MULTIPLICATIVE FORMS AND NONASSOCIATIVE ALGEBRAS

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0. Introduction. In [**1**], we introduced the notion of multiplicative forms on associative algebras $\mathscr{A}$ of finite rank over integral domains D, and obtained a complete classification when $D \subseteq \mathbb{C}$, the complex field. We propose here to remove the hypothesis of associativity, using a refinement of the technique of Schafer [**2**]. In [**1**], it was noted that multiplicative forms extend uniquely under the adjunction of an identity when $\mathscr{A}$ is associative but not unitary; this appears difficult to verify in the general case, so that some mild restriction on $\mathscr{A}$ is required. We shall assume that $\mathscr{A}$ is *biregular*, that is that $\mathscr{A}$ contains elements $\mathbf{e}_L$, $\mathbf{e}_R$ such that the linear maps $\mathbf{x} \mapsto \mathbf{e}_L\mathbf{x}$ and $\mathbf{x} \mapsto \mathbf{x}\mathbf{e}_R$ are bijective on $\mathscr{A}$. We can then (§1) reduce the biregular case to the unitary case, which is handled in §2.

1. The biregular case. Suppose that $D \subseteq \mathbb{C}$ is an integral domain, and that $\mathscr{A}$ is biregular, of finite rank over D; further, assume that $\mathscr{A}$ has no $\mathbf{1}$. We prove the following

PROPOSITION. *Let the binary operation* $*$ *be defined on* $\mathscr{A}$ *by* $\mathbf{a}\mathbf{e}_R * \mathbf{e}_L\mathbf{b} = \mathbf{a}\mathbf{b}$; *then* $\mathbf{e}_L\mathbf{e}_R$ *is an identity element for* $*$, *and* $(\mathscr{A}, *, +)$ *is a unitary D-algebra. Further, if f is a multiplicative form on* $(\mathscr{A}, \cdot, +)$, *then* $f(\mathbf{e}_L\mathbf{e}_R) \neq 0$; *on defining* φ *by* $\varphi(\mathbf{a}) = f(\mathbf{a})/f(\mathbf{e}_L\mathbf{e}_R)$, *we have* $\varphi(\mathbf{a} * \mathbf{b}) = \varphi(\mathbf{a})\varphi(\mathbf{b})$.

Proof. The first sentence is straightforward to verify. Also, as f is not trivial, there is an $\mathbf{a} = \mathbf{b}\mathbf{e}_R = \mathbf{e}_L\mathbf{c}$ such that $f(\mathbf{a}) \neq 0$, so that $f(\mathbf{e}_L) \neq 0 \neq f(\mathbf{e}_R)$. Finally, let $\mathbf{a} = \mathbf{c}\mathbf{e}_R$, $\mathbf{b} = \mathbf{e}_L\mathbf{d}$; then $\mathbf{a} * \mathbf{b} = \mathbf{c}\mathbf{d}$, so that $\varphi(\mathbf{a} * \mathbf{b}) = f(\mathbf{c}\mathbf{d})/f(\mathbf{e}_L\mathbf{e}_R)$; since $\varphi(\mathbf{a}) = f(\mathbf{c})/f(\mathbf{e}_L)$ and $\varphi(\mathbf{b}) = f(\mathbf{d})/f(\mathbf{e}_R)$, we see that φ is multiplicative.

