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Abstract

The Severi degree is the degree of the Severi variety parametrizing plane curves of degree d with δ nodes. Recently, Göttsche and Shende gave two refinements of Severi degrees, polynomials in a variable y, which are conjecturally equal, for large d. At $y = 1$, one of the refinements, the relative Severi degree, specializes to the (non-relative) Severi degree. We give a tropical description of the refined Severi degrees, in terms of a refined tropical curve count for all toric surfaces. We also refine the equivalent count of floor diagrams for Hirzebruch and rational ruled surfaces. Our description implies that, for fixed δ, the refined Severi degrees are polynomials in d and y, for large d. As a consequence, we show that, for $\delta \leqslant 10$ and all $d \geqslant \delta/2 + 1$, both refinements of Göttsche and Shende agree and equal our refined counts of tropical curves and floor diagrams.

1. Introduction

A δ*-nodal curve* is a reduced (not necessarily irreducible) curve with δ simple nodes and no other singularities. The *Severi degree* $N^{d,\delta}$ is the degree of the Severi variety parametrizing plane δ-nodal curves of degree d. Equivalently, $N^{d,\delta}$ is the number of δ-nodal plane curves of degree d through $(d+3)d/2 - \delta$ generic points in the complex projective plane $\mathbb{P}^2$.

Severi degrees are generally difficult to compute. Their study goes back to the midst of the 19th century, when Steiner [Ste48], in 1848, showed that the degree $N^{d,1}$ of the discriminant of $\mathbb{P}^2$ is $3(d-1)^2$. In 1998, Caporaso and Harris [CH98] computed $N^{d,\delta}$ for any d and δ, by their celebrated recursion (involving *relative Severi degrees* $N^{d,\delta}(\alpha, \beta)$ counting curves satisfying tangency conditions to a fixed line), see also [Ran89].

Di Francesco and Itzykson [DFI95], in 1994, conjectured the numbers $N^{d,\delta}$ to be polynomial in d, for fixed δ and d large enough. In 2009, Fomin–Mikhalkin [FM10] showed that, for each $\delta \geqslant 1$, there is a polynomial $N_\delta(d)$ in d with $N^{d,\delta} = N_\delta(d)$, provided that $d \geqslant 2\delta$. The polynomials $N_\delta(d)$ are called *node polynomials*.

More generally, for S a projective algebraic surface, and L a line bundle on S, the Severi degree $N^{(S,L),\delta}$ is the number of δ-nodal curves in the complete linear system $|L|$ through $\dim |L| - \delta$ general points of S . In [Göt98] it was conjectured that the Severi degrees of arbitrary smooth projective surfaces S with a sufficiently ample line bundle L are given by universal polynomials. Specifically the conjecture predicts for each fixed δ, the existence of a polynomial $\widetilde{N}^{(S,L),\delta}$ in the intersection numbers L^2, LK_S, K_S^2, $c_2(S)$ such that $N^{(S,L),\delta} = \widetilde{N}^{(S,L),\delta}$ for L sufficiently ample. We call the $\widetilde{N}^{(S,L),\delta}$ the *curve-counting invariants*. In addition the $\widetilde{N}^{(S,L),\delta}$

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were conjectured to be given by a multiplicative generating function, i.e., there are universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[q]]$, such that

$$\sum_{\delta\geqslant 0} \widetilde{N}^{(S,L),\delta} q^\delta = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{c_2(S)}. \tag{1.1}$$

Furthermore A_1 and A_4 are given explicitly in terms of modular forms. This conjecture was proved by Tzeng [Tze12] in 2010. A second proof was given shortly afterwards by Kool *et al.* [KST11]. In the latter proof, the authors identified the numbers $\widetilde{N}^{(S,L),\delta}$ as coefficients of the generating function of the topological Euler characteristics of relative Hilbert schemes (see § 2). This is motivated by the proposed definition of the Gopakumar Vafa (BPS) invariants in terms of Pandharipande–Thomas invariants in [PT10]. Thus the curve-counting invariants can be viewed as special cases of BPS invariants. By definition for $S = \mathbb{P}^2$ and $L = \mathcal{O}(d)$, the curve-counting invariants coincide with the node polynomials: $\widetilde{N}^{(\mathbb{P}^2,\mathcal{O}(d)),\delta} = N_\delta(d)$.

Inspired by this description, in [GS14] *refined invariants* $\widetilde{N}^{(S,L),\delta}(y)$ are defined as coefficients of a very similar generating function, but with the topological Euler characteristic replaced by the normalized χ_{-y}-genus, a specialization of the Hodge polynomial. They are Laurent polynomials in y, symmetric under $y \mapsto 1/y$. In [GS14] a number of conjectures are made about the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$. In particular they are conjectured to have a multiplicative generating function (as in (1.1)), where now two of the universal power series are explicitly given in terms of Jacobi forms. This fact was proven in the meantime in [GS13] in the case where the canonical divisor K_S is numerically trivial.

In this paper we will concentrate on the case that S is a toric surface, and sometimes we restrict to the case that $S = \mathbb{P}^2$, $L = \mathcal{O}(d)$, and denote $\widetilde{N}^{(\mathbb{P}^2,\mathcal{O}(d)),\delta}(y) = \widetilde{N}^{d,\delta}(y)$. In the case that S is a toric surface and L a toric line bundle, we will change slightly the definition of the Severi degrees. We denote $N^{(S,L),\delta}$ the number of cogenus δ curves in $|L|$ passing though $\dim |L| - \delta$ general points in S, which do not contain a toric boundary divisor as a component. This is done because, as we will see below, with this new definition (and not with the old one) the Severi degrees can be computed via tropical geometry and by a Caporaso–Harris type recursion formula. The Severi degrees as defined before we denote by $N_*^{(S,L),\delta}$, but we will not consider them in the sequel.

If L is δ-very ample (see below for the definition) it is easy to see (Remark 2.1) that $N^{(S,L),\delta} = N_*^{(S,L),\delta}$. In the case $S = \mathbb{P}^2$ it is easy to see that $N^{d,\delta} = N_*^{d,\delta}$. By definition the Caporaso–Harris type recursion of [CH98, Vak00] always computes the invariants $N^{(S,L),\delta}$ for $\mathbb{P}^2$ and rational ruled surfaces.

If S is $\mathbb{P}^2$ or a rational ruled surface, in [GS14] *refined Severi degrees* $N^{(S,L),\delta}(y)$ are defined by a modification of the Caporaso–Harris recursion. These are again Laurent polynomials in y, symmetric under $y \mapsto 1/y$. Again, in the case of $\mathbb{P}^2$, we denote the refined Severi degrees by $N^{d,\delta}(y)$. The recursion specializes to that of [CH98, Vak00] at $y = 1$, so that $N^{(S,L),\delta}(1) = N^{(S,L),\delta}$.

In this paper we will relate the refined Severi degrees $N^{(S,L),\delta}(y)$ and $N^{d,\delta}(y)$ to tropical geometry. Mikhalkin [Mik05] has shown that the Severi degrees of projective toric surfaces can be computed by toric geometry. Fix a lattice polygon Δ in $\mathbb{R}^2$, i.e., Δ is the convex hull of a finite subset of $\mathbb{Z}^2$. Then Δ determines via its normal fan a projective toric surface $X(\Delta)$ and an ample line bundle $L = L(\Delta)$ on $X(\Delta)$ (and $H^0(X(\Delta), L(\Delta))$ can be identified with the vector space with basis $\Delta \cap \mathbb{Z}^2$). Conversely a pair (X, L) of a toric surface and a line bundle on X determines a lattice polygon. We denote by $N^{\Delta,\delta}$ the number of (possibly reducible) cogenus δ

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curves of degree Δ in $(\mathbb{C}^*)^2$ passing through $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$ general points, as defined in [Mik05, Definition 5.1]. By definition $N^{(X(\Delta),L(\Delta)),\delta} = N^{\Delta,\delta}$. The invariants $N^{\Delta,\delta}$ can be computed in tropical geometry.

If $X(\Delta)$ is $\mathbb{P}^2$ or a rational ruled surface, we will in the future also write $N^{\Delta,\delta}(y) := N^{(X(\Delta),L(\Delta)),\delta}(y)$ for the corresponding (refined) Severi degrees as defined in [GS14]. By our definition we then have $N^{\Delta,\delta}(1) = N^{\Delta,\delta}$.

In tropical geometry the Severi degrees $N^{\Delta,\delta}$ can be computed as the count of simple tropical curves C in $\mathbb{R}^2$ through $\dim |L(\Delta)| - \delta$ general points, counted with certain multiplicities $\mathrm{mult}_{\mathbb{C}}(C)$. Roughly speaking, a simple tropical curve is a trivalent graph C immersed in $\mathbb{R}^2$, with some extra data. From this data, one assigns to each vertex v of C a multiplicity $\mathrm{mult}_{\mathbb{C}}(v)$, and defines the multiplicity $\mathrm{mult}_{\mathbb{C}}(C)$ as the product $\prod_{v \text{ vertex of } C} \mathrm{mult}_{\mathbb{C}}(v)$.

For any integer n, and a variable y, we introduce the *quantum number* $[n]_y$ by

$$[n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}} = y^{(n-1)/2} + \cdots + y^{-(n-1)/2}. \tag{1.2}$$

By definition $[n]_1 = n$. We introduce a new polynomial multiplicity $\mathrm{mult}(C; y) \in \mathbb{Z}_{\geqslant 0}[y^{1/2}, y^{-1/2}]$ for tropical curves by $\mathrm{mult}(C; y) = \prod_{v \text{ vertex of } C} [\mathrm{mult}_{\mathbb{C}}(v)]_y$, and define the *tropical refined Severi degrees* $N^{\Delta,\delta}_{\mathrm{trop}}(y)$ as the count of simple tropical curves C in $\mathbb{R}^2$ through $\dim |L(\Delta)| - \delta$ general points with multiplicity $\mathrm{mult}(C; y)$. By definition $N^{\Delta,\delta}_{\mathrm{trop}}(y) \in \mathbb{Z}_{\geqslant 0}[y^{1/2}, y^{-1/2}]$. By definition $[\mathrm{mult}_{\mathbb{C}}(v)]_1 = \mathrm{mult}_{\mathbb{C}}(v)$, and thus we see that $N^{\Delta,\delta}_{\mathrm{trop}}(1) = N^{\Delta,\delta}$.

A priori, $N^{\Delta,\delta}_{\mathrm{trop}}(y)$ should depend on a configuration Π of $\dim |L(\Delta)| - \delta$ general points in $\mathbb{R}^2$ but Itenberg and Mikhalkin show in [IM13] that $N^{\Delta,\delta}_{\mathrm{trop}}(y)$ is a *tropical invariant*, i.e., independent of Π.

We will prove that in the case of the plane and rational ruled surfaces, when the refined Severi degrees have been defined in [GS14], they equal the tropical refined Severi degrees.

Theorem 1.1. *Let $X(\Delta)$ be $\mathbb{P}^2$ or a rational ruled surface or $\mathbb{P}(1,1,m)$. Then the tropical refined Severi degrees satisfy the recursion (2.7) for the refined Severi degrees.*
Thus $N^{\Delta,\delta}_{\mathrm{trop}}(y) = N^{(X(\Delta),L(\Delta)),\delta}(y)$.

We also determine a Caporaso–Harris type recursion formula for $X(\Delta)$ the weighted projective space $\mathbb{P}(1,1,m)$ (cf. Theorem 7.5).

The computation of the Severi degrees via tropical geometry and the proof of the existence of node polynomials $N_\delta(d)$ uses a class of decorated graphs called floor diagrams. The new refined multiplicity $\mathrm{mult}(C; y)$ on tropical curves gives rise to a y-statistics on floor diagrams, which allows us to adapt the arguments to the tropical refined Severi degrees. This statistic is a q-analog of the one of Brugallé and Mikhalkin [BM09], who gave a combinatorial formula for the Severi degrees $N^{d,\delta}$. Theorem 1.1 is a q-analog of their [BM09, Theorem 3.6] for the refined Severi degrees $N^{d,\delta}(y)$.

Using our combinatorial description, we show that the refined Severi degrees become polynomials for sufficiently large degree.

Theorem 1.2. *For fixed $\delta \geqslant 1$, there is a polynomial $N_\delta(d; y) \in \mathbb{Q}[y, y^{-1}, d]$ of degree 2δ in d and δ in y and y^{-1}, such that*

$$N_\delta(d; y) = N^{d,\delta}(y),$$

provided that $d \geqslant \delta$.

We call the $N_\delta(d;y)$ *refined node polynomials.*

The refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ were computed in [GS14] for $\delta \leqslant 10$, and there it was conjectured that the refined Severi degrees $N^{d,\delta}(y)$ agree with the refined invariants $\widetilde{N}^{d,\delta}(y)$ for $d \geqslant \delta/2+1$. If we assume this conjecture, it would follow from Theorem 1.2 that $\widetilde{N}^{d,\delta}(y) = N_\delta(d;y)$, in particular conjecturally the bound on d can be considerably improved. We use the refined Caporaso–Harris recursion formula to compute $N^{d,\delta}(y)$ for $\delta \leqslant 10$ and $d \leqslant 30$. Together with Theorem 1.2 this gives the following.

Corollary 1.3. *For $\delta \leqslant 10$ and any $d \geqslant \delta/2+1$, we have $\widetilde{N}^{d,\delta}(y) = N^{d,\delta}(y) = N_\delta(d,y)$.*

Corollary 1.4. *For $\delta \leqslant 10$ and any $d \geqslant \delta/2+1$, $\widetilde{N}^{d,\delta}(y)$, as a Laurent polynomial in y, has non-negative integral coefficients.*

Our combinatorial description of the Laurent polynomials $N^{d,\delta}(y)$ allows for effective computation of the refined node polynomials; for details see Remark 6.1. For $\delta \leqslant 3$, the polynomials $N_\delta(d;y)$ are explicitly given by Remark 6.1. For $\delta \leqslant 10$ they are given by Theorem 4.4 (proving the formula of Conjecture 2.7 for $\delta \leqslant 10$).

Göttsche and Shende also observed a connection between refined invariants and *real* algebraic geometry. Specifically, they conjectured that $\widetilde{N}^{d,\delta}(-1)$ equals the *tropical Welschinger invariant* $W^{d,\delta}_{\mathrm{trop}}$ (for the definition and details see [IKS09]), for $d \geqslant \delta/3+1$. Furthermore, by definition $N^{d,\delta}(-1) = W^{d,\delta}_{\mathrm{trop}}$, i.e., the refined Severi degree specializes, at $y=-1$ and for all d, to the tropical Welschinger invariant. The numbers $W^{d,\delta}_{\mathrm{trop}}$, in turn, equal counts of real plane curves (i.e., complex plane curves invariant under complex conjugation), counted with a sign, through particular configurations of real points [Shu05, Proposition 6.1]. Indeed, at $y=-1$, the new y-statistic on floor diagrams specializes to the 'real multiplicity' of Brugallé and Mikhalkin [BM09], and Theorem 1.1 becomes [BM09, Theorem 3.9] for the numbers $N^{d,\delta}(-1) = W^{d,\delta}_{\mathrm{trop}}$.

The recursion formula 2.7 simplifies considerably if we specialize $y=-1$. Therefore we have been able to use the recursion to compute $N^{d,\delta}(-1)$ for $\delta \leqslant 15$ and $d \leqslant 45$. As by Theorem 1.2 $N_\delta(d,-1)$ is a polynomial in d of degree at most 2δ, this determines $N_\delta(d,-1)$ for $d \leqslant 15$. On the other hand in [GS14] the $\widetilde{N}^{(S,L),\delta}(-1)$ are computed for all S, L and $\delta \leqslant 14$.

Corollary 1.5. $\widetilde{N}^{d,\delta}(-1) = N_\delta(d,-1) = W^{d,\delta}_{\mathrm{trop}}$ *for $\delta \leqslant 14$ and all $d \geqslant \delta/3+1$.*

We expect our methods to compute refined Severi degrees also for other toric surfaces. Specifically, we expect the argument to generalize to toric surfaces of 'h-transverse' polygons, along the lines of [AB13] (see Remark 5.8). Notice that such surfaces are in general not smooth and are thus outside the realm of the (non-refined) Göttsche conjecture [Göt98].

One may speculate about the meaning of refined Severi degrees at other roots of unity. At $y=-1$, we obtain a (signed) count of complex curves invariant under the involution of complex conjugation, at least in genus 0. This shows the occurrence of a *cyclic sieving phenomenon* [Sag11] of order 2. At least for $y=i$, the imaginary unit, the refined Severi degree again specializes to an integer $N^{\Delta,\delta}(i) \in \mathbb{Z}$. It would be interesting to find a non-tropical enumerative interpretation for these numbers.

This paper is organized as follows. In § 2, we review, following Göttsche and Shende, the refined invariants and refined Severi degrees, the latter for the surfaces $\mathbb{P}^2$, Σ_m, and $\mathbb{P}(1,1,m)$. In § 3, we introduce a refinement of tropical curve enumeration for toric surfaces and extend the notion of refined Severi degrees to this class. In § 4 we discuss various polynomiality and other properties of the refined Severi degrees. In § 5, we refine the floor diagram technique of

Brugallé and Mikhalkin and template decomposition of Fomin and Mikhalkin, and use it in § 6 to prove the results stated in § 4. Finally, in § 7, we introduce tropical refined relative Severi degrees and show that they agree with the refined Severi degrees of the Göttsche and Shende.

2. Refined invariants and refined Severi degrees

In this section we review the definition of the closely related notions of the refined invariants and the refined Severi degrees from [GS14]. In § 3 we will show that the refined Severi degree also has a simple combinatorial interpretation in terms of tropical geometry.

Recall that the Severi degree $N^{d,\delta}$ is the degree of the Severi variety parametrizing δ-nodal plane curves of degree d in $\mathbb{P}^2$. Equivalently, $N^{d,\delta}$ is the number of such curves through $(d+3)d/2-\delta$ generic points in $\mathbb{P}^2$. More generally, given a line bundle L on a surface S, one can define the Severi degree $N^{(S,L),\delta}$ as the number of δ-nodal reduced curves in the complete linear system $|L|=\mathbb{P}(H^0(S,L))$ passing through $\dim|L|-\delta$ general points.

