Bull. Aust. Math. Soc. (First published online 2024), page 1 of 12*
doi:10.1017/S000497272400042X
*Provisional—final page numbers to be inserted when paper edition is published

ARITHMETIC PROPERTIES OF AN ANALOGUE OF t-CORE PARTITIONS

PRANJAL TALUKDAR

(Received 25 March 2024; accepted 29 April 2024)

Abstract

An integer partition of a positive integer n is called t-core if none of its hook lengths is divisible by t. Gireesh *et al.* ['A new analogue of t-core partitions', *Acta Arith.* **199** (2021), 33–53] introduced an analogue $\overline{a}_t(n)$ of the t-core partition function. They obtained multiplicative formulae and arithmetic identities for $\overline{a}_t(n)$ where $t \in \{3, 4, 5, 8\}$ and studied the arithmetic density of $\overline{a}_t(n)$ modulo p_i^j where $t = p_1^{a_1} \cdots p_m^{a_m}$ and $p_i \geq 5$ are primes. Bandyopadhyay and Baruah ['Arithmetic identities for some analogs of the 5-core partition function', *J. Integer Seq.* **27** (2024), Article no. 24.4.5] proved new arithmetic identities satisfied by $\overline{a}_5(n)$. We study the arithmetic densities of $\overline{a}_t(n)$ modulo arbitrary powers of 2 and 3 for $t = 3^\alpha m$ where $\gcd(m, 6)$=1. Also, employing a result of Ono and Taguchi ['2-adic properties of certain modular forms and their applications to arithmetic functions', *Int. J. Number Theory* **1** (2005), 75–101] on the nilpotency of Hecke operators, we prove an infinite family of congruences for $\overline{a}_3(n)$ modulo arbitrary powers of 2.

2020 *Mathematics subject classification*: primary 11P83; secondary 05A17, 11F11.

Keywords and phrases: t-core partition, analogue of t-core partition, theta function, modular form, arithmetic density.

1. Introduction and statement of results

A partition $\pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ of a positive integer n is a nonincreasing sequence of natural numbers such that $\sum_{i=1}^{k} \pi_i = n$. The number of partitions of n is denoted by $p(n)$. The Ferrers–Young diagram of π is an array of nodes with π_i nodes in the ith row. The (i, j) hook is the set of nodes directly to the right of the (i, j) node, together with the set of nodes directly below it, as well as the (i, j) node itself. The hook number, $H(i, j)$, is the total number of nodes on the (i, j) hook. For a positive integer $t \geq 2$, a partition of n is called t-core if none of the hook numbers are divisible by t. We illustrate the Ferrers–Young diagram of the partition $4 + 3 + 1$ of 8 with hook numbers:

$$\begin{array}{llll} \bullet^{6} & \bullet^{4} & \bullet^{3} & \bullet^{1} \\ \bullet^{4} & \bullet^{2} & \bullet^{1} & \\ \bullet^{1} & & & \end{array}$$

It is clear that the partition $4 + 3 + 1$ of 8 is a t-core partition for $t \geq 7$.

The author was partially supported by the Council of Scientific and Industrial Research (CSIR), Government of India, under the CSIR-JRF scheme (Grant No. 09/0796(12991)/2021-EMR-I).

If $c_t(n)$ counts the t-core partitions of n, the generating function of $c_t(n)$ is given by

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty} = \frac{f_t^t}{f_1}$$

(see [6, (2.1)]). Here and throughout the paper, for $|q| < 1$, we define $(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k)$ and for convenience, we set $f_k := (q^k;q^k)_\infty$ for integers $k \geq 1$. Granville and Ono [8] proved that if $t \geq 4$, then $c_t(n) > 0$ for every nonnegative integer n. A survey of t-core partitions can be found in [4].