2. The unitary case. We may now assume that $\mathscr{A}$ contains a $\mathbf{1}$. The multiplicative property is preserved under extension of scalars; so we may also assume that $D = \mathbb{C}$. Given a nontrivial multiplicative f, we may polarise it as in [**2**, p. 778], obtaining a multilinear symmetric form $M = M(f)$ in $n = \deg f$ variables. We denote by $K(M)$ the kernel of M, that is the set of all $\mathbf{a} \in \mathscr{A}$ orthogonal to $\mathscr{A}$ under M. We assert that the vector space $K(M)$ is, in fact, an ideal of $\mathscr{A}$. For, define, as in [**2**, p. 779], $T(\mathbf{x}) = nM(\mathbf{x}, \mathbf{1}, \ldots, \mathbf{1})$, so that T is a linear functional: $\mathscr{A} \to \mathbb{C}$. Then [**2**, p. 780] the bilinear form $B(\mathbf{x}, \mathbf{y}) = T(\mathbf{x}\mathbf{y})$ is a *trace form*, that is $B(\mathbf{x}\mathbf{y}, \mathbf{z}) = B(\mathbf{x}, \mathbf{y}\mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathscr{A}$. We prove that the kernel $K(B)$ of B equals $K(M)$. Indeed, the argument of [**2**, p. 781] shows that $K(B) \subseteq K(M)$; conversely, suppose that $\mathbf{x} \in K(M)$; then $T(\mathbf{x}) = 0$, and $M(\mathbf{x}, \mathbf{y}, \mathbf{1}, \ldots, \mathbf{1}) = 0$ for all $\mathbf{y} \in \mathscr{A}$, so that, by [**2**, p. 780, equation (17)], $T(\mathbf{x}\mathbf{y}) = 0 = B(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in \mathscr{A}$, proving that $K(M) = K(B)$, an ideal of $\mathscr{A}$. It is now clear that M induces a nondegenerate multilinear form M^* on the algebra $\mathscr{A}_1 = \mathscr{A}/K(M)$. Since $f(\mathbf{x}) = M(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x})$, it is also clear that $f(\mathbf{x}+\mathbf{k}) = f(\mathbf{x})$ for all

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$\mathbf{x} \in \mathscr{A}$, $\mathbf{k} \in K(M)$, so that f induces a multiplicative form $f^* : \mathscr{A}_1 \to \mathbb{C}$, which polarises to M^*, a nondegenerate n-linear form. It follows from [2, p. 781–789] that $\mathscr{A}_1$ is a direct sum of simple central alternative $\mathbb{C}$-algebras, and that f^* is composed of factors f_j^*, one from each simple summand. Jacobson [3] has shown that each f_j^* must be a power of the *generic norm*. By the celebrated theorem of Zorn [4, p. 56], the only simple central alternative $\mathbb{C}$-algebras of finite dimension are the total matrix algebras (for which the generic norm is the determinant), and the 8-dimensional Cayley algebras described in [4, p. 44–50]. To summarise, we have proved the following

THEOREM. *Let* $\mathscr{A}$ *be a finite-dimensional unitary* $\mathbb{C}$*-algebra,* $f: \mathscr{A} \to \mathbb{C}$ *a multiplicative form, polarising to* M. *Then* $K(M)$ *is an ideal, and* $\mathscr{A}_1 = \mathscr{A}/K(M)$ *is a semisimple alternative algebra; the induced form* $f^* : \mathscr{A}_1 \to \mathbb{C}$, *given by* $f^*(\mathbf{a}+K(M)) = f(\mathbf{a})$, *is composed of powers of the generic norms of the simple summands of* $\mathscr{A}_1$ (*which are total matrix algebras or* 8-*dimensional Cayley algebras*).

Conversely, if K is any ideal of $\mathscr{A}$ such that $\mathscr{A}/K$ is a semisimple alternative algebra, and g is multiplicative on $\mathscr{A}/K$, then polarisation of g yields a multilinear form $M(g)$ on $\mathscr{A}/K$; then $M(g)$ lifts to a multilinear form M' on $\mathscr{A}$ via $M'(\mathbf{x}_1, \ldots, \mathbf{x}_n) = M(\mathbf{x}_1+K, \ldots, \mathbf{x}_n+K)$. Taking $\mathbf{x}_1 = \mathbf{x}_2 = \ldots = \mathbf{x}_n = \mathbf{x}$, we obtain a multiplicative form $f(\mathbf{x}) = M'(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x})$ on $\mathscr{A}$.

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