2.1 Refined invariants

For a line bundle L on S we denote by $g(L):=L(L+K_S)/2+1$ the arithmetic genus of a curve in $|L|$. For $\delta\geqslant 0$, let $\mathbb{P}^\delta$ be a general δ-dimensional subspace of $|L|$. Let $\mathcal{C}\to\mathbb{P}^\delta$ be the *universal curve*, i.e., $\mathcal{C}$ is the subscheme

$$\mathcal{C}=\{(p,[C]):p\in C\}\subset S\times\mathbb{P}^\delta$$

with a natural map to $\mathbb{P}^\delta$. Here, $[C]$ denotes the curve C viewed as a point of $\mathbb{P}^\delta$. Thus the fiber of $\mathcal{C}\to\mathbb{P}^\delta$ over $[C]\in\mathbb{P}^\delta$ is the curve C. Let $S^{[n]}=\mathrm{Hilb}^n(S)$ be the Hilbert scheme of n points in S. Finally, let $\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)$ be the relative Hilbert scheme

$$\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)=\{([Z],[C]):Z\subset C\}\subset S^{[n]}\times\mathbb{P}^\delta.$$

Here, $[Z]$ is the subscheme Z viewed as a point of $S^{[n]}$ and $Z\subset C$ means that Z is a subscheme of C.

Recall that a line bundle L on S is called δ-very ample if the restriction map $H^0(S,L)\to H^0(L|_Z)$ is surjective for all zero-dimensional subschemes $Z\in S^{[\delta+1]}$. In the introduction we had changed the definition of the Severi degrees for toric surfaces, defining $N^{(S,L),\delta}$ to be the count of δ-nodal curves in $|L|$ through generic points, which do not contain a toric boundary divisor. The count of curves without this condition we denoted $N_*^{(S,L),\delta}$.

Remark 2.1. Let L be δ-very ample on a surface S, then the curves in $|L|$ containing a given curve as a component occur in codimension at least $\delta+1$. In particular if L is a δ-very ample toric line bundle on the toric surface S, then $N^{(S,L),\delta}=N_*^{(S,L),\delta}$.

Proof. Let C be a curve on S. Let Z be any zero-dimensional subscheme of C of length $\delta+1$. Then by δ-very ampleness the canonical restriction map $\rho:H^0(S,L)\to H^0(L|_Z)$ is surjective. The sections s of L such that $Z(s)$ contains C as a component lie in the kernel of ρ, and thus curves having C as a component occur in codimension at least $\delta+1$ in $|L|$. □

We review the definition of the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ in the case where $\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)$ is non-singular of dimension $n+\delta$ for all n. A sufficient condition for this is that L is δ-very ample, see [GS14, Theorem 46].

In their proof [KST11] of the Göttsche conjecture [Göt98, Conjecture 2.1], Kool *et al.* showed, partially based on [PT10], that, if L is δ-very ample, the Severi degrees $N^{(S,L),\delta}$ can be computed from the generating function of their Euler characteristics. Specifically, they show [KST11, Theorem 3.4] that, under this assumption, there exist integers n_r, for $r = 0, \dots, \delta$, such that

$$\sum_{i=0}^{\infty} e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))t^i = \sum_{r=0}^{\delta} n_r t^r (1-t)^{2g(L)-2-2r}. \tag{2.1}$$

Here, $e(-) = \sum_{i\geqslant 0}(-1)^i \operatorname{rk} H^i(-,\mathbb{Z})$ denotes the topological Euler characteristic. Furthermore, they showed that the Severi degree $N^{(S,L),\delta}$ equals the coefficient n_δ in (2.1).

Inspired by this description, Göttsche and Shende [GS14] suggest to replace in (2.1) the Euler characteristic $e(-)$ by the χ_{-y}-genus

$$\chi_{-y}(-) = \sum_{p,q\geqslant 0} (-1)^{p+q} y^q h^{p,q}(-), \tag{2.2}$$

where $h^{p,q}(-)$ are the Hodge numbers. The polynomial χ_{-y} is the *Hodge polynomial* $H(\tilde{x},\tilde{y})(-) = \sum_{p,q\geqslant 0} \tilde{x}^p\, \tilde{y}^q\, h^{p,q}(-)$, at $\tilde{x} = -y$ and $\tilde{y} = -1$. They prove the following.

Proposition 2.2. *Assume* $\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)$ *is non-singular for all* n. *Then there exist polynomials* $n_0(y), \dots, n_{g(L)}(y)$ *such that*

$$\sum_{i=0}^{\infty} \chi_{-y}(\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta))t^n = \sum_{r=0}^{g(L)} n_r(y) t^r ((1-t)(1-ty))^{g(L)-r-1}. \tag{2.3}$$

This is a weak form of an analogue of (2.1). They conjecture that a precise analogue holds.

Conjecture 2.3. Under the assumptions of Proposition 2.2, we have that $n_r(y) = 0$ for $r > \delta$.

Definition 2.4. Under the assumptions of Proposition 2.2 we put $\widetilde{N}^{(S,L),\delta}(y) := n_\delta(y)/y^\delta$, where $n_\delta(y)$ is the polynomial in (2.3). Following [GS14], we call the polynomials $\widetilde{N}^{(S,L),\delta}(y)$ the *refined invariants* of S, L (there they are called normalized refined invariants). It is easy to see from the definition that $\widetilde{N}^{(S,L),\delta}(y)$ is a Laurent polynomial in y, symmetric under $y \mapsto 1/y$.

Finally we extend the definition of the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ to arbitrary L and δ, when the $\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)$ might be singular, or they might not even exist (e.g. if $\delta > \dim |L|$).

Let $f(z) := z(1 - ye^{-z(1-y)})/(1-e^{-z(1-y)}) \in 1 + z\mathbb{Q}[y][[z]]$. Now let S be smooth projective surface, L a line bundle on S. Let $Z_n(S) \subset S \times S^{[n]}$ be the universal family with projections $p : Z_n(S) \to S^{[n]}$, $q : Z_n(S) \to S$. Let $L^{[n]} := p_* q^* L$, a vector bundle of rank n on S, denote $l_1, \dots, l_n$ its Chern roots, and denote $t_1, \dots, t_{2n}$ the Chern roots of the tangent bundle $T_{S^{[n]}}$. The following is proven in [GS14, Proposition 52].

Proposition 2.5. *Assume* $\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)$ *is non-singular for all* n. *Then*

$$\chi_{-y}(\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)) = \operatorname*{res}_{x=0}\bigg[\bigg(\frac{f(x)}{x}\bigg)^{\delta+1} \int_{S^{[n]}} \prod_{i=1}^{2n} f(t_i) \prod_{j=1}^{n} \frac{l_j}{f(l_j+x)}\bigg]. \tag{2.4}$$

(By definition $\prod_{i=1}^{2n} f(t_i) \prod_{j=1}^{n} l_j/f(l_j+x) \in H^*(S^{[n]},\mathbb{Q})[y][[x]]$, *and thus the term in square brackets on the left-hand side of (2.4) is a Laurent series in* x *with coefficients in* $\mathbb{Q}[y]$.*)*

Definition 2.6. Let L be a line bundle on a projective surface S, let $\delta \in \mathbb{Z}_{\geqslant 0}$. The *refined invariants* $\widetilde{N}^{(S,L),\delta}$ are defined by replacing $\chi_{-y}(\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta))$ by the right-hand side of (2.4) in Definition 2.4 and (2.3).

We write $\widetilde{N}^{d,\delta}(y) = \widetilde{N}^{(\mathbb{P}^2,\mathcal{O}(d)),\delta}(y)$ for the refined invariants of $\mathbb{P}^2$.

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At $y=1$ we have $\chi_{-1}(-)=e(-)$, and thus recover the Severi degree as the special case $\widetilde{N}^{(S,L),\delta}(1)=N^{(S,L),\delta}$, for $\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)$ non-singular, from [KST11, Theorem 3.4]. The $\widetilde{N}^{(S,L),\delta}(y)$ satisfy universal polynomiality [GS14]: for each δ, there is a polynomial $\widetilde{N}_\delta(x_1,x_2,x_3,x_4;y)$, such that $\widetilde{N}^{(S,L),\delta}(y)=\widetilde{N}_\delta(L^2,LK_S,K_S^2,e(S);y)$. In particular there exist polynomials $\widetilde{N}_\delta(d;y)$ in d and y such that $\widetilde{N}_\delta(d;y)=\widetilde{N}^{d,\delta}(y)$ for all d,δ. Assuming Conjecture 2.3, these polynomials have a multiplicative generating function: there exist universal power series $A_1,A_2,A_3,A_4\in\mathbb{Q}[y^{\pm1}][[q]]$, such that

$$\sum_{\delta\geqslant 0}\widetilde{N}^{(S,L),\delta}(y)q^\delta=A_1^{L^2}A_2^{LK_S}A_3^{K_S^2}A_4^{e(S)}.$$

More precisely, in [GS14, Conjecture 67] a conjectural generating function for the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ is given. Let

$$\widetilde{\Delta}(y,q):=q\prod_{n=1}^{\infty}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2=q-(2y+2+2y^{-1})q+\cdots,$$

$$\widetilde{DG}_2(y,q):=\sum_{m=1}^{\infty}q^m\sum_{d|m}[d]_y^2\frac{m}{d}=q+(y+4+y^{-1})q^2+\cdots.$$

Denote $D:=q(\partial/\partial q)$.

Conjecture 2.7. There exist universal power series $B_1(y,q)$, $B_2(y,q)$ in $\mathbb{Q}[y,y^{-1}][[q]]$, such that

$$\sum_{\delta\geqslant 0}\widetilde{N}^{(S,L),\delta}(y)(\widetilde{DG}_2)^\delta=\frac{(\widetilde{DG}_2/q)^{\chi(L)}B_1(y,q)^{K_S^2}B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q)\,D\widetilde{DG}_2/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{2.5}$$

Here, to make the change of variables, all functions are viewed as elements of $\mathbb{Q}[y,y^{-1}][[q]]$.

Equivalently, letting

$$g(y,t)=t-(y+4+y^{-1})t^2+(y^2+14y+30+14y^{-1}+y^{-2})t^3+\cdots$$

be the compositional inverse of $\widetilde{DG}_2$, (2.5) says

$$\sum_{\delta\geqslant 0}\widetilde{N}^{(S,L),\delta}(y)t^\delta=(t/g(y,t))^{\chi(L)}\frac{B_1(y,q)^{K_S^2}B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q)D\widetilde{DG}_2/q^2)^{\chi(\mathcal{O}_S)/2}}\bigg|_{q=g(y,t)}. \tag{2.6}$$

In [GS14] this conjecture is proven modulo q^{11} and the power series $B_1(y,q)$, $B_2(y,q)$ are determined modulo q^{11} (the result can be found directly after [GS14, Conjecture 67]). Here we list $B_1(y,q)$, $B_2(y,q)$ for completeness modulo q^6:

$$\begin{aligned}B_1(y,q)&=1-q-((y^2+3y+1)/y)q^2+((y^4+10y^3+17y^2+10y+1)/y^2)q^3\\&\quad-((18y^4+87y^3+135y^2+87y+18)/y^2)q^4\\&\quad+((12y^6+210y^5+728y^4+1061y^3+728y^2+210y+12)/y^3)q^5+O(q^6),\\B_2(y,q)&=\frac{1}{(1-yq)(1-q/y)}(1+3q-((3y^2+y+3)/y)q^2\\&\quad+((y^4+8y^3+18y^2+8y+1)/y^2)q^3-((13y^4+53y^3+76y^2+53y+13)/y^2)q^4\\&\quad+((7y^6+100y^5+316y^4+455y^3+316y^2+100y+7)/y^3)q^5+O(q^6)).\end{aligned}$$

This gives a formula for the $\widetilde{N}^{(S,L),\delta}(y)$ as explicit polynomials of degree at most δ in L^2, LK_S, K_S^2, $\chi(\mathcal{O}_S)$ proven for $\delta\leqslant 10$. The $\widetilde{N}^{d,\delta}(y)$ are obtained from this by specifying $\chi(L)=\binom{d+2}{2}$, $LK_S=-3d$, $K_S^2=9$, $\chi(\mathcal{O}_S)=1$, giving them as polynomials of degree at most 2δ in d.

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2.2 Refined Severi degrees

Throughout this section we take S to be $\mathbb{P}^2$, a rational ruled surface, or a weighted projective space $\mathbb{P}(1,1,m)$. In the case $S=\mathbb{P}^2$, let H be a line in $\mathbb{P}^2$; in the case where S is a rational ruled surface $\Sigma_m=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-m))$, let H be the class of a section with $H^2=m$, and let E be the class of the section with $E^2=-m$ and F the class of a fibre on Σ_m. We denote H the class of a line in $\mathbb{P}(1,1,m)$ with $H^2=m$. For a rational ruled surface Σ_m we can also allow m to be negative. In this case $\Sigma_m=\Sigma_{-m}$, but the role of H and E is exchanged. Therefore below, in the case of Σ_m, we actually represent two different recursion formulas.

Caporaso and Harris showed that the Severi degrees $N^{d,\delta}$ satisfy a recursion formula [CH98]. A similar recursion formula computes the Severi degrees $N^{(S,L),\delta}$ on rational ruled surfaces [Vak00]. In [GS14] a refined Caporaso–Harris type recursion formula is used to define Laurent polynomials $N^{(S,L),\delta}(y)$, which the authors call *refined Severi degrees*. By definition for $y=1$ these polynomials specialize to the Severi degrees: $N^{(S,L),\delta}(1)=N^{(S,L),\delta}$. We now briefly review this recursion and also extend it to $\mathbb{P}(1,1,m)$.

By a *sequence* we mean a collection $\alpha=(\alpha_1,\alpha_2,\dots)$ of non-negative integers, almost all of which are zero. For two sequences α, β we define $|\alpha|=\sum_i\alpha_i$, $I\alpha=\sum_i i\alpha_i$, $\alpha+\beta=(\alpha_1+\beta_1, \alpha_2+\beta_2,\dots)$, and $\binom{\alpha}{\beta}=\prod_i\binom{\alpha_i}{\beta_i}$. We write $\alpha\leqslant\beta$ to mean $\alpha_i\leqslant\beta_i$ for all i. We write e_k for the sequence whose kth element is 1 and all other ones 0. We usually omit writing down trailing zeros.

For sequences α, β, and $\delta\geqslant 0$, let $\gamma(L,\beta,\delta)=\dim|L|-HL+|\beta|-\delta$. The relative Severi degree $N^{(S,L),\delta}(\alpha,\beta)$ is the number of δ-nodal curves in $|L|$ not containing H, through $\gamma(L,\beta,\delta)$ general points, and with α_k given points of contact of order k with H, and β_k arbitrary points of contact of order k with H.

Definition 2.8 [GS14, Recursion 76, Proposition 78]. Recall the definition of the quantum numbers $[n]_y=(y^{n/2}-y^{-n/2})/(y^{1/2}-y^{-1/2})$. Let L be a line bundle on S and let α, β be sequences with $I\alpha+I\beta=HL$, and let $\delta\geqslant 0$ be an integer. We define the *refined relative Severi degrees* $N^{(S,L),\delta}(\alpha,\beta)(y)$ recursively as follows: if $\gamma(L,\beta,\delta)>0$, then

$$\begin{aligned}N^{(S,L),\delta}(\alpha,\beta)(y)&=\sum_{k:\beta_k>0}[k]_y\cdot N^{(S,L),\delta}(\alpha+e_k,\beta-e_k)(y)\\&\quad+\sum_{\alpha',\beta',\delta'}\left(\prod_i[i]_y^{\beta'_i-\beta_i}\right)\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}N^{(S,L-H),\delta'}(\alpha',\beta')(y).\end{aligned}\tag{2.7}$$

Here the second sum runs through all α',β',δ' satisfying the condition

$$\begin{gathered}\alpha'\leqslant\alpha,\quad \beta'\geqslant\beta,\quad I\alpha'+I\beta'=H(L-H),\\ \delta'=\delta+g(L-H)-g(L)+|\beta'-\beta|-1=\delta-H(L-H)+|\beta'-\beta|.\end{gathered}\tag{2.8}$$

Initial conditions: if $\gamma(L,\beta,\delta)=0$ we have $N^{(S,L),\delta}(\alpha,\beta)(y)=0$ unless we are in one of the following cases.

(i) In the case $S=\mathbb{P}^2$ we put $N^{H,0}((1),(0))(y)=1$.

(ii) In the case $S=\Sigma_m$, let F be the class of a fibre of the ruling; we put $N^{kF,0}((k),(0))(y)=1$.

(iii) In the case $S=\mathbb{P}(1,1,m)$, $L=dH$, we put and $N^{H,0}((1),(0))(y)=1$.

We abbreviate $N^{(S,L),\delta}(y):=N^{(S,L),\delta}((0),(LH))(y)$, and, in the case $S=\mathbb{P}^2$, $N^{d,\delta}(\alpha,\beta)(y):=N^{(\mathbb{P}^2,\mathcal{O}(d)),\delta}(\alpha,\beta)(y)$, $N^{d,\delta}(y):=N^{d,\delta}((0),(d))(y)$. The refined relative Severi degrees are Laurent polynomials in $y^{1/2}$, symmetric under $y\mapsto 1/y$.

Remark 2.9. As mentioned in the beginning of this section, for S a Hirzebruch surface this recursion is defined for $m \in \mathbb{Z}$; in this case $\Sigma_{-m} = \Sigma_m$ but the class H on Σ_{-m} is the class E on Σ_m. For $m \in \mathbb{Z}$, we will write $N^{(\Sigma_m,L),\delta}(\alpha,\beta)(y)$ for the invariants obtained by this recursion. Below in Theorem 7.5 we will see that $N^{(\Sigma_m,L),\delta}(y) = N^{(\Sigma_{-m},L),\delta}(y)$. In general we do not have $N^{(\Sigma_m,L),\delta}(\alpha,\beta)(y) = N^{(\Sigma_{-m},L),\delta}(\alpha,\beta)(y)$, because (expressed on Σ_m) the first counts curves with contact conditions along H and the second with contact conditions along E.

Remark 2.10. The recursions for the refined Severi degrees are chosen so that they specialize at $y = 1$ to the recursion for the usual Severi degrees. Furthermore the recursions of [IKS09] (see [GS14, Recursion 92]) for the tropical Welschinger numbers $W^{(S,L),\delta}_{\mathrm{trop}}(\alpha,\beta)$ are obtained by specializing instead to $y = -1$. Thus (as already noted in [GS14, Proposition 93]) we get:

$$\begin{aligned} N^{(S,L),\delta}(\alpha,\beta)(1) &= N^{(S,L),\delta}(\alpha,\beta), \quad N^{(S,L),\delta}(1) = N^{(S,L),\delta},\\ N^{(S,L),\delta}(\alpha,\beta)(-1) &= W^{(S,L),\delta}_{\mathrm{trop}}(\alpha,\beta), \quad N^{(S,L),\delta}(-1) = W^{(S,L),\delta}_{\mathrm{trop}}. \end{aligned} \tag{2.9}$$

This result also follows from Theorem 7.5 below, which says that the tropical refined Severi degrees are equal to the Severi degrees.