For an integral power series $F(q) = \sum_{n=0}^{\infty} a(n)q^n$ and $0 \leq r < M$, we define the arithmetic density $\delta_r(F, M; X)$ by

$$\delta_r(F, M; X) := \frac{\#\{0 \leq n \leq X : a(n) \equiv r \pmod{M}\}}{X}.$$

An integral power series F is called *lacunary modulo M* if

$$\lim_{X\to\infty} \delta_0(F, M; X) = 1,$$

that is, almost all of the coefficients of F are divisible by M. Arithmetic densities of $c_t(n)$ modulo arbitrary powers of 2, 3 and for primes greater than or equal to 5 are studied by Jindal and Meher [10].

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is given by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

In Ramanujan's notation, the Jacobi triple product identity [3, page 35, entry 19] takes the shape

$$f(a, b) = (-a; ab)_\infty(-b; ab)_\infty(ab; ab)_\infty.$$

Ramanujan introduced two special cases of $f(a, b)$:

$$\varphi(-q) := f(-q, -q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{f_1^2}{f_2},$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1.$$

In this notation, the generating function of $c_t(n)$ may be rewritten as

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{f^t(-q^t)}{f(-q)}. \tag{1.1}$$

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

Recently, Gireesh *et al.* [7] considered a new function $\overline{a}_t(n)$ by substituting $\varphi(-q)$ for $f(-q)$ in the generating function of $c_t(n)$ (in (1.1)), namely

$$\sum_{n=0}^{\infty} \overline{a}_t(n)q^n = \frac{\varphi^t(-q^t)}{\varphi(-q)} = \frac{f_2 f_t^{2t}}{f_1^2 f_{2t}^t}. \tag{1.2}$$

They proved several multiplicative formulae and arithmetic identities for $\overline{a}_t(n)$ for $t = 2, 3, 4$ and 8 using Ramanujan's theta functions and q-series techniques. Using the theory of modular forms, they studied the divisibility of $\overline{a}_t(n)$ modulo arbitrary powers of primes greater than 5. More precisely, they proved the following theorem.

THEOREM 1.1. *Let $t = p_1^{a_1} \cdots p_m^{a_m}$ where the p_i are prime numbers greater than or equal to 5. Then for every positive integer j,*

$$\lim_{X\to\infty} \frac{\#\{0 \le n \le X : \overline{a}_t(n) \equiv 0 \pmod{p_i^j}\}}{X} = 1.$$

They also deduced a Ramanujan type congruence for $\overline{a}_5(n)$ modulo 5 by using an algorithm developed by Radu and Sellers [14]. Bandyopadhyay and Baruah [2] proved some new identities connecting $\overline{a}_5(n)$ and $c_5(n)$. They also found a recurrence relation for $\overline{a}_5(n)$.

Recently, Cotron *et al.* [5, Theorem 1.1] proved a strong result regarding lacunarity of eta-quotients modulo arbitrary powers of primes under certain conditions. We observe that the eta-quotients associated with $\overline{a}_t(n)$ do not satisfy these conditions, which makes the problem of studying lacunarity of $\overline{a}_t(n)$ more interesting. In this article, we study the arithmetic densities of $\overline{a}_t(n)$ modulo arbitrary powers of 2 and 3 where $t = 3^\alpha m$. To be specific, we prove the following theorems.

THEOREM 1.2. *Let $k \ge 1$, $\alpha \ge 0$ and $m \ge 1$ be integers with* $\gcd(m, 6) = 1$. *Then the set* $\{n \in \mathbb{N} : \overline{a}_{3^\alpha m}(n) \equiv 0 \pmod{2^k}\}$ *has arithmetic density* 1.

THEOREM 1.3. *Let $k \ge 1$, $\alpha \ge 0$ and $m \ge 1$ be integers with* $\gcd(m, 6) = 1$. *Then the set* $\{n \in \mathbb{N} : \overline{a}_{3^\alpha m}(n) \equiv 0 \pmod{3^k}\}$ *has arithmetic density* 1.

