_1, \ldots, P_k$ such that, setting $P_\alpha = P_0$ and $P_\beta = P_{k+1}$, we have that $\text{height}_R(P_i + P_{i+1}) \leqslant 1$ (i.e. $\text{height}_R(P_i + P_{i+1}) = 1$ if $P_i \neq P_{i+1}$) for all $0 \leqslant i \leqslant k$. Accordingly, let P_α and P_β be two arbitrary minimal prime ideals in B. Then we have prime ideals $\tilde{P}_\alpha$ and $\tilde{P}_\beta$ in C lying over P_α and P_β, respectively, and $\tilde{P}_\alpha$ and $\tilde{P}_\beta$ are minimal in C as well, by the going-up theorem. The graph associated to C is connected, hence there exists a sequence of minimal prime ideals $\tilde{P}_1, \ldots, \tilde{P}_k$ in C so that, setting $\tilde{P}_0 = \tilde{P}_\alpha$ and $\tilde{P}_{k+1} = \tilde{P}_\beta$,

$$\text{height}_C(\tilde{P}_i + \tilde{P_{i+1}}) = 1 \quad \text{for } 0 \leqslant i \leqslant k.$$

Let $P_1, \ldots, P_k$ be the pullback of $\tilde{P}_1, \ldots, \tilde{P}_k$ in B. We let $P_0 = P_\alpha$ and $P_{k+1} = P_\beta$. To show that the graph associated to B is connected, it is enough to show that

$$\text{height}_B(P_i + P_{i+1}) \leqslant 1 \quad \text{for all } i \leqslant k.$$

This amounts to showing that if $P_i \neq P_{i+1}$, then $\text{height}_B(P_i + P_{i+1}) = 1$. Accordingly, we assume that $P_i \neq P_{i+1}$. Since $\text{height}_C(\tilde{P}_i + \tilde{P}_{i+1}) = 1$, there exists a prime ideal $\tilde{Q}$ in C with height 1 containing $\tilde{P}_i$ and $\tilde{P}_{i+1}$. Let Q be the pullback of $\tilde{Q}$ in B. Since $\dim(B) = \dim(C) = d-1$ and C is catenary, equidimensional and $\text{height}_C(\tilde{Q}) = 1$, we will have a chain of prime ideals

$$\tilde{Q} \subsetneq \tilde{Q}_1 \subsetneq \cdots \subsetneq \tilde{Q}_{d-2}.$$

Taking the pullback of this chain in B, we will have

$$Q \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_{d-2}.$$

Hence, $\text{height}_B(Q) \leqslant 1$, i.e. $\text{height}_B(P_i + P_{i+1}) \leqslant 1$. This shows that the graph associated to B is connected and completes the proof of Proposition 2.2 and Theorem 1.4. □

In conclusion, we give an application of our results to projective schemes over a field. For any projective scheme X of dimension d over a field k, we can write X as $\text{Proj}(k[x_0, \ldots, x_n]/I)$ for some n and I homogeneous in $k[x_0, \ldots, x_n]$, i.e. we have an embedding $X \hookrightarrow \mathbb{P}^n_k$. Let A denote the local ring $(k[x_0, \ldots, x_n]/I)_{(x_0, \ldots, x_n)}$. Since A is a local ring containing a field, we can consider the Lyubeznik numbers of A.

Our Main Theorem provides some supporting evidence for a positive answer to the open question of whether $\lambda_{i,j}(A)$ the Lyubeznik numbers of the above ring A depend only on the integers i, j and the scheme X but are independent of the embedding $X \hookrightarrow \mathbb{P}^n_k$ (see [Lyu02, p. 133]). Indeed, we have the following theorem which is a direct consequence of our Main Theorem.

Theorem 2.7. *Let X be an arbitrary projective scheme of dimension d over a field k. Under some embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$, we can write $X = \text{Proj}(R)$, where $R = k[x_0, \ldots, x_n]/I$ with some homogeneous ideal I in the polynomial ring $k[x_0, \ldots, x_n]$. Let $A := R_{(x_0, \ldots, x_n)}$. Then $\lambda_{d+1,d+1}(A)$ does not depend on the choice of n and I, i.e. it does not depend on the embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$. In other words, it is a numerical invariant on X. Indeed, let k^{sep} be the separable closure of k and let $X_1, \ldots, X_s$ be the d-dimensional irreducible components of $X \times_k k^{\text{sep}}$. Let Γ_X be the graph on vertices $X_0, \ldots, X_s$ and X_i, X_j are joined by an edge if and only if $\dim(X_i \cap X_j) = d-1$. Then $\lambda_{d+1,d+1}(A)$ is equal to the number of connected components of Γ_X.*

Acknowledgements

The result of this paper is from my thesis. I would like to thank my advisor Professor Gennady Lyubeznik for suggesting this problem to me. I also wish to thank Professor Craig Huneke for suggesting a simplification in the proof of Proposition 2.1. Thanks are also due to the referee for his or her useful suggestions.

References

BB05 M. Blickle and R. Bondu, *Local cohomology multiplicities in terms of étale cohomology*, Ann. Inst. Fourier (Grenoble) **55** (2005), 2239–2256.

BS98 M. Brodman and R. Sharp, *Local cohomology* (Cambridge University Press, Cambridge, 1998).

HH94 M. Hochster and C. Huneke, *Indecomposable canonical modules and connectedness*, in *Commutative algebra: syzygies, multiplicities, and birational algebra*, South Hadley, MA, 1992, Contemporary Mathematics, vol. 159 (American Mathematical Society, Providence, RI, 1994), 197–208.

Kaw00 K. I. Kawasaki, *On the Lyubeznik number of local cohomology modules*, Bull. Nara Univ. Ed. Natur. Sci. **49** (2000), 5–7.

Kaw02 K. I. Kawasaki, *On the highest Lyubeznik number*, Math. Proc. Cambridge Philos. Soc. **132** (2002), 409–417.

Lyu93 G. Lyubeznik, *Finiteness properties of local cohomology modules*, Invent. Math. **113** (1993), 41–55.

Lyu02 G. Lyubeznik, *A partial survey of local cohomology*, in *Local cohomology and its applications*, Guanajuato, 1999, Lecture Notes in Pure and Applied Mathematics, vol. 226 (Dekker, New York, 2002), 121–154.

Lyu06 G. Lyubeznik, *On some local cohomology invariants of local rings*, Math. Z. **254** (2006), 627–640.

GS98 R. Garcia Lopez and C. Sabbah, *Topological computation of local cohomology multiplicities*, Collect. Math. **49** (1998), 317–324.

Wal01 U. Walther, *On the Lyubeznik numbers of a local ring*, Proc. Amer. Math. Soc. **129** (2001), 1631–1634.

Wenliang Zhang wlzhang@math.umn.edu
Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

https://doi.org/10.1112/S0010437X06002387 Published online by Cambridge University Press