According to [KS13], if the general $\mathbb{P}^\delta \subset |L|$ contains no non-reduced curves and no curves containing components with negative self intersection, the Severi degrees are computed by the universal formulas. We expect the same for refined Severi degrees.

Conjecture 2.11 [GS14]. Let S be $\mathbb{P}^2$ or a rational ruled surface, let L be a line bundle, and assume $\mathbb{P}^\delta \subset |L|$ contains no non-reduced curves and no curves containing components with negative self intersection. Then the refined Severi degrees are computed by the universal formulas: $N^{(S,L),\delta}(y) = \widetilde{N}^{(S,L),\delta}(y)$. Explicitly, we have the following.

(i) On $\mathbb{P}^2$ we have $N^{d,\delta}(y) = \widetilde{N}^{d,\delta}(y)$, for $d \geqslant \delta/2 + 1$.
(ii) Assume $c + d > 0$. We have $N^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y) = \widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y)$, for $c, d \geqslant \delta/2$.
(iii) On $S = \Sigma_m$ with $m > 0$, assume $d + c > 0$. Then $N^{(S,cF+dH),\delta}(y) = \widetilde{N}^{(S,cF+dH),\delta}(y)$ for $\delta \leqslant \min(2d, c)$.

Remark 2.12. For $m \geqslant 2$ the weighted projective space $\mathbb{P}(1,1,m)$ is singular, so [GS14, Conjecture 2.11] does not apply. In fact the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ have not even been defined in this case.

We instead compare the refined Severi degrees $N^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ to the corresponding refined invariants $\widetilde{N}^{(\Sigma_m,dH),\delta}(y)$ on the minimal resolution Σ_m of $\mathbb{P}(1,1,m)$.

We obtain the following conjectures.

Conjecture 2.13. There is a polynomial $N_\delta(d,m;y)$ of degree 2δ in d and δ in m, such that $N^{(\mathbb{P}(1,1,m),dH),\delta}(y) = N_\delta(d,m;y)$ for $\delta \leqslant min(2d-2, 2m-1)$.

Note that Theorem 4.2(3) implies a weaker form of Conjecture 2.13, assuming stronger bounds.

Conjecture 2.14. There exist power series $C_1, C_2, C_3 \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that

$$\sum_{\delta\geqslant 0} N_\delta(d,m;y)(\widetilde{DG_2})^\delta = \left(\sum_{\delta\geqslant 0} \widetilde{N}^{(\Sigma_m,dH),\delta}(y)\,(\widetilde{DG_2})^\delta\right) C_1^{(m+2)d} C_2^{m+2} C_3.$$

Remark 2.15. We have used the Caporaso–Harris recursion to compute $N^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ for $\delta \leqslant 6$, $d \leqslant 5$ and $m \leqslant 5$. The results confirm Conjectures 2.13, 2.14. Furthermore, assuming these conjectures, they determine C_1, C_2, C_3 modulo q^7. We list them modulo q^6. Conjecturally this gives in particular $N^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ for $\delta \leqslant 5$, $d \geqslant 4$, $m \geqslant 3$:

$$\begin{aligned}
C_1 &= 1 - ((y^2+3y+1)/y)q + ((6y^2+11y+6)/y)q^2 - ((4y^4+36y^3+60y^2+36y+4)/y^2)q^3 \\
&\quad + ((y^6+54y^5+243y^4+373y^3+243y^2+54y+1)/y^3)q^4 \\
&\quad - ((41y^6+525y^5+1723y^4+2478y^3+1723y^2+525y+41)/y^3)q^5 + O(q^6), \\
C_2 &= \frac{1}{(1-qy)(1-q/y)}(1+2q-((2y^2+2y+2)/y)q^2 \\
&\quad + ((y^4+6y^3+11y^2+6y+1)/y^2)q^3 - ((10y^4+38y^3+56y^2+38y+10)/y^2)q^4 \\
&\quad + ((7y^6+79y^5+241y^4+339y^3+241y^2+79y+7)/y^3)q^5 + O(q^6)), \\
C_3 &= 1+2q-((4y^2+6y+4)/y)q^2 + ((20y^2+32y+20)/y)q^3 - ((19y^4+100y^3+170y^2 \\
&\quad +100y+19)/y^2)q^4 + ((4y^6+154y^5+564y^4+824y^3+564y^2+154y+4)/y^3)q^5 + O(q^6).
\end{aligned}$$

Denote by $N_0^{(S,L),\delta}$ the irreducible Severi degrees, i.e., the number of irreducible δ-nodal curves in $|L| \neq |E|$ passing though $\dim|L| - \delta$ general points. In particular it is clear that $N_0^{(S,L),\delta} \geqslant 0$ and $N_0^{(S,L),\delta} = 0$ if $\delta > g(L)$. In [Get97] it is noted in the case $S = \mathbb{P}^2$, and in [Vak00] for rational ruled surfaces, that the $N_0^{(S,L),\delta}$ can be expressed by a formula in terms of the Severi degrees $N^{(S,L),\delta}$. In [GS14] *irreducible refined Severi degrees* $N_0^{(S,L),\delta}(y)$ are defined by the same formula

$$\sum_{L,\delta} \frac{z^{\dim|L|-\delta}}{(\dim|L|-\delta)!} v^L N_0^{(S,L),\delta}(y) = \log\left(1 + \sum_{L,\delta} \frac{z^{\dim|L|-\delta}}{(\dim|L|-\delta)!} v^L N^{(S,L),\delta}(y)\right). \tag{2.10}$$

Here $\{v^L\}_{L \text{ effective, } L\neq E}$ are elements of the Novikov ring, i.e., $v^{L_1}v^{L_2} = v^{L_1+L_2}$. Evidently $N_0^{(S,L),\delta}(y)$ is a Laurent polynomial in y invariant under $y \mapsto 1/y$, and $N_0^{(S,L),\delta}(1) = N_0^{(S,L),\delta}$.

We will show below that $N_0^{(S,L),\delta}(y)$ is a count of irreducible tropical curves with Laurent polynomials in y with non-negative integer coefficients as multiplicities, see Theorem 4.14. In particular, $N_0^{(S,L),\delta}(y) \in \mathbb{Z}_{\geqslant 0}[y^{\pm 1}]$. Furthermore, $N_0^{(S,L),\delta}(y) = 0$, if $\delta > g(L)$.

3. Refined tropical curve counting

We now define a refinement of Severi degrees for any toric surface, by introducing a 'y-weight' into Mikhalkin's tropical curve enumeration. For the surfaces $S = \Sigma_m$ and $S = \mathbb{P}(1,1,m)$, the new invariants agree with the refined Severi degrees defined via the recursion in Definition 2.8. We extend our definition to the case of tangency conditions in § 7. We denote tropical curves and classical curves with the same notation C, as it usually will be clear which curves we are talking about.

Definition 3.1. A *metric graph* is a non-empty graph whose edges e have a *length* $l(e) \in \mathbb{R}_{>0} \cup \{\infty\}$.

An *abstract tropical curve* C is a metric graph with all vertices of valence 1 or at least 3 such that, for an edge e of C, we have length $l(e) = \infty$ precisely when e is adjacent to a leaf (i.e., a 1-valent vertex) of C. We conventionally remove the (infinitely far away) leaf vertices from C.

Note that we do not require the underlying graph of a metric graph to be connected. Connectedness will correspond to the irreducibility of algebraic curves. Let Δ be a lattice polygon in $\mathbb{R}^2$. A non-zero vector $u \in \mathbb{Z}^2$ is *primitive* if its entries are coprime.

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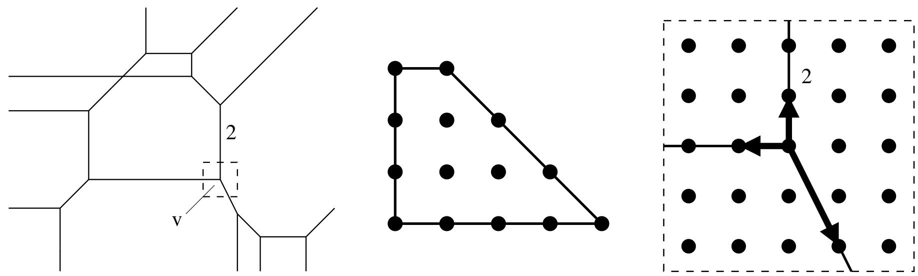

Figure 1. A tropical curve (left) of degree Δ (middle) and a balanced vertex (right).

Definition 3.2. A *(parametrized) tropical curve of degree* Δ is an abstract tropical curve C, together with a continuous map $h : C \to \mathbb{R}^2$ satisfying the following.

(i) (Rational slope) The map h is affine linear on each edge e of C, i.e., $h|_e(t) = t \cdot v + a$ for some non-zero $v \in \mathbb{Z}^2$ and $a \in \mathbb{R}^2$. If V is a vertex of the edge e and we parametrize e starting at V, then we call the above v the *direction vector* of e starting at V, and we write $v = v(V, e) \in \mathbb{Z}^2$. The lattice length of $v(V, e)$ (i.e., the greatest integral common divisor of the entries of $v(V, e)$) is the *weight* $\omega(e)$ of e. We call the integral vector $u(V, e) = (1/\omega(e))v(V, e)$ the *primitive direction vector* of e.

(ii) (Balancing) Each vertex V of C is *balanced*, i.e.,

$$\sum_{e:\, V \in \partial e} v(V, e) = 0.$$

(iii) (Degree) For each primitive vector $u \in \mathbb{Z}^2$, the total weight of the unbounded edges with primitive direction vector u equals the lattice length of an edge of $\partial\Delta$ with outer normal vector u (if there is no such edge, we require the total weight to be zero).

Example 3.3. In Figure 1 (left), is an example of a (parametrized) tropical curve of degree Δ, pictured to its right. One edge is of weight 2, all others have weight 1 (omitted in the drawing). All vertices of C are balanced, for vertex v this means that $2\binom{0}{1} + \binom{-1}{0} + \binom{1}{-2} = 0$. The place where two edges in this graph cross is not a 4-valent vertex but the image of two edges of the underlying abstract tropical curve.

In order to define the tropical analogs of the Severi degree and its refinement, we recall the following tropical notions (cf. [Mik05, § 2]). We sometimes abuse notation and simply write C for the parametrized tropical curve (C, h) if no confusion can occur.

Definition 3.4. (i) We say that a tropical curve (C, h) is *irreducible* if the underlying topological space of C has exactly one component. The *genus* $g(C, h)$ of an irreducible tropical curve (C, h) is the genus (i.e., the first Betti number) of the underlying topological space of C.

(ii) The *dual subdivision* Δ_C of the parametrized tropical plane curve (C, h) is the unique subdivision of Δ whose 2-faces Δ_v correspond to the vertices v of $h(C)$ such that the (images of) edges e of C are orthogonal to the edges $e^\perp \in \mathbb{R}^2$ of Δ_C and, further, that the lattice length of $e^\perp$ equals $\omega(e)$; see Figure 2.

(iii) The tropical curve (C, h) is *nodal* if its dual subdivision Δ_C consists only of triangles and parallelograms.

(iv) We say that (C, h) is *simple* if all vertices of C are 3-valent, the self-intersections of h are disjoint from vertices, and the inverse image under h of self-intersection points consists of exactly two points of C.

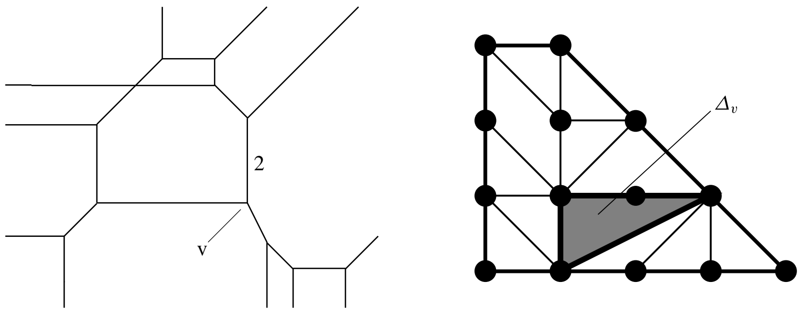

Figure 2. The dual subdivision of the curve of Example 3.3. The triangle Δ_v is dual to the vertex v.

(v) The *number of nodes* $\delta(C,h)$ of a nodal irreducible tropical curve of degree Δ is $\delta(C,h) = |\Delta^0 \cap \mathbb{Z}^2| - g(C,h)$, where $|\Delta^0 \cap \mathbb{Z}^2|$ is the number of interior lattice points of Δ.

(vi) Let (C,h) be a nodal tropical curve with irreducible components $(C_1,h_1),\ldots,(C_t,h_t)$ (i.e., C_i are the components of C and h_i are the restrictions of h to C_i), of degrees $\Delta_1,\ldots,\Delta_t$ and number of nodes $\delta_1,\ldots,\delta_t$, respectively. (Note that the Minkowski sum $\Delta_1+\cdots+\Delta_t$ equals Δ.) The *number of nodes* of (C,h) is

$$\delta(C,h) = \sum_{i=1}^{t} \delta_i + \sum_{i<j} \mathcal{M}(\Delta_i,\Delta_j),$$

where $\mathcal{M}(\Delta_i,\Delta_j) := \frac{1}{2}(\mathrm{Area}(\Delta_i+\Delta_j) - \mathrm{Area}(\Delta_i) - \mathrm{Area}(\Delta_j))$ is the *mixed area* of Δ_i and Δ_j. Here, $\mathrm{Area}(-)$ is the *normalized* area, given by twice the Euclidean area in $\mathbb{R}^2$.

Equivalently, $\delta(C,h)$ is the number of parallelograms, counted with their Euclidean area, of the dual subdivision Δ_C plus the number of edges of Δ_C, counted with their lattice length minus 1 (thus edges of length 1 do not contribute), if (C,h) is simple.

Example 3.3 *(continued).* The tropical curve of Example 3.3 has genus 1 as it is the image of a trivalent genus-1 graph. It is not the union of two tropical curves and thus irreducible. Its number of nodes is thus equal to $|\Delta^0 \cap \mathbb{Z}^2| - g = 3-1 = 2$. The two tropical nodes are 'visible' as the pair of edges crossing transversely as well as the edge of weight 2. (In general, a transverse intersection of two edges e and e' contributes $|u(V,e)\wedge u(V',e')|$ to $\delta(C)$, for any adjacent vertices V and V', while an edge of multiplicity m contributes $m-1$ to $\delta(C)$.)

Definition 3.4(v) is motivated by the classical degree–genus formula. In Definition 3.4(vi), the formula for $\delta(C,h)$ is chosen according to Bernstein's theorem [Ber75], so that Theorem 3.11 holds.

In [Mik05], Mikhalkin assigns to a 3-valent vertex v of a simple tropical curve (C,h) the *(Mikhalkin) vertex multiplicity*

$$\mathrm{mult}_{\mathbb{C}}(v) = \mathrm{Area}(\Delta_v). \tag{3.1}$$

To the tropical curve (C,h), he assigns the *(Mikhalkin) multiplicity*

$$\mathrm{mult}_{\mathbb{C}}(C,h) = \prod_v \mathrm{mult}_{\mathbb{C}}(v) = \prod_v \mathrm{Area}(\Delta_v), \tag{3.2}$$

the product running over the 3-valent vertices v of (C,h) and Δ_v is the triangle in the subdivision Δ_C dual to v (cf., Definition 3.4 and Figure 2). If v has adjacent edges e_1,e_2, and e_3, then the

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vertex multiplicity $\mathrm{mult}_{\mathbb{C}}(v)$ equals the Euclidean area of the parallelogram spanned by any two of the direction vectors starting at v.

Example 3.3 *(continued).* The dual subdivision of the tropical curve of Example 3.3 consists of 2 triangles of (normalized) area 2 and 9 triangles of area 1. The Mikhalkin multiplicity is thus $\mathrm{mult}_{\mathbb{C}}(C) = 2^2 \cdot 1^9 = 4$. (The quadrangle does not contribute to $\mathrm{mult}_{\mathbb{C}}(C)$.)

We associate to a simple tropical curve (C, h) a refined weight. Recall that, for an integer n, we denote by

$$[n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}} = y^{(n-1)/2} + \cdots + y^{-(n-1)/2}$$

the *quantum number* of n. In particular, $[n]_1 = n$. We can think about $[n]_y$ as a (shifted) q-analog of n.

Definition 3.5. The *refined vertex multiplicity* of a 3-valent vertex v of a simple tropical curve (C, h) is

$$\mathrm{mult}(v; y) = [\mathrm{Area}(\Delta_v)]_y. \tag{3.3}$$

The *refined multiplicity* of a simple tropical curve (C, h) is

$$\mathrm{mult}(C, h; y) = \prod_v [\mathrm{Area}(\Delta_v)]_y, \tag{3.4}$$

the product running over the 3-valent vertices of (C, h).

Example 3.3 *(continued).* The refined multiplicity of vertex v of the tropical curve of Example 3.3 is $[\mathrm{Area}(\Delta_v)]_y = [2]_y = y^{1/2} + y^{-1/2}$. As the dual subdivision consists of 2 triangles of area 2 and 9 triangles of area 1, the refined multiplicity of (C, h) is

$$\mathrm{mult}(C, h; y) = (y^{1/2} + y^{-1/2})^2 \cdot 1^9 = y + 2 + y^{-1}.$$

(Again, the quadrangle does not contribute.)

Remark 3.6. We emphasize that we define the refined weight only for simple tropical curves. This is sufficient for our purposes as we only consider tropical curves passing through generic points and such curves are necessarily simple (by [Mik05, Definition 4.7]).

We now define the tropical refinement of Severi degrees. For smooth toric surfaces, these invariants conjecturally agree with the refined invariants $\widetilde{N}^{(X(\Delta), L(\Delta)), \delta}(y)$, provided $L(\Delta)$ is sufficiently ample, see Conjecture 3.14.

As with classical curve counting, we require the configuration of tropical points to be in *tropically generic position*; the precise definition is given in [Mik05, Definition 4.7]. Roughly, tropically generic means there are no tropical curves of unexpectedly small degree passing through the points. By [Mik05, Proposition 4.11], the set of such points configurations is open and dense in the space of point configurations in $\mathbb{R}^2$.

An important example of a tropically generic point configuration is the following. The combinatorics of tropical curves passing through such configurations is essentially given by the floor diagrams of § 5.

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DEFINITION 3.7 [Bru13]. Let Δ be a lattice polygon. A point configuration $\Pi = \{(x_1, y_1), \ldots, (x_N, y_N)\}$ in $\mathbb{R}^2$ is called *vertically stretched with respect to* Δ if, for every tropical curve C of degree Δ, we have

$$\min_{i \neq j} |y_i - y_j| > \max_{i \neq j} |x_i - x_j| \cdot |\text{maximal slope of an edge of } C| \cdot (\text{number of edges of } C), \quad (3.5)$$

where for the maximal slope we only consider non-vertical edges of C.