The fact that the action of Hecke algebras on spaces of modular forms of level 1 modulo 2 is locally nilpotent was first observed by Serre and proved by Tate (see [15–17]). Later, this result was generalised to higher levels by Ono and Taguchi [13]. We observe that the eta-quotient associated to $\overline{a}_3(n)$ is a modular form whose level is in the list of Ono and Taguchi. We use a result of Ono and Taguchi to prove the following congruences for $\overline{a}_3(n)$.

THEOREM 1.4. *Let n be a nonnegative integer. Then there exists an integer $c \ge 0$ such that for every $d \ge 1$ and distinct primes $p_1, \dots, p_{c+d}$ coprime to 6,*

$$\overline{a}_3\Big(\frac{p_1 \cdots p_{c+d} \cdot n}{24}\Big) \equiv 0 \pmod{2^d}$$

whenever n is coprime to $p_1, \dots, p_{c+d}$.

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

The paper is organised as follows. In Section 2, we state some preliminaries from the theory of modular forms. Then we prove Theorems 1.2–1.4 using the properties of modular forms in Sections 3–5, respectively. We mention some directions for future study in the concluding section.

2. Preliminaries

In this section, we recall some basic facts and definitions for modular forms (for more details, see [11, 12]).

First, we define the matrix groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod N \right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N \text{ and } b \equiv c \equiv 0 \pmod N \right\},$$

where N is a positive integer. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some N, and the smallest N with this property is called the level of Γ. For instance, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N.

Let $\mathbb{H}$ denote the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d}.$$

We identify ∞ with $1/0$ and define

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{r}{s} = \frac{ar + bs}{cr + ds},$$

where $r/s \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, if $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$, $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, we define the slash operator $|_\ell$ by

$$(f|_\ell \gamma)(z) := (\det(\gamma))^{\ell/2} (cz + d)^{-\ell} f(\gamma z).$$

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

DEFINITION 2.1. Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if the following hold:

(1) for all $z \in \mathbb{H}$ and all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$,

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^{\ell} f(z);$$

(2) if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_{\ell}\gamma)(z)$ has a Fourier expansion of the form

$$(f|_{\ell}\gamma)(z) = \sum_{n\geq 0} a_{\gamma}(n) q_N^n,$$

where $q_N := e^{2\pi i z/N}$.

For a positive integer ℓ, the complex vector space of modular forms of weight ℓ with respect to a congruence subgroup Γ is denoted by $M_{\ell}(\Gamma)$.

DEFINITION 2.2 [12, Definition 1.15]. If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_{\ell}(\Gamma_1(N))$ has Nebentypus character χ if

$$f\Big(\frac{az+b}{cz+d}\Big) = \chi(d)(cz+d)^{\ell} f(z)$$

for all $z \in \mathbb{H}$ and all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_{\ell}(\Gamma_0(N), \chi)$.

The relevant modular forms for the results of this paper arise from eta-quotients. The Dedekind eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_{\infty} = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_{\delta}},$$

where N is a positive integer and r_{δ} is an integer. Now, we recall two important theorems from [12, page 18] that will be used later.

THEOREM 2.3 [12, Theorem 1.64]. *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_{\delta}}$ is an eta-quotient such that $\ell = 1/2 \sum_{\delta|N} r_{\delta} \in \mathbb{Z}$, $\sum_{\delta|N} \delta r_{\delta} \equiv 0 \pmod{24}$ and $\sum_{\delta|N} (N/\delta) r_{\delta} \equiv 0 \pmod{24}$, then $f(z)$ satisfies*

$$f\Big(\frac{az+b}{cz+d}\Big) = \chi(d)(cz+d)^{\ell} f(z)$$

for every $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. Here the character χ is defined by

$$\chi(d) := \Big(\frac{(-1)^{\ell} s}{d}\Big), \quad \textit{where } s := \prod_{\delta|N} \delta^{r_{\delta}}.$$

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

Let f be an eta-quotient that satisfies the conditions of Theorem 2.3 and suppose that the associated weight ℓ is a positive integer. If $f(z)$ is holomorphic at all the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$. The necessary criterion for determining orders of an eta-quotient at cusps is given by the following theorem.