The notion of a vertically stretched point configuration for a fixed polygon Δ is well defined, as (3.5) depends only on Π and the finitely many combinatorial types of tropical curves of degree Δ. Our definition of a vertically stretched point configuration is slightly more restricted than in [BM09, § 5] but has the advantage of being explicit. It is sufficient for the floor decomposition techniques of tropical curves [Bru13].

DEFINITION 3.8. Fix a lattice polygon Δ and $\delta \geqslant 0$.

(i) The *tropical refined Severi degree* $N^{\Delta,\delta}_{\text{trop}}(y)$ of the pair $(X(\Delta), L(\Delta))$ is

$$N^{\Delta,\delta}_{\text{trop}}(y) := \sum_{(C,h)} \text{mult}(C, h; y), \quad (3.6)$$

where the sum is over all δ-nodal tropical curves (C, h) of degree Δ passing through $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$ tropically generic points.

(ii) The *tropical irreducible refined Severi degree* of $(X(\Delta), L(\Delta))$ is

$$N^{\Delta,\delta}_{0,\text{trop}}(y) := \sum_{(C,h)} \text{mult}(C, h; y), \quad (3.7)$$

the sum ranging over all irreducible tropical curves of degree Δ with δ nodes passing through $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$ tropically generic points.

In the special case, when $S = \mathbb{P}^2$ and $\Delta = \Delta(\mathcal{O}(d))$, we simply write $N^{d,\delta}_{\text{trop}}(y)$, respectively $N^{d,\delta}_{0,\text{trop}}(y)$, for the tropical refined Severi degree, respectively tropical irreducible refined Severi degree of $\mathbb{P}^2$.

By Theorem 4.14, the tropical irreducible refined Severi degree agrees with its non-tropical version defined in (2.10) for $\mathbb{P}^2$, Hirzebruch surfaces and rational ruled surfaces. Itenberg and Mikhalkin showed that both refined tropical enumerations give indeed invariants.

THEOREM 3.9 [IM13, Theorem 1]. *The sum (3.7), and thus $N^{\Delta,\delta}_{0,\text{trop}}(y)$, are independent of the tropical point configuration, as long as the configuration is generic.*

COROLLARY 3.10. *The sum in (3.6), and thus $N^{\Delta,\delta}_{\text{trop}}(y)$, are independent of the tropical point configuration, as long as the configuration is generic.*

Proof. The refined Severi degree can be expressed in terms of the irreducible refined Severi degrees, which are, by Theorem 3.9, independent of the specific location of the points.

Specifically, let $\Pi \subset \mathbb{R}^2$ be a tropically generic set of $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$ points. Then (see also [AB13, § 2.3])

$$N^{\Delta,\delta}_{\text{trop}}(y) = \sum_{\Pi = \cup \Pi_i} \sum_{(\Delta_i, \delta_i)} \prod_i N^{\Delta_i,\delta_i}_{0,\text{trop}}(y), \quad (3.8)$$

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where the first sum is over all partitions of Π, and the second sum is over all pairs (Δ_i, δ_i) which satisfy

$$\begin{gathered} |\Pi_i| = |\Delta_i \cap \mathbb{Z}^2| - 1 - \delta_i \quad \text{for all } 1 \leqslant i \leqslant t, \\ \Delta = \Delta_1 + \cdots + \Delta_t \quad \text{(Minkowski sum)}, \\ \delta = \sum_{i=1}^{t} \delta_i + \sum_{1 \leqslant i < j \leqslant t} \mathcal{M}(\Delta_i, \Delta_j). \end{gathered} \tag{3.9}$$

Here, again $\mathcal{M}(\Delta_i, \Delta_j) = \frac{1}{2}(\text{Area}(\Delta_i + \Delta_j) - \text{Area}(\Delta_i) - \text{Area}(\Delta_j))$ is the mixed area of the polygons Δ_i and Δ_j. □

At $y = 1$, we recover Mikhalkin's (complex) correspondence theorem.

Theorem 3.11 (Mikhalkin's (complex) correspondence theorem [Mik05, Theorem 1]). *For any lattice polygon Δ both the following hold.*

(i) *The (tropical) Severi degree $N^{\Delta,\delta}_{\text{trop}}(1)$ equals the (classical) Severi degree $N^{\Delta,\delta}$.*

(ii) *The (tropical) irreducible Severi degree $N^{\Delta,\delta}_{0,\text{trop}}(1)$ equals the irreducible (classical) Severi degree $N^{\Delta,\delta}_0$.*

At $y = -1$, we recover Mikhalkin's real correspondence theorem. The *classical Welschinger invariant* $W^{\Delta,\delta}(\Pi)$ and the *irreducible classical Welschinger invariant* $W^{\Delta,\delta}_0(\Pi)$ count real curves, respectively irreducible real curves, of degree Δ with δ nodes through the real point configuration Π, counted with Welschinger sign. For positive genus, unlike for Severi degrees, both invariants depend on the point configuration Π, even for generic Π. For details see [Mik05, § 7.3].

Theorem 3.12 (Mikhalkin's real correspondence theorem [Mik05, Theorem 6]). *For any lattice polygon Δ both the following hold.*

(i) *The (tropical) Welschinger invariant $W^{\Delta,\delta}_{\text{trop}}$ equals the (classical) Welschinger invariant $W^{\Delta,\delta}(\Pi)$ for some real point configuration Π.*

(ii) *The irreducible (tropical) Welschinger invariant $W^{\Delta,\delta}_{0,\text{trop}}$ equals the irreducible (classical) Welschinger invariant $W^{\Delta,\delta}_0(\Pi)$ for some real point configuration Π.*

Remark 3.13. The tropical refined Severi degrees $N^{\Delta,\delta}_{\text{trop}}(y)$ thus interpolate between Severi degrees and Welschinger invariants. Similarly, the refined irreducible Severi degrees $N^{\Delta,\delta}_{0,\text{trop}}(y)$ interpolate between irreducible (classical) Severi degrees and irreducible (classical) Welschinger invariants.

In Theorem 7.5 we will show that the (tropical) refined Severi degrees $N^{\Delta,\delta}_{\text{trop}}(y)$ coincide with the refined Severi degrees defined above in the case of $\mathbb{P}^2$, Σ_m and $\mathbb{P}(1,1,m)$. Therefore the following is a generalization of Conjecture 2.11.

Conjecture 3.14. Let Δ be a convex lattice polygon, such that $S = X(\Delta)$ is a smooth surface and $L = L(\Delta)$ a δ-very ample line bundle. Then the (tropical) refined Severi degrees are computed by the universal formulas:

$$N^{\Delta,\delta}_{\text{trop}}(y) = \widetilde{N}^{(S,L),\delta}(y).$$

In [KS13, Corollary 6] the following is proven (without the restriction on toric surfaces) for the non-refined invariants; we expect the same is true also in the refined case.

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Conjecture 3.15. Let $S = X(\Delta)$ be a classical toric del Pezzo surface, and $L = L(\Delta)$. Assume the following loci have codimension more than δ in $|L|$:

(i) the non-reduced curves;

(ii) the curves with a (-1) curve as a component.

Then

$$N^{\Delta,\delta}_{\text{trop}}(y) = \widetilde{N}^{(S,L),\delta}(y).$$

4. From refined tropical curve counting to refined invariants

In this section, we state and show a few properties of refined tropical Severi degrees and relate them to the refined invariants and refined Severi degrees of § 2. Specifically, we state that the refined Severi degrees agree with the tropical refined Severi degrees in § 4.1, discuss the polynomiality of refined Severi degrees in the parameters of Δ in § 4.2, conjecture the polynomiality of their coefficients (as Laurent polynomials in y) in § 4.3, discuss implications for the conjectures of Göttsche and Shende in § 4.4, and irreducible refined Severi degrees in § 4.5.

4.1 Refined Severi degress equal tropical refined Severi degrees

The refined tropical Severi degrees constructed in the last section agree with the refined Severi degrees as defined by Göttsche and Shende's recursion 2.8. In fact, achieving this was one of the main motivations in the construction of the tropical analog. We postpone the proof until § 7 as it relies on a generalization of tropical Severi degrees incorporating tangency conditions. The following theorem is a special case of Theorem 7.5. It will allow us to use in the following the refined Severi degrees and the tropical refined Severi degrees interchangeably.

Theorem 4.1. *For all polygons Δ, with $X(\Delta) = \mathbb{P}(1, 1, m)$ or $X(\Delta) = \Sigma_m$, the refined tropical Severi degrees satisfy (2.7) with $L = L(\Delta)$. Therefore, the refined Severi degrees defined via the recursion 2.8 and the tropical refined Severi degrees agree:*

$$N^{\Delta,\delta}(y) = N^{\Delta,\delta}_{\text{trop}}(y).$$

4.2 Refined node polynomials

We will now prove Conjecture 3.14 for the projective plane $\mathbb{P}^2$ and $\delta \leqslant 10$, for $\mathbb{P}^1 \times \mathbb{P}^1$ for $\delta \leqslant 6$ and for all Hirzebruch surfaces Σ_m for $\delta \leqslant 2$ and $\mathbb{P}(1, 1, m)$ for $\delta \leqslant 2$.

First we state the existence of refined node polynomials $N_\delta(d; y)$, $N_\delta(c, d, m; y)$, $N_\delta(d, m; y)$, refining some results of [FM10, AB13]. The proof of the following theorem is in § 6.

Theorem 4.2. *For fixed $\delta \geqslant 1$ we have the following.*

(i) ($\mathbb{P}^2$) *There is a polynomial $N_\delta(d; y) \in \mathbb{Q}[y^{\pm1}][d]$ of degree at most 2δ in d such that, for $d \geqslant \delta$,*

$$N_\delta(d; y) = N^{d,\delta}_{\text{trop}}(y).$$

(ii) (*Hirzebruch surface*) *There is a polynomial $N_\delta(c, d, m; y) \in \mathbb{Q}[y^{\pm1}][c, d, m]$ of degree at most δ in c, m and 2δ in d such that, for $c + m \geqslant 2\delta$ and $d \geqslant \delta$*

$$N_\delta(c, d, m; y) = N^{(\Sigma_m, cF+dH),\delta}_{\text{trop}}(y).$$

(iii) $(\mathbb{P}(1,1,m))$ *There is a polynomial* $N_\delta(d,m;y) \in \mathbb{Q}[y^{\pm 1}][d,m]$ *of degree at most* 2δ *in* d *and* δ *in* m *such that, for* $d \geqslant \delta$ *and* $m \geqslant 2\delta$,

$$N_\delta(d,m;y) = N_{\text{trop}}^{(\mathbb{P}(1,1,m),dH),\delta}(y).$$

Note that by Theorem 4.1 we can replace $N_{\text{trop}}^{d,\delta}(y)$, $N_{\text{trop}}^{(\mathbb{P}(1,1,m),dH),\delta}(y)$, $N_{\text{trop}}^{(\Sigma_m,cF+dH),\delta}(y)$ *by* $N^{d,\delta}(y)$, $N^{(\mathbb{P}(1,1,m),dH),\delta}(y)$, $N^{(\Sigma_m,cF+dH),\delta}(y)$ *respectively.*

We call the polynomials $N_\delta(d;y)$, $N_\delta(c,d,m;y)$, and $N_\delta(d,m;y)$ *refined node polynomials.*

Remark 4.3. Theorem 4.2 generalizes to toric surfaces from 'h-transverse' polygons with bounds exactly as in Theorems 1.2 and 1.3 in [AB13]. The argument of [AB13] generalizes to the refined setting by replacing all (Mikhalkin) weights by refined weights. As the argument is long and technical, we do not reproduce it here and restrain ourselves to more manageable cases.

Theorem 4.4. (i) $(\mathbb{P}^2)$ *For* $\delta \leqslant 10$ *and* $d \geqslant \delta/2+1$ *we have*

$$\widetilde{N}^{d,\delta}(y) = N_\delta(d;y) = N^{d,\delta}(y).$$

(ii) *For* $\delta \leqslant 6$ *and* $c,d \geqslant \delta/2$, *we have*

$$\widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y) = N_\delta(c,d,0;y) = N^{(\mathbb{P}_1\times\mathbb{P}_1,cF+dH),\delta}(y).$$

(iii) *(Hirzebruch surfaces) For* $\delta \leqslant 2$ *and* $d \geqslant 1$, $c \geqslant \delta$ *we have*

$$\widetilde{N}^{(\Sigma_m,cF+dH),\delta}(y) = N_\delta(c,d,m;y) = N^{(\Sigma_m,cF+dH),\delta}(y).$$

(iv) $(\mathbb{P}(1,1,m))$ *For* $\delta \leqslant 2$ *and* $d \geqslant 2$ *and* $m \geqslant 1$ *we have*

$$N_\delta(d,m;y) = N^{(\mathbb{P}(1,1,m),d),\delta}(y),$$

and $N_\delta(d,m;y)$ *is given by Conjecture 2.14 and Remark 2.15.*

Proof. In [GS14] we have computed $\widetilde{N}^{(S,L),\delta}(y)$ for all (S,L) and all $\delta \leqslant 10$. It is a polynomial of degree δ in the intersection numbers L^2, LK_S, K_S^2 and $\chi(\mathcal{O}_S)$.

(i) In the case $(S,L) = (\mathbb{P}^2, \mathcal{O}(d))$ this gives $\widetilde{N}^{d,\delta}(y)$ as a polynomial of degree 2δ in d. Using the Recursion 2.7 we compute $N^{d,\delta}(y)$ for all $\delta \leqslant 10$ and all $d \leqslant 30$. We find that $N^{d,\delta}(y) = \widetilde{N}^{d,\delta}(y)$ for $\delta \leqslant 10$, and $\delta/2+1 \leqslant d \leqslant 30$. We also know by Theorems 4.2 and 4.1 that $N_\delta(d)$ is a polynomial of degree 2δ in d, and that $N_\delta(d) = N^{d,\delta}(y)$ for $d \geqslant \delta$. Thus for $0 \leqslant \delta \leqslant 10$ the two polynomials $N_\delta(d)$ and $\widetilde{N}^{d,\delta}(y)$ of degree 2δ in d have the same value for $\delta \leqslant d \leqslant 30$. Thus they are equal.

(ii) Is very similar to (i). Using the recursion 2.7 we compute $N^{(\mathbb{P}_1\times\mathbb{P}_1,cF+dH),\delta}(y)$ for $c \leqslant 18$ and $d \leqslant 12$ and $\delta \leqslant 6$. We find that in this realm $N^{(\mathbb{P}_1\times\mathbb{P}_1,cF+dH),\delta}(y) = \widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y)$ for $c,d \geqslant \delta/2$. We know by Theorem 4.2 and symmetry, that $N_\delta(c,d,0;y)$ is a polynomial of bidegree (δ,δ) in c,d. Thus for $0 \leqslant \delta \leqslant 6$, the two polynomials $N_\delta(c,d,0;y)$ and $\widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y)$ have the same value, whenever $18 \geqslant c \geqslant 2\delta$, $12 \geqslant d \geqslant \delta$. Thus they are equal.

(iii) This case is again similar. We compute $N^{(\Sigma_m,cF+dH),\delta}(y)$ for $c \leqslant 6$ and $d \leqslant 6$, $m \leqslant 4$ and $\delta \leqslant 2$. The claim follows in the same way as before.

(iv) We compute $N^{(\mathbb{P}(1,1,m)dH),\delta}(y)$ for $d \leqslant 6$, $m \leqslant 6$ and $\delta \leqslant 2$. The claim follows in the same way as before. ☐

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Corollary 4.5. *The coefficients of the refined invariants* $\widetilde{N}^{(S,L),\delta}(y)$ *are non-negative, i.e.,*

$$\widetilde{N}^{(S,L),\delta}(y) \in \mathbb{Z}_{\geqslant 0}[y^{\pm 1}]$$

provided:

- *either* $S = \mathbb{P}^2$, $L = dH$, $\delta \leqslant 10$, *and* $d \geqslant d/2 + 1$
- *or* $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L = cF + dH$, $\delta \leqslant 6$, *and* $c, d \geqslant \delta/2$;
- $S = \Sigma_m$, $L = cF + dH$, $\delta \leqslant 2$, *and* $d \geqslant 1$, $c \geqslant \delta$;
- $S = \mathbb{P}(1, 1, m)$, $L = dH$, $\delta \leqslant 2$, *and* $d \geqslant 2$, $m \geqslant 1$.

Proof. For any convex lattice polygon, the tropical refined Severi degree $N^{\Delta,\delta}_{\text{trop}}(y)$ is a Laurent polynomial in y with non-negative coefficients. The corollary follows from Theorem 4.4. □

Conjecture 4.6. For any smooth projective surface S and δ-very ample line bundle L on S, the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ have non-negative coefficients.

We have the following evidence for this conjecture. In [GS13] Conjecture 2.7 is proven for S an abelian or K3 surface, and the positivity of $\widetilde{N}^{(S,L),\delta}(y)$ follows for all line bundles L on S. If S is a toric surface and L is δ-very ample on S, then Conjecture 4.6 is implied by Conjecture 3.14. Numerical computations give in all examples considered that Conjecture 4.6 is true. Comparing with (2.6) numerical checks confirm that, in the realm checked, for $l > \delta$ all the coefficients of $(t/g(y,t))^l$ of degree at most δ in t are positive. If L is δ-very ample we expect $\chi(L) > \delta$ and also $\chi(L)$ that is large with respect to K_S^2 and LK_S. Therefore we would expect that all coefficients of the left-hand side of (2.6) of degree at most δ in t are non-negative.

4.3 Coefficient polynomiality of tropical refined Severi degrees

The tropical refined Severi degrees $N^{d,\delta}_{\text{trop}}(y)$ of $\mathbb{P}^2$, as Laurent polynomial in y, have non-negative integral coefficients. Furthermore, for fixed δ, these coefficients behave polynomially in d, for sufficiently large d, by Theorem 4.2. In this section, we conjecture that particular coefficients of the tropical refined Severi degree are polynomial in d *independent of* δ (Conjecture 4.9). We also give enumerative meaning to the first leading coefficient (Proposition 4.11). For simplicity, we consider only $\mathbb{P}^2$ in this section. Throughout this section, we fix the number of nodes $\delta \geqslant 1$.

Notation 4.7. We denote the coefficients of the tropical refined Severi degree by

$$N^{d,\delta}_{\text{trop}}(y) = p^\delta_{d,0} \cdot y^\delta + p^\delta_{d,1} \cdot y^{\delta-1} + p^\delta_{d,2} \cdot y^{\delta-2} + \cdots + p^\delta_{d,\delta} \cdot y^0 + \cdots + p^\delta_{d,0} \cdot y^{-\delta}$$

for $p^\delta_{d,0}, p^\delta_{d,1}, \ldots, p^\delta_{d,\delta} \in \mathbb{Z}_{\geqslant 0}$.

Similarly, we write the coefficients of the refined node polynomial as

$$N_\delta(d; y) = p^\delta_0(d) \cdot y^\delta + p^\delta_1(d) \cdot y^{\delta-1} + p^\delta_2(d) \cdot y^{\delta-2} + \cdots + p^\delta_\delta(d) \cdot y^0 + \cdots + p^\delta_0(d) \cdot y^{-\delta}$$

for polynomials $p^\delta_0(d), p^\delta_1(d), \ldots, p^\delta_\delta(d) \in \mathbb{Z}[d]$.

From Theorem 4.2, the following is immediate.

Corollary 4.8. *For* $0 \leqslant i \leqslant \delta$, *we have* $p^\delta_i(d) = p^\delta_{d,i}$, *whenever* $d \geqslant \delta$.

Conjecturally, we have the lower bound $d \geqslant \delta/2+1$ (cf., Conjecture 2.11), which still depends on δ. We conjecture that for the leading coefficients of the refined Severi degree, this dependence disappears.

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Conjecture 4.9. For $0 \leqslant i \leqslant \delta$, we have $p_i^\delta(d) = p_{d,i}^\delta$, whenever $d \geqslant i+2$.

In other words, the larger the order of the coefficients of the refined Severi degree, the sooner the polynomiality kicks in. This conjecture was predicted as part of [GS14, Conjecture 89], where in addition a formula for the coefficients $p_i^\delta(d)$ was conjectured. Proposition 4.11 below gives a new proof for $i = 0$.

Remark 4.10. (i) Conjecture 4.9 is part of [GS14, Conjecture 89(1)].

(ii) More precisely, this conjecture says that $p_i^\delta(d)$ is a polynomial of degree 2δ in d, which is divisible by $\binom{\binom{d-1}{2}-3i}{\delta-i}$. Moreover, [GS14, Conjectures 86, 87] give a conjectural formula for the quotient $p_i^\delta(d)/\binom{\binom{d-1}{2}-3i}{\delta-i}$ in terms of the $\widetilde{N}^{d,\delta}(y)$ with $\delta \leqslant 3i$. Thus, assuming these conjectures, Theorem 4.4 gives a formula for $p_i^\delta(d)$ for $i \leqslant 3$.

(iii) Computational evidence suggests that for $d \geqslant 2$ the bound in Conjecture 4.9 is optimal: $p_i^\delta(d) = p_{d,i}^\delta$, if and only if $d \geqslant i+2$. We checked this for $d \leqslant 14$, $\delta \leqslant 11$.

We give a formula for the leading coefficient of the refined Severi degree. This result was also obtained in [GS14, Proposition 83] (for refined Severi degrees) and [IM13, Proposition 2.11] (for tropical refined Severi degrees).

Proposition 4.11. *The leading coefficients of $N_{\mathrm{trop}}^{d,\delta}(y)$ is given by*

$$p_{d,0}^\delta = \binom{\binom{d-1}{2}}{\delta} \quad \text{for } d \geqslant 1.$$

The formula could be interpreted as the number of ways to choose δ of the $\binom{d-1}{2}$ nodes of a genus 0 nodal curve C of degree d, i.e., as the number of δ-nodal curves obtained as partial resolutions of C.

We prove this proposition in § 6.

The same formulas hold for the coefficients of the tropical irreducible refined Severi degrees $N_{0,\mathrm{trop}}^{d,\delta}(y)$. Again we can write $N_{0,\mathrm{trop}}^{d,\delta}(y) = p_{d,0}^{\delta,0} y^\delta + p_{d,1}^{\delta,0} y^{\delta-1} + \cdots + p_{d,1}^{\delta,0} y^{-\delta+1} + p_{d,0}^{\delta,0} y^{-\delta}$. Assuming Conjecture 4.9, a similar result also holds for the $p_{d,i}^{\delta,0}$, because of the following lemma.

Lemma 4.12. *Assuming Conjecture 4.9, we have $p_{d,i}^{\delta,0} = p_{d,i}^\delta$ if $d \geqslant i+2$.*

Proof. If we specialize the formula (3.8) to $N_{\mathrm{trop}}^{d,\delta}(y)$, we express $N_{\mathrm{trop}}^{d,\delta}(y) - N_{0,\mathrm{trop}}^{d,\delta}(y)$ as a sum of products $\prod_{i=1}^t N_{0,\mathrm{trop}}^{d_i,\delta_i}(y)$, with $t \geqslant 2$, $d = d_1 + \cdots + d_t$ and

$$\delta = \sum_{i=1}^t \delta_i + \frac{1}{2} \sum_{1 \leqslant i < j \leqslant t} ((d_i + d_j)^2 - d_i^2 - d_j^2).$$

It is an easy exercise to see that for given d the rightmost sum is minimal if $t = 2$ and $\{d_1, d_2\} = \{1, d-1\}$, and the corresponding sum is $d-1$. Thus in all summands for $N_{\mathrm{trop}}^{d,\delta}(y) - N_{0,\mathrm{trop}}^{d,\delta}(y)$ we have $\delta - \sum_i \delta_i \geqslant d-1$. As the $N_{0,\mathrm{trop}}^{d_i,\delta_i}(y)$ have degree at most δ_i in y, y^{-1}, we see that $p_{d,i}^\delta = p_{d,i}^{\delta,0}$ for $i < d-1$. □

The argument also shows that $N_{\mathrm{trop}}^{d,\delta}(y) = N_{0,\mathrm{trop}}^{d,\delta}(y)$ if $\delta \leqslant d-2$. Thus we obtain the following corollary.

Corollary 4.13. *For $\delta \leqslant d-2$ we have $N_{0,\mathrm{trop}}^{d,\delta}(y) = N_\delta(d; y)$.*

4.4 Numerical evidence for Göttsche and Shende's conjectures

Theorems 4.2 and 4.4 provide strong evidence for Conjectures 2.7, 2.11: on $\mathbb{P}^2$ and rational ruled surfaces, for L sufficiently ample with respect to δ, $N^{(S,L),\delta}(y)$ is indeed given by a node polynomial in L^2, LK_S, K_S^2 and $\chi(\mathcal{O}_S)$. Furthermore, if δ is not too large, we show that this polynomial coincides with $\widetilde{N}^{(S,L),\delta}(y)$. Unfortunately in the case of rational ruled surfaces we only prove this for $\delta \leqslant 2$. There is, however, more and stronger numerical evidence, even if it does not lead to a proof of formulas for higher δ. Below we list briefly some of this evidence.

(i) In [GS14] the $N^{d,\delta}(y)$ have been computed for $d \leqslant 17$ and $\delta \leqslant 32$. Assuming Conjectures 2.7, 2.11 this determines the power series $B_1(y,q)$ and $B_2(y,q)$ modulo q^{29}, and thus all the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ as polynomials in L^2, LK_S, K_S^2, $\chi(\mathcal{O}_S)$ for all S, L and all $\delta \leqslant 28$. Denote for the moment $\widehat{N}^{(S,L),\delta}(y)$ the refined invariants obtained this way (and $\widehat{N}^{d,\delta}(y)$ the corresponding invariants of $\mathbb{P}^2$). For $\delta \leqslant 10$ (where the $\widetilde{N}^{(S,L),\delta}(y)$ have been computed in [GS14]) $\widehat{N}^{(S,L),\delta}(y) = \widetilde{N}^{(S,L),\delta}(y)$.

The computation mentioned above gives $N^{d,\delta}(y) = \widehat{N}^{d,\delta}(y)$ for $d \leqslant 17$ and $\delta \leqslant \min(2d-2, 28)$.

(ii) We have also computed the $N^{d,\delta}(y)$ for $d \leqslant 20$, $\delta \leqslant 20$, again within this realm $N^{d,\delta}(y) = \widehat{N}^{d,\delta}(y)$ for $\delta \leqslant 2d-2$.

(iii) We computed $N^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y)$ for arbitrary δ and $c, d \leqslant 8$. We find in this realm $N^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y) = \widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y)$ for $\delta \leqslant \min(2c, 2d)$.

(iv) We computed $N^{(\Sigma_m, cF+dH),\delta}(y)$ for $m \leqslant 10$, $\delta \leqslant 10$, $d \leqslant 6$, $c \leqslant 10$. We find in this realm $N^{(\Sigma_m, cF+dH),\delta}(y) = \widetilde{N}^{(\Sigma_m, cF+dH),\delta}(y)$ if $\delta \leqslant \min(2d, c)$.

4.5 On the relation with irreducible refined Severi degrees

We show that the irreducible refined Severi degree, formally defined in (2.10) for $\mathbb{P}^2$, Hirzebruch surfaces and rational ruled surfaces, agrees with the refined enumeration of irreducible tropical curves. It therefore follows that also the irreducible refined Severi degree has non-negative coefficients.

Theorem 4.14. *The tropical irreducible refined Severi degree $N_{0,\mathrm{trop}}^{\Delta,\delta}(y)$ agrees with the irreducible refined Severi degree defined in (2.10).*

The refined multiplicity of an irreducible tropical curve by definition has non-negative integer coefficients in $y^{\pm 1}$. Therefore, we have shown the following.

Corollary 4.15. *$N_0^{\Delta,\delta}(y)$ has non-negative integer coefficients.*

Proof of Theorem 4.14. Recall the relation (3.8) between tropical refined Severi degrees and their tropical irreducible analog

$$N_{\mathrm{trop}}^{\Delta,\delta}(y) = \sum_{\Pi=\cup\Pi_i} \sum_{(\Delta_i,\delta_i)} \prod_i N_{0,\mathrm{trop}}^{\Delta_i,\delta_i}(y), \tag{4.1}$$

where the first sum is over all partitions of Π, and the second sum is over all pairs (Δ_i, δ_i) which satisfy (cf. (3.9))

$$\begin{gathered} |\Pi_i| = |\Delta_i \cap \mathbb{Z}^2| - 1 - \delta_i, \quad \text{for all } 1 \leqslant i \leqslant t, \\ \Delta = \Delta_1 + \cdots + \Delta_t \quad \text{(Minkowski sum)}, \\ \delta = \sum_{i=1}^{t} \delta_i + \sum_{1\leqslant i<j\leqslant t} \mathcal{M}(\Delta_i, \Delta_j). \end{gathered} \tag{4.2}$$

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Here, again $\mathcal{M}(\Delta_i, \Delta_j) = \frac{1}{2}(\mathrm{Area}(\Delta_i + \Delta_j) - \mathrm{Area}(\Delta_i) - \mathrm{Area}(\Delta_j))$ is the mixed area of the polygons Δ_i and Δ_j.

Any collection of lattice polygons $\Delta_1, \Delta_2, \dots, \Delta_t, \Delta$ and non-negative integers $\delta_1, \dots, \delta_t, \delta$ satisfying the second and third condition of (4.2) also satisfy

$$\sum_{i=1}^{t} (\dim \Delta_i - \delta_i) = \dim \Delta - \delta,$$

where we write $\dim \Delta = |\Delta \cap \mathbb{Z}^2| - 1$. Indeed, both sides equal the number of point conditions of a tropical curve of degree Δ with δ nodes which has irreducible components of degrees Δ_i with δ_i nodes, respectively. Furthermore, we have $\mathrm{mult}(C; y) = \prod_{i=1}^{t} \mathrm{mult}(C_i; y)$.

The exponential generating functions of the tropical refined Severi degrees $N^{\Delta,\delta}_{\mathrm{trop}}(y)$ and the tropical irreducible refined Severi degree $N^{\Delta,\delta}_{0,\mathrm{trop}}(y)$ thus satisfy

$$\exp\Bigg(\sum_{\Delta,\delta} \frac{z^{\dim \Delta - \delta}}{(\dim \Delta - \delta)!} v^{\Delta} N^{\Delta,\delta}_{0,\mathrm{trop}}(y)\Bigg) = 1 + \sum_{\Delta,\delta} \frac{z^{\dim \Delta - \delta}}{(\dim \Delta - \delta)!} v^{\Delta} N^{\Delta,\delta}_{\mathrm{trop}}(y), \tag{4.3}$$

where we define $v^{\Delta} \cdot v^{\Delta'} := v^{\Delta + \Delta'}$ for lattice polygons Δ and Δ', and both sums are over all lattice polygons Δ (up to translation) and $\delta \geqslant 0$. Comparing (4.3) and (2.10) and using Theorem 4.1, the result follows. □

5. *y*-Weighted floor diagrams and templates

Floor diagrams are purely combinatorial representations of tropical curves. They exist for all 'h-transverse' polygons Δ. We focus mostly on the cases $S = \mathbb{P}^2$, Σ_m, and $\mathbb{P}(1,1,m)$, all of whose moment polygons are h-transverse. More specifically, if we consider tropical curves through a vertically stretched point configuration (see Definition 3.7) the tropical curves are uniquely encoded by a 'marking' of a floor diagram and, vice versa, every marked floor diagram corresponds to a tropical curve. This gives a purely combinatorial way to compute refined Severi degrees for toric surfaces with h-transverse polygons. Floor diagrams were invented (in the unrefined setting) by Brugallé and Mikhalkin [BM07, BM09].

5.1 Floor diagrams

We now briefly review the marked floor diagrams of Brugallé and Mikhalkin [BM07, BM09] for surfaces $S = \mathbb{P}^2$, $S = \mathbb{P}(1,1,m)$, and $S = \Sigma_m$, with some emphasis on the $\mathbb{P}^2$ case. We present them in the notation of Ardila and Block [AB13], following Fomin and Mikhalkin [FM10]. In each case, we fix a polygon Δ (cf. Figure 3):

- ($\mathbb{P}^2$ case) $\Delta = \mathrm{conv}((0,0),(0,d),(d,0))$, for $d \geqslant 1$; or
- (Σ_m case) $\Delta = \mathrm{conv}((0,0),(0,d),(c,d),(c+md,0))$, for $c, d, m \geqslant 1$; or
- ($\mathbb{P}(1,1,m)$ case) $\Delta = \mathrm{conv}((0,0),(0,d),(dm,0))$, for $d, m \geqslant 1$. In this case, set $c = 0$.

Definition 5.1. A Δ-*floor diagram* $\mathcal{D}$ consists of:

(i) a graph on a vertex set $\{1, \dots, d\}$, possibly with multiple edges, directed such that if $i \to j$ is an edge, then $i < j$;

(ii) a sequence $(s_1, \dots, s_d)$ of non-negative integers such that $s_1 + \cdots + s_d = c$ (if $S = \mathbb{P}(1,1,m)$ then all s_i equal 0);

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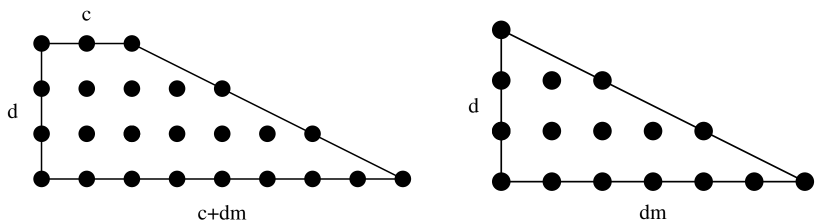


Figure 3. Lattice polygons of the Hirzebruch surface Σ_m with line bundle $L = dH + cF$ (left) and $\mathbb{P}(1,1,m)$ with $L = dH$ (right). In both cases $m = 2$. For $S = \mathbb{P}^2$, we set $c = 0$ and $m = 1$.

subject to the following condition (Divergence condition): for each vertex j of $\mathcal{D}$, we have

$$\operatorname{div}(j) \stackrel{\text{def}}{=} \sum_{\substack{\text{edges } e \\ j \stackrel{e}{\to} k}} \operatorname{wt}(e) - \sum_{\substack{\text{edges } e \\ i \stackrel{e}{\to} j}} \operatorname{wt}(e) \leqslant m + s_j.$$

The last condition says that at every vertex of $\mathcal{D}$ the total weight of the outgoing edges is larger by at most $m + s_j$ than the total weight of the incoming edges.

We loosely think of Δ as the *degree* of the floor diagram $\mathcal{D}$. If $S = \mathbb{P}^2$, we say that $\mathcal{D}$ is of degree d. A floor diagram is *connected* if its underlying graph is. If $\mathcal{D}$ is connected its *genus* is the genus of the underlying graph. A connected floor diagram $\mathcal{D}$ of degree Δ and genus g has *cogenus* $\delta(\mathcal{D})$ equal to the number of interior lattice points in Δ minus g.

If $\mathcal{D}$ is not connected, there are lattice polygons $\Delta_1, \Delta_2, \ldots$ such that their Minkowski sum equals $\Delta_1 + \Delta_2 + \cdots = \Delta$ and the Δ_i are the degrees of the connected components of $\mathcal{D}$. Let $\delta_1, \delta_2, \ldots$ be the cogenera of the connected components. Similarly to the case of tropical curves, we define the *cogenus*

$$\delta(\mathcal{D}) = \sum_i \delta_i + \sum_{i<j} \mathcal{M}(\Delta_i, \Delta_j),$$

where again $\mathcal{M}(\Delta_i, \Delta_j) := \frac{1}{2}(\operatorname{Area}(\Delta_i + \Delta_j) - \operatorname{Area}(\Delta_i) - \operatorname{Area}(\Delta_j))$ is the *mixed area* of Δ_i and Δ_j. As before, $\operatorname{Area}(-)$ is the normalized area, given by twice the Euclidean area in $\mathbb{R}^2$.

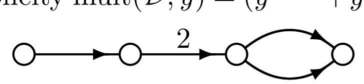
The refined multiplicity of tropical curves (see Definition 3.5) translates to floor diagram as follows, yielding a purely combinatorial formula for the refined Severi degrees for Σ_m and $\mathbb{P}(1,1,m)$ in Definition 5.6.

Definition 5.2. We define the *refined multiplicity* $\operatorname{mult}(\mathcal{D}, y)$ of a floor diagram $\mathcal{D}$ as

$$\operatorname{mult}(\mathcal{D}, y) = \prod_{\text{edges } e} ([\operatorname{wt}(e)]_y)^2.$$

Notice that the weight $\operatorname{mult}(\mathcal{D}, y)$ is a Laurent polynomial in y with positive integral coefficients. We draw floor diagrams using the convention that vertices in increasing order are arranged left to right. Edge weights of 1 are omitted.