THEOREM 2.4 [12, Theorem 1.64]. *Let c, d and N be positive integers with $d \mid N$ and* $\gcd(c, d) = 1$. *If f is an eta-quotient satisfying the conditions of Theorem 2.3 for N, then the order of vanishing of $f(z)$ at the cusp c/d is*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,N/d)d\delta}.$$

We now recall a deep theorem of Serre [see 12, page 43] that will be used in proving Theorems 1.2 and 1.3.

THEOREM 2.5 [12, page 43]. *Let $g(z) \in M_k(\Gamma_0(N), \chi)$ have Fourier expansion*

$$g(z) = \sum_{n=0}^{\infty} b(n)q^n \in \mathbb{Z}[[q]].$$

Then for a positive integer r, there is a constant $\alpha > 0$ such that

$$\#\{0 < n \le X : b(n) \not\equiv 0 \pmod{r}\} = \mathcal{O}\Big(\frac{X}{(\log X)^\alpha}\Big).$$

Equivalently,

$$\lim_{X\to\infty}\frac{\#\{0 < n \le X : b(n) \not\equiv 0 \pmod{r}\}}{X} = 0.$$

Finally, we recall the definition of Hecke operators. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then the action of the Hecke operator T_m on $f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty}\Big(\sum_{d|\gcd(n,m)}\chi(d)d^{\ell-1}a\Big(\frac{nm}{d^2}\Big)\Big)q^n.$$

In particular, if $m = p$ is prime, then

$$f(z)|T_p := \sum_{n=0}^{\infty}\Big(a(pn) + \chi(p)p^{\ell-1}a\Big(\frac{n}{p}\Big)\Big)q^n. \tag{2.1}$$

We note that $a(n) = 0$ unless n is a nonnegative integer.

3. Proof of Theorem 1.2

Putting $t = 3^\alpha m$ in (1.2),

$$\sum_{n=0}^{\infty}\overline{a}_{3^\alpha m}(n)q^n = \frac{f_2 f_{3^\alpha m}^{2\cdot 3^\alpha m}}{f_1^2 f_{2\cdot 3^\alpha m}^{3^\alpha m}}. \tag{3.1}$$

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

We define

$$A_{\alpha,m}(z) := \frac{\eta^2\,(2^3 3^{\alpha+1} mz)}{\eta\,(2^4 3^{\alpha+1} mz)}.$$

For any prime p and positive integer j, $(q;q)_\infty^{p^j} \equiv (q^p;q^p)_\infty^{p^{j-1}} \pmod{p^j}$. Hence, for any integer $k \geq 1$,

$$A_{\alpha,m}^{2^k}(z) = \frac{\eta^{2^{k+1}}(2^3 3^{\alpha+1} mz)}{\eta^{2^k}(2^4 3^{\alpha+1} mz)} \equiv 1 \pmod{2^{k+1}}. \tag{3.2}$$

Next we define

$$B_{\alpha,m,k}(z) := \frac{\eta\,(48z)\eta^{2\cdot 3^\alpha m}(2^3 3^{\alpha+1} mz)}{\eta^2(24z)\eta^{3^\alpha m}(2^4 3^{\alpha+1} mz)} A_{\alpha,m}^{2^k}(z) = \frac{\eta\,(48z)\eta^{2\cdot 3^\alpha m + 2^{k+1}}(2^3 3^{\alpha+1} mz)}{\eta^2(24z)\eta^{3^\alpha m + 2^k}(2^4 3^{\alpha+1} mz)}.$$