Example 5.3. An example of a floor diagram for $\mathbb{P}^2$ of degree $d = 4$, genus $g = 1$, cogenus $\delta = 2$, divergences $1, 1, 0, -2$, and multiplicity $\operatorname{mult}(\mathcal{D}; y) = (y^{-1/2} + y^{1/2})^2 = y^{-1} + 2 + y$ is drawn below.



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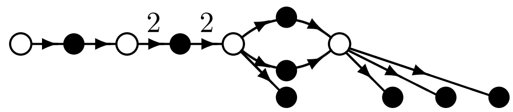

Figure 4. The result of applying Steps 1–3 to the floor diagram of Example 5.3.

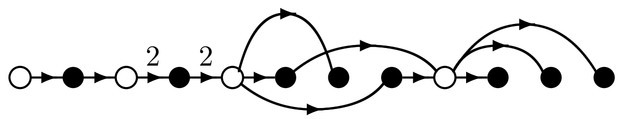

Figure 5. A marking of the floor diagram of Figure 4.

To a floor diagram we associate a last statistic, as in [FM10, § 1]. Notice that this statistic is independent of y.

Definition 5.4. A *marking* of a floor diagram $\mathcal{D}$ is defined by the following four step process.

Step 1: For each vertex j of $\mathcal{D}$ create s_j new indistinguishable vertices and connect them to j with new edges directed towards j.

Step 2: For each vertex j of $\mathcal{D}$ create $m + s_j - \operatorname{div}(j)$ new indistinguishable vertices and connect them to j with new edges directed away from j. This makes the divergence of vertex j equal to m.

Step 3: Subdivide each edge of the original floor diagram $\mathcal{D}$ into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Denote the resulting graph $\widetilde{\mathcal{D}}$. See, for example, Figure 4.

Step 4: Linearly order the vertices of $\widetilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram $\mathcal{D}$ such that, as before, each edge is directed from a smaller vertex to a larger vertex. See, for example, Figure 5.

The extended graph $\widetilde{\mathcal{D}}$ together with the linear order on its vertices is called a *marked floor diagram*, or a *marking* of the original floor diagram $\mathcal{D}$.

We want to count marked floor diagrams up to equivalence. Two markings $\widetilde{\mathcal{D}}_1$, $\widetilde{\mathcal{D}}_2$ of a floor diagram $\mathcal{D}$ are *equivalent* if there exists an automorphism of weighted graphs which preserves the vertices of $\mathcal{D}$ and maps $\widetilde{\mathcal{D}}_1$ to $\widetilde{\mathcal{D}}_2$. The *number of markings* $\nu(\mathcal{D})$ is the number of marked floor diagrams $\widetilde{\mathcal{D}}$ up to equivalence.

Example 5.5. The floor diagram $\mathcal{D}$ of Example 5.3 has $\nu(\mathcal{D}) = 3 + 4 = 7$ markings (up to equivalence): in step 3 the extra 1-valent vertex connected to the third white vertex from the left can be inserted in three ways between the third and fourth white vertex (up to equivalence) and in four ways right of the fourth white vertex (again up to equivalence).

With these two statistics, we define a purely combinatorial notion of refined Severi degrees for $S = \mathbb{P}^2$, $S = \Sigma_m$, and $S = \mathbb{P}(1, 1, m)$. The combinatorial invariants agree with the tropical refined Severi degree $N^{\Delta,\delta}_{\text{trop}}(y)$ of § 3 (Theorem 5.7). They also agree conjecturally with the refined invariants of Göttsche and Shende if S is smooth and the line bundle is sufficiently ample (cf. Conjecture 3.14 and Theorem 4.4).

See Remark 5.8 for a discussion how to generalize to a much larger family of toric surfaces corresponding to 'h-transverse' Δ. Denote by $\mathbf{FD}(\Delta, \delta)$ the set of Δ-floor diagrams $\mathcal{D}$ with cogenus δ.

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Definition 5.6. Fix $\delta \geqslant 0$ and let Δ be as above. We define the *combinatorial refined Severi degree* $N^{\Delta,\delta}_{\mathrm{comb}}(y)$ to be the Laurent polynomial in y given by

$$N^{\Delta,\delta}_{\mathrm{comb}}(y) = \sum_{\mathcal{D}\in \mathbf{FD}(\Delta,\delta)} \mathrm{mult}(\mathcal{D}; y) \cdot \nu(\mathcal{D}). \tag{5.1}$$

Theorem 5.7. *For Δ as in Definition 5.6 and $\delta \geqslant 0$, the combinatorial refined Severi degree and the tropical refined Severi degree agree:*

$$N^{\Delta,\delta}_{\mathrm{comb}}(y) = N^{\Delta,\delta}_{\mathrm{trop}}(y).$$

Proof. Let $\Pi \subset \mathbb{R}^2$ be a vertically stretched (Definition 3.7) configuration of $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$ tropical points. In [BM09, Proposition 5.9], Brugallé and Mikhalkin construct an explicit bijection between the set of parametrized tropical curves of degree Δ with δ nodes passing through Π and the set of marked Δ-floor diagrams of cogenus δ. This bijection is y-weight preserving. $\square$

In what follows, we will usually write N instead of N_{comb} even while referring to the combinatorial defined refined Severi degree if no confusion can occur.

Remark 5.8. We expect the results in this section to also hold for toric surfaces from 'h-transverse' polygons Δ. Brugallé and Mikhalkin [BM09] construct marked floor diagrams for this class of polygons. One can define a notion of *combinatorial refined Severi degrees for any toric surface from an 'h-transverse polygon'*: simply replace the multiplicity of a 'Δ-floor diagram' $\mathcal{D}$ in [AB13, Equation (Severi1)] by the y-weight

$$\mathrm{mult}(\mathcal{D}, y) = \prod_{\text{edges } e} ([\mathrm{wt}(e)]_y)^2.$$

Theorem 5.7 can then be extended to the more general setting. We omit the details here to avoid too many technicalities.

5.2 Templates

The following gadget was introduced by Fomin and Mikhalkin [FM10].

Definition 5.9. A *template* Γ is a directed graph (possibly with multiple edges) on vertices $\{0, \ldots, l\}$, where $l \geqslant 1$, with edge weights $\mathrm{wt}(e) \in \mathbb{Z}_{>0}$, satisfying the following.

(i) If $i \to j$ is an edge, then $i < j$.

(ii) Every edge $i \xrightarrow{e} i + 1$ has weight $\mathrm{wt}(e) \geqslant 2$. (No 'short edges.').

(iii) For each vertex j, $1 \leqslant j \leqslant l - 1$, there is an edge 'covering' it, i.e., there exists an edge $i \to k$ with $i < j < k$.

Every template Γ comes with some numerical data associated with it. Its *length* $\ell(\Gamma)$ is the number of vertices minus 1. Its *cogenus* $\delta(\Gamma)$ is

$$\delta(\Gamma) = \sum_{i \xrightarrow{e} j} [(j - i)\mathrm{wt}(e) - 1]. \tag{5.2}$$

We define its *y-multiplicity* $\mathrm{mult}(\Gamma, y)$ to be

$$\mathrm{mult}(\Gamma, y) = \prod_{\text{edges } e} ([\mathrm{wt}(e)]_y)^2.$$

See Figure 6 for examples.

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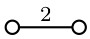
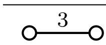
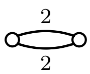
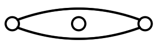
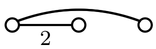
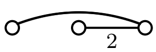
Γ	$\delta(\Gamma)$	$\ell(\Gamma)$	$\mathrm{mult}(\Gamma; y)$	$\varepsilon_0(\Gamma)$	$\varepsilon_1(\Gamma)$	$\varkappa(\Gamma)$	$k_{\min}(\Gamma)$
	1	1	$y^{-1}+2+y$	0	0	(2)	2
	1	2	1	1	1	(1,1)	1
	2	1	$y^{-2}+2y^{-1}+3+2y+y^2$	0	0	(3)	3
	2	1	$y^{-2}+4y^{-1}+6+4y+y^2$	0	0	(4)	4
	2	2	1	1	1	(2,2)	2
	2	2	$y^{-1}+2+y$	0	1	(3,1)	3
	2	2	$y^{-1}+2+y$	1	0	(1,3)	2
	2	3	1	1	1	(1,1,1)	1
	2	3	1	1	1	(1,2,1)	1

Figure 6. The templates with $\delta(\Gamma) \leqslant 2$.

For $1 \leqslant j \leqslant \ell(\Gamma)$, let $\varkappa_j = \varkappa_j(\Gamma)$ denote the sum of the weights of edges $i \to k$ with $i < j \leqslant k$. So $\varkappa_j(\Gamma)$ equals the total weight of the edges of Γ from a vertex left of j to a vertex right of or equal to j. Define

$$k_{\min}(\Gamma) = \max_{1 \leqslant j \leqslant l}(\varkappa_j - j + 1).$$

For $S = \mathbb{P}^2$, this makes $k_{\min}(\Gamma)$ the smallest positive integer k such that Γ can appear in a floor diagram on $\{1, 2, \dots\}$ with left-most vertex k when the floor diagram is composed into templates as explained in § 5.3 (see Example 5.10). Lastly, set

$$\varepsilon_0(\Gamma) = \begin{cases} 1 & \text{if all edges starting at 0 have weight 1,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\varepsilon_1(\Gamma) = \begin{cases} 1 & \text{if all edges arriving at } l \text{ have weight 1,} \\ 0 & \text{otherwise.} \end{cases}$$

Figure 6 (taken from Fomin–Mikhalkin [FM10]) shows all templates Γ with $\delta(\Gamma) \leqslant 2$.

Notice that, for each δ, there are only a finite number of templates with cogenus δ. At $y = 1$, we recover Fomin and Mikhalkin's template multiplicity $\prod_e \mathrm{wt}(e)^2$. It is clear that $\mathrm{mult}(\Gamma, y)$ is a Laurent polynomial with positive integral coefficients.

5.3 Decomposition into templates

A floor diagram $\mathcal{D}$ with d vertices decomposes into an ordered collection $(\Gamma_1, \dots, \Gamma_s)$ of templates as follows. If $S = \mathbb{P}^2$ or $\mathbb{P}(1, 1, m)$, then we set as before $c = 0$. We treat $S = \mathbb{P}^2$ as the special case of $\mathbb{P}(1, 1, m)$ for $m = 1$.

First, add an additional vertex 0 ($<$1) to $\mathcal{D}$ and connect it to every vertex j of $\mathcal{D}$ by s_j many new edges of weight 1 from 0 to j for each $1 \leqslant j \leqslant d$. (For $S = \mathbb{P}^2$ and $S = \mathbb{P}(1, 1, m)$, there is nothing to do, as $s_j = 0$ for all j.) Second, add an additional vertex $d + 1$ ($> d$), together with

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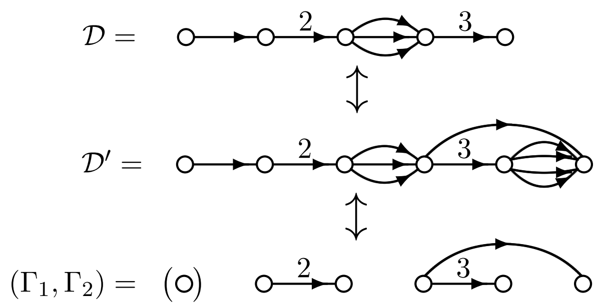
$m + s_j - \operatorname{div}(j)$ new edges of weight 1 from j, for each $1 \leqslant j \leqslant d$. The *divergence sequence* of the resulting diagram $\mathcal{D}'$ is $\mathbf{a} := (c, m, \ldots, m) \in \mathbb{Z}_{\geqslant 0}^{d+1}$, after we remove the (superfluous) last entry. Now remove all *short edges* from $\mathcal{D}'$, that is, all edges of weight 1 between consecutive vertices. The result is an ordered collection of templates $(\Gamma_1, \ldots, \Gamma_s)$, listed left to right. We also keep track of the initial vertices $k_1, \ldots, k_s$ of these templates.

Conversely, given the collection of templates $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_s)$, the initial vertices $k_1, \ldots, k_s$, and the divergence sequence $(c, m, \ldots, m) \in \mathbb{Z}_{\geqslant 0}^{d+1}$, this process is easily reversed. To recover $\mathcal{D}'$, we first place the templates at their starting points k_i in the interval $[0, \ldots, M]$, and add in all short edges we removed from $\mathcal{D}'$. More precisely, we need to add $(a_0 + \cdots + a_{j-1} - \varkappa_{j-k_i}(\Gamma_i))$ short edges between $j-1$ and j, where Γ_i is the template containing j. The sequence s records the number s_j of edges between vertices 0 and j. Finally, we remove the first and last vertices and their incident edges to obtain $\mathcal{D}$.

Example 5.10. An example for $S = \mathbb{P}^2$ of the decomposition of a floor diagram into templates is illustrated below. Here, $k_1 = 2$ and $k_2 = 4$ and all $s_j = 0$. We see that $k_{\min} = 2$ for the left template because a floor diagram for $\mathbb{P}^2$ cannot have an edge of weight 2 adjacent to its leftmost vertex because of the divergence condition.



We record, for each ordered template collection $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_s)$, all valid 'positions' k_i that can occur in the template decomposition of a Δ-floor diagram by the lattice points in a polytope. There are two cases. If $S = \mathbb{P}^2$, we set

$$A_{\mathbf{\Gamma}}(d) = \{\mathbf{k} \in \mathbb{R}^s : k_i \geqslant k_{\min}(\Gamma_i), \\ k_i + \ell(\Gamma_i) \leqslant k_{i+1}\ (1 \leqslant i < s), k_s + \ell(\Gamma_s) \leqslant d + \varepsilon_1(\Gamma_s)\}. \tag{5.3}$$

If $S = \mathbb{P}(1,1,m)$ or $S = \Sigma_m$, we set

$$A_{\mathbf{\Gamma}}(d) = \{\mathbf{k} \in \mathbb{R}^s : k_1 \geqslant 1 - \varepsilon_0(\Gamma_1),\ k_i \geqslant k_{\min}(\Gamma_i), \\ k_i + \ell(\Gamma_i) \leqslant k_{i+1}\ (1 \leqslant i < s), k_s + \ell(\Gamma_s) \leqslant d + \varepsilon_1(\Gamma_s)\}. \tag{5.4}$$

The first inequality in (5.3) says that, due to the divergence condition, templates cannot appear too early in a floor diagram. The first inequality in (5.4) says that the first starting position can be 0 precisely when all outgoing edges of the first vertex of Γ_1 have weight 1. The second, respectively third, inequality in (5.3) and (5.4) say that templates cannot overlap, respectively cannot hang over at the end of the floor diagram.

We note that the lattice points in $A_{\mathbf{\Gamma}}(d)$ in (5.4) record all template positions if the divergence at the first vertex is at least 2δ: the quantity $\varkappa_j(\Gamma)$ is maximal, for a given $\delta(\Gamma) = \delta$, when Γ is the template with two vertices and δ edges between them, each with weight 2, and $j = 1$. The condition $\operatorname{div}(1) \geqslant 2\delta$ implies then that every collection of lattice points in the polytope can be the sequence of positions of templates, and vice versa. We always make the assumption $\operatorname{div}(1) \geqslant 2\delta$ in § 6, where we prove polynomiality of the refined Severi degrees for parameters in this regime (cf. Theorem 4.2).

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5.4 Multiplicity, cogenus, and markings

The refined multiplicity, cogenus, and markings of a floor diagram behave well under template decomposition, as in the unrefined case. If a floor diagram $\mathcal{D}$ has template decomposition $\boldsymbol{\Gamma}$, then by definition

$$\operatorname{mult}(\mathcal{D}; y) = \prod_{i=1}^{s} \operatorname{mult}(\Gamma_i; y).$$

Furthermore, the decomposition of § 5.3 is cogenus preserving, i.e., $\delta(\mathcal{D}) = \sum_{i=1}^{s} \delta(\Gamma_i)$ (see [AB13, § 3.3.2]). The number of markings of floor diagrams is expressible in terms of the number of 'markings of the templates': in Step 4 in Definition 5.4, instead of linearly ordering $\tilde{\mathcal{D}}$, we can order each template individually. To make this precise, associate to each template Γ a polynomial $P_\Gamma(c, m; k)$ in k, which depends also on the parameters c and m of the polygon Δ (cf. Figure 3), counting the number of markings for $k \geqslant k_{\min}$. Specifically, let $\Gamma_{(c,m,k)}$ denote the graph obtained from Γ by first adding

$$c + (k + j - 1)m - \varkappa_j(\Gamma)$$

short edges between $j - 1$ and j, making the divergence of all vertices m, and then subdividing each of the resulting graphs by introducing a new vertex for each edge. Let $P_\Gamma(c, m; k)$ be the number of linear extensions, up to equivalence, of the vertex poset of the graph $\Gamma_{(c,m,k)}$ extending the vertex order of Γ. Then

$$\nu(\mathcal{D}) = \prod_{i=1}^{s} P_{\Gamma_i}(c, m; k_i).$$

We can summarize the previous discussion in the following proposition.

Proposition 5.11. *The combinatorial refined Severi degree for:*

(i) $S = \mathbb{P}^2$, *any* $\delta \geqslant 1$ *and* $d \geqslant 1$; *or*

(ii) $S = \mathbb{P}(1, 1, m)$ *and* $m, d \geqslant 1$ *and* $m \geqslant 2\delta$; *or*

(iii) $S = \Sigma_m$, $\delta \geqslant 1$ *and* $m, c, d \geqslant 1$ *and* $m + c \geqslant 2\delta$

is given by

$$N^{\Delta,\delta}_{\text{comb}}(y) = \sum_{\boldsymbol{\Gamma}:\ \sum_i \delta(\Gamma_i) = \delta} \left[\left(\prod_{i=1}^{s} \operatorname{mult}(\Gamma_i, y) \right) \sum_{\mathbf{k} \in A_{\boldsymbol{\Gamma}}(d) \cap \mathbb{Z}^s} \left(\prod_{i=1}^{s} P_{\Gamma_i}(c, m; k_i) \right) \right], \tag{5.5}$$

the first sum running over all templates collections $\boldsymbol{\Gamma} = (\Gamma_1, \ldots, \Gamma_s)$ *with* $\sum_{i=1}^{s} \delta(\Gamma_i) = \delta$.

For $y = 1$ and $S = \mathbb{P}^2$, expression (5.5) specializes to [FM10, (5.13)]. For $y = 1$ and $S = \Sigma_m$, respectively $S = \mathbb{P}(1, 1, m)$, expression (5.5) specializes to [AB13, Proposition 3.3].