In view of (3.1) and (3.2),

$$B_{\alpha,m,k}(z) \equiv \frac{\eta\,(48z)\eta^{2\cdot 3^\alpha m}(2^3 3^{\alpha+1} mz)}{\eta^2(24z)\eta^{3^\alpha m}(2^4 3^{\alpha+1} mz)} \equiv \frac{f_{48} f_{2^3 3^{\alpha+1} m}^{2\cdot 3^\alpha m}}{f_{24}^2 f_{2^4\cdot 3^{\alpha+1} m}^{3^\alpha m}} \equiv \sum_{n=0}^{\infty} \overline{a}_{3^\alpha m}(n) q^{24n} \pmod{2^{k+1}}. \tag{3.3}$$

Next, we will show that $B_{\alpha,m,k}(z)$ is a modular form. Applying Theorem 2.3, we find that the level of $B_{\alpha,m,k}(z)$ is $N = 2^4 3^{\alpha+1} mM$, where M is the smallest positive integer such that

$$2^4 3^{\alpha+1} mM\Big(\frac{-2}{24} + \frac{1}{48} + \frac{2\cdot 3^\alpha m + 2^{k+1}}{2^3 3^{\alpha+1} m} + \frac{-3^\alpha m - 2^k}{2^4 3^{\alpha+1} m}\Big) \equiv 0 \pmod{24},$$

which implies

$$3\cdot 2^k M \equiv 0 \pmod{24}.$$

Therefore, $M = 4$ and the level of $B_{\alpha,m,k}(z)$ is $N = 2^6 3^{\alpha+1} m$.

The representatives for the cusps of $\Gamma_0(2^6 3^{\alpha+1} m)$ are given by fractions c/d where $d \mid 2^6 3^{\alpha+1} m$ and $\gcd(c, 2^6 3^{\alpha+1} m) = 1$ (see [5, Proposition 2.1]). By Theorem 2.4, $B_{\alpha,m,k}(z)$ is holomorphic at a cusp c/d if and only if

$$-2\frac{\gcd(d,24)^2}{24} + \frac{\gcd(d,48)^2}{48} + (3^\alpha m + 2^k)\Big(2\frac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{2^3 3^{\alpha+1} m} - \frac{\gcd(d, 2^4 3^{\alpha+1} m)^2}{2^4 3^{\alpha+1} m}\Big) \geq 0.$$

Equivalently, $B_{\alpha,m,k}(z)$ is holomorphic at a cusp c/d if and only if

$$L := 3^\alpha m(-4G_1 + G_2 + 4G_3 - 1) + 2^k(4G_3 - 1) \geq 0,$$

where

$$G_1 = \frac{\gcd(d,24)^2}{\gcd(d, 2^4 3^{\alpha+1} m)^2}, \quad G_2 = \frac{\gcd(d,48)^2}{\gcd(d, 2^4 3^{\alpha+1} m)^2}, \quad G_3 = \frac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{\gcd(d, 2^4 3^{\alpha+1} m)^2}.$$

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

Let d be a divisor of $2^6 3^{\alpha+1} m$. We can write $d = 2^{r_1} 3^{r_2} t$ where $0 \le r_1 \le 6$, $0 \le r_2 \le \alpha + 1$ and $t \mid m$. We now consider two cases depending on r_1.

Case 1: $0 \le r_1 \le 3$, $0 \le r_2 \le \alpha + 1$. Then $G_1 = G_2$, $1/3^{2\alpha} t^2 \le G_1 \le 1$ and $G_3 = 1$. Therefore, $L = 3^{\alpha+1} m(1 - G_1) + 3 \cdot 2^k \ge 3 \cdot 2^k$.

Case 2: $4 \le r_1 \le 6$, $0 \le r_2 \le \alpha + 1$. Then $G_2 = 4G_1$, $1/4 \cdot 3^{2\alpha} t^2 \le G_1 \le 1/4$ and $G_3 = 1/4$, which implies $L = 0$.

Hence, $B_{\alpha,m,k}(z)$ is holomorphic at every cusp c/d. The weight of $B_{\alpha,m,k}(z)$ is $\ell = 1/2(3^\alpha m + 2^k - 1)$ which is a positive integer and the associated character is given by

$$\chi_1(\bullet) = \left(\frac{(-1)^\ell 3^{(\alpha+1)(3^\alpha m + 2^k) - 1} m^{3^\alpha m + 2^k}}{\bullet} \right).$$

Thus, $B_{\alpha,m,k}(z) \in M_\ell(\Gamma_0(N), \chi)$ where ℓ, N and χ are as above. By Theorem 2.5, the Fourier coefficients of $B_{\alpha,m,k}(z)$ are almost all divisible by $r = 2^k$. From (3.3), this holds for $\overline{a}_{3^\alpha m}(n)$ also. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

We proceed along the same lines as in the proof of Theorem 1.2. Here we define

$$C_{\alpha,m}(z) := \frac{\eta^3(2^4 3^{\alpha+1} m z)}{\eta(2^4 3^{\alpha+2} m z)}.$$

By the binomial theorem, for any integer $k \ge 1$,

$$C_{\alpha,m}^{3^k}(z) = \frac{\eta^{3^{k+1}}(2^4 3^{\alpha+1} m z)}{\eta^{3^k}(2^4 3^{\alpha+2} m z)} \equiv 1 \pmod{3^{k+1}}. \tag{4.1}$$

Next we define

$$\begin{aligned} D_{\alpha,m,k}(z) &:= \frac{\eta(48z) \eta^{2 \cdot 3^\alpha m}(2^3 3^{\alpha+1} m z)}{\eta^2(24z) \eta^{3^\alpha m}(2^4 3^{\alpha+1} m z)} C_{\alpha,m}^{3^k}(z) \\ &= \frac{\eta(48z) \eta^{2 \cdot 3^\alpha m}(2^3 3^{\alpha+1} m z) \eta^{3^{k+1} - 3^\alpha m}(2^4 3^{\alpha+1} m z)}{\eta^2(24z) \eta^{3^k}(2^4 3^{\alpha+2} m z)}. \end{aligned}$$

From (3.1) and (4.1),

$$D_{\alpha,m,k}(z) \equiv \frac{\eta(48z) \eta^{2 \cdot 3^\alpha m}(2^3 3^{\alpha+1} m z)}{\eta^2(24z) \eta^{3^\alpha m}(2^4 3^{\alpha+1} m z)} \equiv \frac{f_{48} f_{2^3 3^{\alpha+1} m}^{2 \cdot 3^\alpha m}}{f_{24}^2 f_{2^4 \cdot 3^{\alpha+1} m}^{3^\alpha m}} \equiv \sum_{n=0}^{\infty} \overline{a}_{3^\alpha m}(n) q^{24n} \pmod{3^{k+1}}. \tag{4.2}$$

We now prove that $D_{\alpha,m,k}(z)$ is a modular form. Applying Theorem 2.3, we find that the level of $D_{\alpha,m,k}(z)$ is $N = 2^4 3^{\alpha+2} m M$, where M is the smallest positive integer such that

$$2^4 3^{\alpha+2} m M \left(\frac{-2}{24} + \frac{1}{48} + \frac{2 \cdot 3^\alpha m}{2^3 3^{\alpha+1} m} + \frac{3^{k+1} - 3^\alpha m}{2^4 3^{\alpha+1} m} + \frac{-3^k}{2^4 3^{\alpha+2} m} \right) \equiv 0 \pmod{24},$$

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

which gives $8 \cdot 3^k M \equiv 0 \pmod{24}$. Therefore, $M = 1$ and the level of $D_{\alpha,m,k}(z)$ is $N = 2^4 3^{\alpha+2} m$.