6. Polynomiality proofs

We now use floor diagrams and templates to prove Theorem 4.2 and Proposition 4.11. The argument for the former is based on the combinatorial formula (5.5). Our technique is a q-analog extension of Fomin and Mikhalkin's method [FM10, § 5] for the $\mathbb{P}^2$ and Ardila and Block's [AB13] for Σ_m and $\mathbb{P}(1, 1, m)$. The method provides an algorithm to compute refined node polynomials for any δ; see Remark 6.1 for a list for $\delta \leqslant 2$ for $\mathbb{P}^2$. By Theorem 7.5 the verbatim statement of the theorem below also holds for refined Severi degrees.

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Theorem 4.2. *For fixed $\delta \geqslant 1$ we have the following.*

(i) ($\mathbb{P}^2$) *There is a polynomial $N_\delta(d;y) \in \mathbb{Q}[y^{\pm 1}][d]$ of degree at most 2δ in d such that, for $d \geqslant \delta$,*

$$N_\delta(d;y) = N_{\mathrm{trop}}^{d,\delta}(y).$$

(ii) *(Hirzebruch surface) There is a polynomial $N_\delta(c,d,m;y) \in \mathbb{Q}[y^{\pm 1}][c,d,m]$ of degree at most δ in c, m and 2δ in d such that, for $c+m \geqslant 2\delta$ and $d \geqslant \delta$*

$$N_\delta(c,d,m;y) = N_{\mathrm{trop}}^{(\Sigma_m, cF+dH),\delta}(y).$$

(iii) ($\mathbb{P}(1,1,m)$) *There is a polynomial $N_\delta(d,m;y) \in \mathbb{Q}[y^{\pm 1}][d,m]$ of degree at most 2δ in d and δ in m such that, for $d \geqslant \delta$ and $m \geqslant 2\delta$,*

$$N_\delta(d,m;y) = N_{\mathrm{trop}}^{(\mathbb{P}(1,1,m),dH),\delta}(y).$$

Proof of Theorem 4.2. The proof for $S = \mathbb{P}^2$ is essentially the proof of [FM10, Theorem 5.1], generalized to refined multiplicities. For $S = \Sigma_m$ and $S = \mathbb{P}(1,1,m)$ our argument is a special (but now refined) case of the proof of [AB13, Theorem 1.2]. We first want to show that, for $S = \mathbb{P}(1,1,m)$, respectively $S = \Sigma_m$, and fixed δ, the expression in (5.5) is polynomial in d and m, respectively c, d and m for appropriately large values of c, d and m. As before, for $S = \mathbb{P}(1,1,m)$, we set $c = 0$. The case $S = \mathbb{P}^2$ we treat at the end.

The number of template collections $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_s)$ with fixed cogenus $\sum_{i=1}^s \delta(\Gamma_i) = \delta$ is finite. The factor $\prod_{i=1}^s \mathrm{mult}(\Gamma_i, y)$ is simply a Laurent polynomial in y; it thus remains to show that the second sum in (5.5) is polynomial for appropriately large d and m, and also c if $S = \Sigma_m$.

Since, for each template Γ_i and any j, we have $\varkappa_j(\Gamma_i) \leqslant 2\delta \leqslant c+m$, each individual template Γ_i can 'float freely' between $k_i = \varepsilon_0(\Gamma_i)$ and $d - \ell(\Gamma_i) + \varepsilon_1(\Gamma_i)$. Thus, as $c+m \geqslant 2\delta$, the valid starting positions k_i of all templates are given by the inequalities of $A_{\mathbf{\Gamma}}(d)$ as in (5.4).

If $d \geqslant \delta$ then $A_{\mathbf{\Gamma}}(d)$ is non-empty as

$$\varepsilon_0(\Gamma_1) + \ell(\Gamma_1) + \cdots + \ell(\Gamma_s) - \varepsilon_1(\Gamma_s) \leqslant \delta.$$

In fact, the combinatorial type of $A_{\mathbf{\Gamma}}(d)$ does not change if $d \geqslant \delta$: it is always combinatorially equivalent to a simplex. The inequalities are given by $A \cdot \mathbf{k} \leqslant b(d)$ for a unimodular matrix A and a vector $b(d)$ of linear forms in d.

For each lattice point $(k_1, \dots, k_s)$ in $A_{\mathbf{\Gamma}}(d)$, the number of markings $P_{\Gamma_i}(c,m;k_i)$ of Γ_i at position k_i is polynomial in k_i, c and m provided that $c+m \geqslant 2\delta$ [FM10, Lemma 5.8]. Thus, for $\mathbf{k} \in A_{\mathbf{\Gamma}}(d) \cap \mathbb{Z}^s$,

$$\prod_{i=1}^s P_{\Gamma_i}(c,m;k_i) \tag{6.1}$$

is a polynomial in $c, m, k_1, \dots k_s$. From the explicit description of $P_{\Gamma_i}(c,m;k_i)$, it is not hard to see that the degree of $P_{\Gamma_i}(c,m;k_i)$ in k_i, in c, and in m is bounded above by the number of edges of Γ_i and thus by $\delta(\Gamma_i)$. Hence, if $c+m \geqslant 2\delta$, the number (6.1) of markings of the template collection $\mathbf{\Gamma}$ is of degree at most δ in c and in m, and at most $\delta(\Gamma_i)$ in k_i.

By [AB13, Lemma 4.9], the second sum in (5.5) is a piecewise polynomial in c, d, and m: the second sum is a 'discrete integral' of a polynomial over the facet-unimodular polytope $A_{\mathbf{\Gamma}}(d)$. But for $c+m \geqslant 2\delta$ and $d \geqslant \delta$, the combinatorial type of $A_{\mathbf{\Gamma}}(d)$ does not change; $A_{\mathbf{\Gamma}}(d)$ is a dilation of a unit simplex by the (non-negative) number

$$d - (\varepsilon_0(\Gamma_1) + \ell(\Gamma_1) + \cdots + \ell(\Gamma_s) - \varepsilon_1(\Gamma_s)).$$

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Hence the second sum in (5.5) is polynomial in c, d, and m for $c+m \geqslant 2\delta$ and $d \geqslant \delta$. This polynomial is of degree at most δ in c and in m. As the number s of templates in the template collection $\boldsymbol{\Gamma}$ is bounded by δ, we (discretely) integrate over at most δ dimensions in (5.5), and thus the degree of the refined Severi degree in d is at most $\delta(\Gamma_1)+\cdots+\delta(\Gamma_s)+s \leqslant 2\delta$.

To conclude the result for $S=\mathbb{P}(1,1,m)$ set $c=0$.

For $S=\mathbb{P}^2$, the proof is identical to the proof of [Blo11, Theorem 1.3]; we only need to replace $\operatorname{mult}(\Gamma_i)(=\operatorname{mult}(\Gamma_i;1))$ by $\operatorname{mult}(\Gamma_i;y)$ throughout (e.g., (5.5) at $y=1$ becomes [Blo11, (3.1)]). The proof to further reduce the threshold value for polynomiality in d of $N^{d,\delta}(y)$ from 2δ to δ (as in the theorem) relies on another statistic '$s(\Gamma)$' [Blo11, p. 13]. The two key [Blo11, Lemmas 4.2 and 4.3] only involve the markings of a floor diagram and are thus verbatim in the refined case. The degree bound follows as in the case of $\mathbb{P}(1,1,m)$ (with $m=1$). For $S=\mathbb{P}^2$, the degree bound 2δ in d is tight: a template collection $\boldsymbol{\Gamma}$ with each Γ_j a template with $\delta(\Gamma_j)=1$ for $1 \leqslant j \leqslant \delta$ contributes to $N_\delta(d;y)$ in degree 2δ in d. □

Remark 6.1. Expression (5.5) gives, in principle, an algorithm to compute refined node polynomials. The algorithm of [Blo11, § 3], based on the algorithm of Fomin and Mikhalkin [FM10, § 5], easily adapts to the refined case. Below we show $N_\delta(d;y)$, for $S=\mathbb{P}^2$ and for $\delta \leqslant 2$ as computed by this method. (Note that Theorem 4.4 determines (by another method) the $N_\delta(d;y)$ for $\delta \leqslant 10$.) We have

$$\begin{aligned} N_1(d;y) &= \tfrac{1}{2}yd^2-\tfrac{3}{2}yd+y+2d^2-3d+1+\tfrac{1}{2}y^{-1}d^2-\tfrac{3}{2}y^{-1}d+y^{-1},\\ N_2(d;y) &= \tfrac{1}{8}y^2d^4-\tfrac{3}{4}y^2d^3+\tfrac{11}{8}y^2d^2-\tfrac{3}{4}y^2d+yd^4-\tfrac{9}{2}yd^3+2yd^2+\tfrac{21}{2}yd-9y+\tfrac{9}{4}d^4-\tfrac{15}{2}d^3\\ &\quad -\tfrac{3}{4}d^2+21d-15+y^{-1}d^4-\tfrac{9}{2}y^{-1}d^3+2y^{-1}d^2+\tfrac{21}{2}y^{-1}d-9y^{-1}+\tfrac{1}{8}y^{-2}d^4\\ &\quad -\tfrac{3}{4}y^{-2}d^3+\tfrac{11}{8}y^{-2}d^2-\tfrac{3}{4}y^{-2}d. \end{aligned}$$

Proof of Proposition 4.11. To a floor diagram $\mathcal{D}$, we associated the new statistic

$$i(\mathcal{D})=\sum_{e\in\mathcal{D}}\operatorname{wt}(e)(\operatorname{len}(e)-1).$$

It captures how much of the cogenus is contributed by edges of length greater than 1. By degree considerations, one can see that a floor diagram $\mathcal{D}$ contributes only to the coefficients $p^{\delta}_{d,i}$ of $N^{d,\delta}(y)$ with $i(\mathcal{D}) \leqslant i$. To compute $p^{\delta}_{d,0}$, it thus suffices to consider only the floor diagrams of degree d with cogenus δ and $i(\mathcal{D})=0$. Furthermore, each such floor diagram has $\operatorname{mult}(\mathcal{D};y)$ a degree δ polynomial in y and y^{-1} with leading coefficient 1. It thus suffices to show that the number of *marked* floor diagrams with $d(\mathcal{D})=d$, $\delta(\mathcal{D})=\delta$, and $i(\mathcal{D})=0$ equals $\binom{\binom{d-1}{2}}{\delta}$.

Each such marked floor diagram arises as follows. Let $\mathcal{D}_0$ be the unique floor diagram of degree d and cogenus 0 ($\mathcal{D}_0$ has one edge of weight 1 between vertex 1 and 2, two edges of weight 1 between vertex 2 and 3, and so on). The genus of $\mathcal{D}_0$ is $\binom{d-1}{2}$. Subdivide each edge of $\mathcal{D}_0$ by introducing a new vertex and order all vertices linearly, extending the linear order of the d original vertices. Call a cycle in $\mathcal{D}_0$ of length 2 *contractible* if the two midpoints corresponding to the two edges are adjacent in the linear order. Choose δ contractible cycles and 'contract' each cycle by identifying the two edges and the two midpoints to obtain the graph $\mathcal{D}_1$. To each edge in $\mathcal{D}_1$ assign a weight equal to the number of edges of $\mathcal{D}_0$ that were identified in obtaining $\mathcal{D}_1$. Note that $\mathcal{D}_1$ comes with a linear order on its vertices and is thus a marked floor diagram with $\delta(\mathcal{D}_1)=\delta$ and $i(\mathcal{D}_1)=0$, and all such marked floor diagrams arise this way. (One can see this by example: if a 'white' vertex j of $\mathcal{D}_1$ has four outgoing edges of weights 1, 2, 1, and 3, with the order determined by the order of the four adjacent 'black' vertices, the unique choice of cycles to contract in $\mathcal{D}_0$ between vertices j and $j+1$ is to contract the second, fourth, and fifth cycle, where the order of the cycles is determined by the order of the midpoints.) □

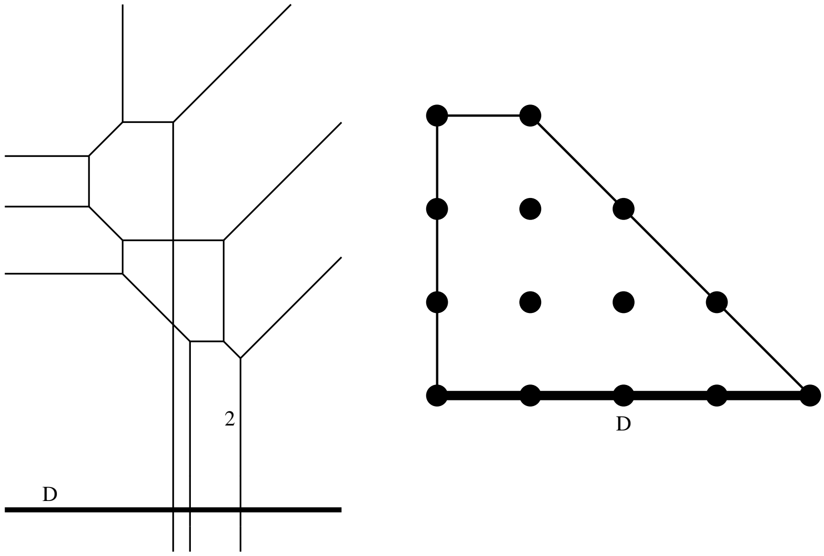

Figure 7. A tropical curve C with tangency $\beta = (2, 1)$ to a tropical divisor D. The tropical divisor corresponds to the bottom horizontal edge (also denoted D) of the polygon Δ of the Hirzebruch surface Σ_1.

7. Refined relative Severi degrees

In this section, we generalize tropical refined Severi degrees to include tropical tangency conditions. We then show that, in the case of the surfaces $S = \Sigma_m$ and $S = \mathbb{P}(1, 1, m)$, the resulting invariants satisfy the recursion of Göttsche and Shende (Definition 2.8) and thus both invariants agree. Our definitions are a refinement of [GM07] for $S = \mathbb{P}^2$ and [IKS09] for arbitrary toric surfaces.

Throughout this section, $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ denote infinite sequences of non-negative integers with only finitely many non-zero entries. Recall the notations $|\alpha| = \sum_{i\geqslant 1} \alpha_i$ and $I\alpha = \sum_{i\geqslant 1} i\alpha_i$.

Definition 7.1. Let Δ be a lattice polygon and $h : C \to \mathbb{R}^2$ a parametrized tropical curve of degree Δ (see Definition 3.2). Again we simply write C instead of $h : C \to \mathbb{R}^2$. Let D be an edge of Δ and $l(D)$ its lattice length.

(i) The *tropical boundary divisor* of D is a (classical) line in $\mathbb{R}^2$ parallel to D and sufficiently far in the direction dual to D (so that all intersections with C are orthogonal). Abusing notation, we denote the tropical boundary divisor by D also.

(ii) We say that the tropical curve C is *tangent to D of order β* if the partition of edge weights of the unbounded edges of C orthogonal to D is β (i.e., if there are β_1 such edges of weight 1, β_2 of weight 2, and so on).

See Figure 7 for an example. Throughout, we fix the following data:

(i) a tropical boundary divisor D (corresponding to an edge D of Δ);

(ii) two sequences α and β with $I\alpha + I\beta$ equal the lattice length $l(D)$ of the edge D; and

(iii) a tropically generic point configuration Π of $n = |\Delta \cap \mathbb{Z}^2| - 1 - \delta - I(\alpha + \beta) + |\alpha| + |\beta|$ points with precisely $|\alpha|$ points on D.

The number of points n is chosen so that the resulting curve count is non-zero and finite (unless δ is very large).

As in the classical case, we distinguish two types of tangencies: tangencies to D at a *fixed* point (i.e., a point in Π), the number of such of multiplicity i we denote by α_i. The other type of tangency to D is at unspecified or *free* points; we denote the number of such of multiplicity i by β_i. The following is a refinement of [GM07, Definition 4.1].

Definition 7.2. (i) A tropical curve C passing through Π is (α, β)-*tangent to* D if precisely $\alpha_i + \beta_i$ unbounded edges of C are orthogonal to and intersect D and have multiplicity i and, further, α_i of the edges pass through $\Pi \cap D$.

(ii) The subdivision of Δ dual to the tropical curve C is the *combinatorial type* of C.

(iii) The *refined relative multiplicity* $\operatorname{mult}_{\alpha,\beta}(C; y)$ of a tropical curve (α, β)-tangent to D is

$$\operatorname{mult}_{\alpha,\beta}(C; y) = \frac{1}{\prod_{i\geqslant 1}([i]_y)^{\alpha_i}} \cdot \operatorname{mult}(C; y). \tag{7.1}$$

(iv) The *tropical refined relative Severi degree* $N^{\Delta,\delta}_{\text{trop}}(\alpha, \beta)(y)$ is the number of δ-nodal tropical curves C of degree Δ passing through Π that are (α, β)-tangent to D, counted with multiplicity $\operatorname{mult}_{\alpha,\beta}(C; y)$.

(v) The *tropical refined relative irreducible Severi degree* $N^{\Delta,\delta}_{0,\text{trop}}(\alpha, \beta)(y)$ is the number of irreducible tropical curves C of degree Δ with δ nodes passing through Π that are (α, β)-tangent to D, counted with multiplicity $\operatorname{mult}_{\alpha,\beta}(C; y)$.

Both $N^{\Delta,\delta}_{\text{trop}}(\alpha, \beta)(y)$ and $N^{\Delta,\delta}_{0,\text{trop}}(\alpha, \beta)(y)$ in general depend on the tropical boundary divisor D. To simplify notation, we suppress this dependence. We discuss the cases $S = \mathbb{P}(1, 1, m)$ and $S = \Sigma_m$ in detail later and will always choose D to be a horizontal line $y = \mathit{const}$, for $\mathit{const} \ll 0$, cf. Figure 7.

Theorem 7.3. *The tropical refined relative Severi degree $N^{\Delta,\delta}_{\text{trop}}(\alpha, \beta)(y)$ and the refined relative irreducible Severi degree $N^{\Delta,\delta}_{0,\text{trop}}(\alpha, \beta)(y)$ are independent of the tropical point configuration if it is generic.*

Proof. The invariance of the tropical refined relative irreducible Severi degree $N^{\Delta,\delta}_{0,\text{trop}}(\alpha, \beta)(y)$ follows from a rather straightforward modification of Itenberg and Mikhalkin's proof [IM13, Theorem 1] of the independence of the refined irreducible Severi degree $N^{\Delta,\delta}_0$. We are brief here, in order to not repeat a lengthy argument. The modification with respect to [IM13] is to allow combinatorial types of tropical curves with arbitrary tangency conditions to one tropical divisor. The result then follows from the observation that Itenberg and Mikhalkin's argument also holds in this setting.

Let $\Pi = \{p_1, p_2, \ldots, p_n\}$ be a configuration of $n = |\Delta \cap \mathbb{Z}^2| - 1 - I(\alpha + \beta) + |\alpha| + |\beta|$ tropical points. It suffices to show the invariance if we smoothly perturb the points Π to $\Pi(t) = \{p_1, \ldots, p_{k-1}, p_k(t), p_{k+1}, \ldots, p_n\}$, for some $1 \leqslant k \leqslant n$, and all $t \in [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ and $\Pi(0) = \Pi$ such that $\Pi(t)$ is tropically generic for $t \neq 0$.

Fix an irreducible tropical curve $h : C \to \mathbb{R}^2$ with genus g with $\Pi \subset h(C)$ that is (α, β)-tangent to D. Let $S^{\pm}(t)$ be the set of tropical curves $h^{\pm}(t) : C \to \mathbb{R}^2$ that are (α, β)-tangent to D with $\Pi(\varepsilon) \subset h^{\pm}(\varepsilon)(C)$ for $\pm\varepsilon \in [0, t]$ that deform to h, i.e., with $h^{\pm}(0) = h$.

In the following, we conclude that

$$\sum_{C^+\in S^+(t)} \operatorname{mult}_{\alpha,\beta}(C^+; y) = \sum_{C^-\in S^-(t)} \operatorname{mult}_{\alpha,\beta}(C^-; y). \tag{7.2}$$

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If $h : C \to \mathbb{R}^2$ has no 4-valent vertex, then for $t > 0$ small enough, $|S^+(t)| = |S^-(t)| = 1$ and the combinatorial types of C^+ and C^- agree and (7.2) follows. Otherwise, every 4-valent vertex of h is perturbed as shown in [IM13, Figure 6] because for $t > 0$ small enough the combinatorial type of $h(t)$ changes only locally around the 4-valent vertex. (The detailed argument is given by Itenberg and Mikhalkin in the proof of [IM13, Lemma 3.3]; their proof also holds if we fix multiplicity of unbounded edges of h (to incorporate the β-tangency conditions) as well as point conditions on these edges very far away (to incorporate the α-tangency conditions).) The refined relative multiplicity $\mathrm{mult}_{\alpha,\beta}$ on both sides of (7.2) equals $1/\prod_{i\geqslant 1}([i]_y)^{\alpha_i}$ times the refined (non-relative) multiplicity $\mathrm{mult}(C; y)$. Thus, to show that the difference between both sides of (7.2) is zero it suffices to show that

$$\sum_{C^+\in S^+(t)} \mathrm{mult}(C^+; y) = \sum_{C^-\in S^-(t)} \mathrm{mult}(C^-; y). \tag{7.3}$$

As the tropical curves on both sides of this equation differ only locally around the 4-valent vertices of h, the argument to prove (7.3) is identical to the proof of [IM13, Lemma 3.3]. The invariance of the tropical refined relative Severi degree $N^{\Delta,\delta}_{\mathrm{trop}}(\alpha, \beta)(y)$ then follows from (4.1). □

Remark 7.4. The tropical refined relative Severi degree $N^{\Delta,\delta}_{\mathrm{trop}}(\alpha, \beta)(y)$ is a symmetric (under $y \leftrightarrow y^{-1}$) Laurent polynomial in $y^{1/2}$ with non-negative integer coefficients (not in y in general). As before, one may ask what the coefficients of $N^{\Delta,\delta}_{\mathrm{trop}}(\alpha, \beta)(y)$ count.

Theorem 7.5. *For all polygons Δ, with $X(\Delta) = \mathbb{P}(1, 1, m)$ or $X(\Delta) = \Sigma_m$, the refined relative tropical Severi degrees satisfy (2.7) with $L = L(\Delta)$. Therefore, the refined relative Severi degrees defined via the recursion 2.8 and the tropical refined relative Severi degrees agree:*

$$N^{\Delta,\delta}_{\mathrm{trop}}(y) = N^{\Delta,\delta}_{\mathrm{trop}}(y).$$

Proof. Our proof follows closely and extends the argument of Gathmann and Markwig's proof of [GM07, Theorem 4.3], where they proved this result in the non-refined case (i.e., $y = 1$) for the surface $S = \mathbb{P}^2$. Instead of points in a horizontal strip, we consider points in a vertical strip. The Gathmann–Markwig proof rests on an observation of Mikhalkin [Mik05, Lemma 4.20] that holds for any toric surface. We use it in generalizing their argument.

Fix a small $\varepsilon > 0$ and a large real number M. Consider a tropical generic point configuration $\Pi = \{p_1, p_2, \ldots, p_n\}$ such that the following hold.

(i) The x-coordinates of all p_i (including those on the divisor D) are within the interval $(-\varepsilon, \varepsilon)$.
(ii) The point p_1 is not on the divisor D but its y-coordinate is less than $-M$.
(iii) All points $p_i \neq p_1$ not lying on D have y-coordinate in the interval $(-\varepsilon, \varepsilon)$.

Let C be a tropical curve of degree Δ with δ nodes. Then C is of the following form.

(i) All vertices of C have x-coordinate in $(-\varepsilon, \varepsilon)$.
(ii) There are constants a and b, depending only on Δ, with $-N < a < b < -\varepsilon$ so that C has no vertices in the strip $\mathbb{R} \times [a, b]$; all edges in this strip are vertical.

See Figure 8 for an illustration. This follows directly from the verbatim argument in [GM07]; note that their argument rests on [Mik05, Lemma 4.20] which applies to arbitrary Δ, so in particular to $S = \mathbb{P}(1, 1, m)$ and $S = \Sigma_m$.

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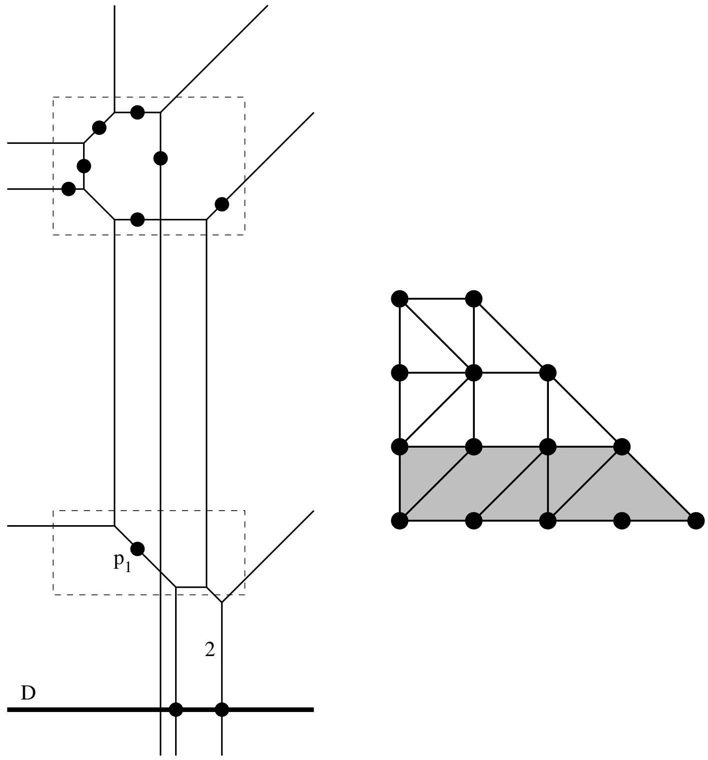

Figure 8. A $((1,1),(1))$-tangent curve C to a tropical divisor D. All point conditions are in a small vertical strip, p_1 is far from all other points. The curve C decomposes into the 'upper' part C' and the 'lower' part containing p_1. C' is $((0),(3))$-tangent to D. C and C' have $\delta = 2$ and $\delta' = 1$ as can be see from the dual subdivision on the right. The shaded part of the polygon is the difference between the degree Δ of C and Δ' of C'.

There are two cases.

Case 1: p_1 lies on a vertical edge with weight $k \geqslant 1$. Then all edges of C with y-coordinates $\leqslant -\varepsilon$ are vertical by the Gathmann–Markwig argument. We can move p_1 down onto the divisor D and obtain a tropical curve with one more 'fixed' tangency condition. The weight of C is $[k]_y$ times the weight of this new curve. The total contribution of tropical curves through Π with p_1 on a vertical edge is thus

$$\sum_{k:\beta_k>0} [k]_y \cdot N^{\Delta,\delta}_{\text{trop}}(\alpha + e_k, \beta - e_k)(y).$$

Case 2: p_1 does not lie on a vertical half-ray. Then C can be broken into two pieces: let C' be the curve with bounded edges in the vicinity of the points $p_2, \ldots, p_n$ that do not lie on D. The other piece, containing p_1, consists of the bounded edges of C in the vicinity of p_1, one unbounded edge in direction $(-1, 0)$ and $(1, m)$, respectively, and some vertical edges. See Figure 8 for an illustration of this decomposition. By construction, the degree Δ' of C' is the lattice polygon obtained from Δ by removing a horizontal strip of width one at the bottom of Δ.

Next, we determine in how many ways C' can be extended to a tropical curve of degree Δ that is (α, β)-tangent to the divisor D and passes through Π. We know that C' is (α', β') tangent to D, for some $\alpha' \leqslant \alpha$ and $\beta' \geqslant \beta$. There are $\binom{\alpha}{\alpha'}$ ways to choose which vertical edges of C through a point in $\Pi \cap D$ belong to C'. Similarly, there are $\binom{\beta'}{\beta}$ ways to choose which

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vertical edges of C' intersecting D but not containing a point in Π belong to C (for more details see [GM07]).

To show that the tangency conditions α' and β' satisfy $I\alpha' + I\beta' = H(L-H)$, recall that degree Δ' of C' is the polygon obtained from Δ by removing from Δ the bottom strip of lattice width 1. Furthermore, $I\alpha' + I\beta'$ equals the lattice length of the bottom edge of Δ'. We argue for each surface separately.

(a) $S = \mathbb{P}^2$: here H is the class of a line and L is the class of a degree d curve. Thus, we have $H(L-H) = d-1$, the length of the bottom edge of Δ'.

(b) $S = \Sigma_m$: in this case, we defined H as the class of a section with $H^2 = m$. Then $H(L-H) = c + (d-1)m$. Recall that $\Delta = \operatorname{conv}((0,0),(0,d),(c,d),(c+dm,0))$. The bottom edge of Δ' has lattice length $c+(d-1)m = H(L-H)$.

(c) $S = \mathbb{P}(1,1,m)$: here H is the class of a line, and we have $H(L-H) = (d-1)m$. As $\Delta = \operatorname{conv}((0,0),(0,d),(dm,0))$, $H(L-H)$ is precisely the lattice length of Δ'.

Next, we relate the y-multiplicities of C and C'. We have

$$\operatorname{mult}(C;y) = \prod_{i\geqslant 1}([i]_y)^{\alpha_i - \alpha'_i + \beta'_i - \beta_i} \operatorname{mult}(C';y)$$

and, therefore,

$$\operatorname{mult}_{\alpha,\beta}(C;y) = \frac{1}{\prod_{i\geqslant 1}([i]_y)^{\alpha_i}} \operatorname{mult}(C;y) = \prod_{i\geqslant 1}([i]_y)^{\beta'_i - \beta_i} \operatorname{mult}_{\alpha',\beta'}(C';y).$$

Now, we show that the cogenus δ' of C' satisfies

$$\delta - \delta' = I\alpha' + I\beta' - |\beta' - \beta|.$$

By definition, $\delta - \delta'$ counts the number of parallelograms in the horizontal bottom strip of width 1 in the dual subdivision Δ_C, where we count each parallelogram with its Euclidean area. This number equals the number of unbounded edges of C' that intersect D and are unbounded in C. But this number is precisely the length of the upper edge of the width 1 strip minus the number of edges of C', that become bounded as edges in C, and thus equals $I\alpha' + I\beta' - |\beta' - \beta|$.

The recursive formula now follows: by the balancing condition, the (α',β')-tangent curve C' can be completed to a (α,β)-tangent curve C with $p_1 \in C \backslash C'$ in a unique way, once we choose which vertical edges of C through a point in $\Pi \cap D$ belong to C' and which vertical edges of C' intersecting D but not containing a point in C belong to C (giving $\binom{\alpha}{\alpha'} \cdot \binom{\beta'}{\beta}$ choices).

Checking the initial conditions is trivial. □

Remark 7.6. Note that in the case of rational ruled surfaces Σ_m, the above proof works also if we allow m to be negative. Then $\Sigma_m = \Sigma_{-m}$, but with the role of E and H exchanged (this corresponds to exchanging the top and the bottom edge of Δ). Expressed on Σ_m, the proof thus also shows the recursion (2.7), with the same initial conditions, but everywhere with H replaced by E and α, β specifying contacts along E instead along H.

7.1 Refined relative node polynomials for plane curves

We now extend the floor diagram technique to refined relative Severi degree for $S = \mathbb{P}^2$. Then we show a polynomiality result (Theorem 7.8) about refined relative Severi degrees of $\mathbb{P}^2$, refining the result of [Blo12, Theorem 1.1]. We expect a similar, more technical argument to work also for

$S=\Sigma_m$ and $S=\mathbb{P}(1,1,m)$, but restrict ourselves to $\mathbb{P}^2$ for simplicity. The following definitions are a quite straightforward refinement of [FM10, § 3.2].

As we are only concerned with $S=\mathbb{P}^2$, we denote by $\mathbf{FD}(d,\delta)$ the set of Δ-floor diagrams with $\Delta=\operatorname{conv}((0,0),(0,d),(d,0))$ and cogenus δ, for any $d\geqslant 1$. Also, let $\mathbf{FD}_{\mathrm{conn}}(d,\delta)$ denote the collection of connected such floor diagrams. Let α and β be two sequences of non-negative integers with only finitely many non-zero entries.

To each floor diagram $\mathcal{D}\in\mathbf{FD}(d,\delta)$, there is a statistic $\nu_{\alpha,\beta}(\mathcal{D})$, counting the number of '(α,β)-markings' of $\mathcal{D}$ as in the non-relative case. The precise definition is given in [Blo12, Definition 2.3], a reformulation of [FM10, Definition 3.13]. Intuitively, $\nu_{\alpha,\beta}(\mathcal{D})$ counts the number of tropical curves of degree Δ that satisfy the following.

(i) They are (α,β)-tangent to $D=\{y=const\}$, where $const\ll 0$ (so D is a very far down horizontal line).

(ii) They 'correspond' to the floor diagram $\mathcal{D}$ (in the sense of [FM10, Theorem 3.17]).

(iii) They pass through a vertically stretched point configuration.

The refined relative Severi degree of $\mathbb{P}^2$ can be expressed purely combinatorially in terms of the y-weighted floor diagrams of § 5: to simplify the formula, define (cf., [FM10, (3.6)] for the unrefined setting)

$$\operatorname{mult}_\beta(\mathcal{D},y)=\prod_{i\geqslant 1}([i]_y)^{\beta_i}\cdot\operatorname{mult}(\mathcal{D},y).$$

Proposition 7.7. (i) *For any $d\geqslant 1$ and $\delta\geqslant 1$, the refined relative Severi degree of $\mathbb{P}^2$ is given by*

$$N^{d,\delta}(\alpha,\beta)(y)=\sum_{\mathcal{D}\in\mathbf{FD}(d,\delta)}\operatorname{mult}_\beta(\mathcal{D},y)\cdot\nu_{\alpha,\beta}(\mathcal{D}).$$

(ii) *For any $d\geqslant 1$ and $\delta\geqslant 1$, the refined irreducible relative Severi degree of $\mathbb{P}^2$ is computed by*

$$N_0^{d,\delta}(\alpha,\beta)(y)=\sum_{\mathcal{D}\in\mathbf{FD}_{conn}(d,\delta)}\operatorname{mult}_\beta(\mathcal{D},y)\cdot\nu_{\alpha,\beta}(\mathcal{D}).$$

Proof. We first prove part (ii). We may assume, by Theorem 7.3, that the tropical point configuration is vertically stretched. By [FM10, Theorem 3.17], there is a bijection f between irreducible tropical curves of degree d and cogenus δ that are (α,β)-tangent to D and (α,β)-marked floor diagrams $\mathcal{D}$ with $\mathcal{D}\in\mathbf{FD}_{\mathrm{conn}}(d,\delta)$, where we used that these tropical curves have genus $g=\binom{d-1}{2}-\delta$. By [FM10, Theorem 3.17] (see also [FM10, Theorem 3.7] for the non-relative case but with more details), the map f preserves the unrefined multiplicity ($y=1$) for any such tropical curve C with corresponding floor diagram $\mathcal{D}$:

$$\operatorname{mult}_{\alpha,\beta}(C,1)=\operatorname{mult}_\beta(\mathcal{D},1).$$

By definition of the refined multiplicities of tropical curves and floor diagrams (Definitions 3.5 and 5.2 and (7.1)), the bijection f preserves also the refined multiplicities,

$$\operatorname{mult}_{\alpha,\beta}(C,y)=\operatorname{mult}_\beta(\mathcal{D},y),$$

as $(1/\prod_{i\geqslant 1}([i]_y)^{\alpha_i})\operatorname{mult}(C,y)=\prod_{i\geqslant 1}([i]_y)^{\beta_i}\cdot\operatorname{mult}(\mathcal{D},y)$, and part (ii) follows.

Part (i) follows from part (ii) by a straightforward refined extension of the inclusion–exclusion procedure of [FM10, § 1] that was used to conclude [FM10, Corollary 1.9] (the non-relative unrefined count of reducible curves via floor diagrams) from [FM10, Theorem 1.6] (the non-relative unrefined count of irreducible curves via floor diagrams). □

Theorem 7.8. *For any $\delta \geqslant 1$, there is a polynomial*

$$N_\delta(\alpha;\beta;y) \in \mathbb{Q}[y^{\pm 1}][\alpha_1,\ldots,\alpha_\delta;\beta_1,\ldots,\beta_\delta]$$

in α_i and β_i with coefficients in $\mathbb{Q}[y^{\pm 1}]$ such that, for any α and β with $|\beta| \geqslant \delta$, we have

$$N^{d,\delta}(\alpha;\beta)(y) = \prod_{i\geqslant 1}([i]_y)^{\beta_i} \frac{(|\beta|-\delta)!}{\beta_1!\beta_2!\cdots} \cdot N_\delta(\alpha;\beta;y).$$

The coefficients of the polynomial $N_\delta(\alpha;\beta;y)$ are preserved under the transformation $y \leftrightarrow y^{-1}$.

We call $N_\delta(\alpha;\beta;y)$ the *refined relative node polynomial* of $\mathbb{P}^2$.

Proof. The proof is identical to the non-refined argument in [Blo12, Theorem 1.1] but with the unrefined multiplicity of a floor diagram replaced by the refined multiplicity of the present paper. □

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