The representatives for the cusps of $\Gamma_0(2^4 3^{\alpha+2} m)$ are given by fractions c/d where $d \mid 2^4 3^{\alpha+2} m$ and $\gcd(c, 2^4 3^{\alpha+2} m) = 1$. From Theorem 2.4, $D_{\alpha,m,k}(z)$ is holomorphic at a cusp c/d if and only if

$$-2\frac{\gcd(d,24)^2}{24} + \frac{\gcd(d,48)^2}{48} + 2 \cdot 3^{\alpha} m \frac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{2^3 3^{\alpha+1} m} + (3^{k+1} - 3^{\alpha} m)\frac{\gcd(d, 2^4 3^{\alpha+1} m)^2}{2^4 3^{\alpha+1} m} - 3^k \frac{\gcd(d, 2^4 3^{\alpha+2} m)^2}{2^4 3^{\alpha+2} m} \geq 0.$$

Equivalently, $D_{\alpha,m,k}(z)$ is holomorphic at a cusp c/d if and only if

$$L := 3^{\alpha+1} m(-4G_1 + G_2 + 4G_3 - G_4) + 3^k(9G_4 - 1) \geq 0,$$

where

$$G_1 = \frac{\gcd(d,24)^2}{\gcd(d, 2^4 3^{\alpha+2} m)^2}, \quad G_2 = \frac{\gcd(d,48)^2}{\gcd(d, 2^4 3^{\alpha+2} m)^2},$$
$$G_3 = \frac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{\gcd(d, 2^4 3^{\alpha+2} m)^2}, \quad G_4 = \frac{\gcd(d, 2^4 3^{\alpha+1} m)^2}{\gcd(d, 2^4 3^{\alpha+2} m)^2}.$$

Let d be a divisor of $2^4 3^{\alpha+2} m$. We write $d = 2^{r_1} 3^{r_2} t$ where $0 \leq r_1 \leq 4$, $0 \leq r_2 \leq \alpha + 2$ and $t \mid m$. We now consider four cases depending on the values of r_1 and r_2.

Case 1: $0 \leq r_1 \leq 3$, $0 \leq r_2 \leq \alpha + 1$. Then $G_1 = G_2$, $1/3^{2\alpha} t^2 \leq G_1 \leq 1$ and $G_3 = G_4 = 1$. Hence, we have $L = 3^{\alpha+2} m(1 - G_1) + 8 \cdot 3^k \geq 8 \cdot 3^k$.

Case 2: $0 \leq r_1 \leq 3$, $r_2 = \alpha + 2$. Then $G_1 = G_2$, $1/3^{2(\alpha+1)} t^2 \leq G_1 \leq 1/3^{2(\alpha+1)}$ and $G_3 = G_4 = 1/9$. Therefore, $L = 3^{\alpha+2} m(1/9 - G_1) \geq 0$.

Case 3: Let $r_1 = 4$, $0 \leq r_2 \leq \alpha + 1$. Then $G_2 = 4G_1$, $1/4 \cdot 3^{(\alpha+1)} t^2 \leq G_1 \leq 1/4$, $G_4 = 4G_3$ and $G_3 = 1/4$. Hence, we have $L = 8 \cdot 3^k$.

Case 4: Let $r_1 = 4$, $r_2 = \alpha + 2$. Then $G_2 = 4G_1$, $1/4 \cdot 3^{(\alpha+1)} t^2 \leq G_1 \leq 1/4 \cdot 3^{2(\alpha+1)}$, $G_4 = 4G_3$ and $G_3 = 1/36$. Therefore, $L = 0$.

Therefore, $D_{\alpha,m,k}(z)$ is holomorphic at every cusp c/d. The weight of $D_{\alpha,m,k}(z)$ is $\ell = 1/2(3^{\alpha} m - 1) + 3^k$ which is a positive integer and the associated character is given by

$$\chi_2(\bullet) = \left(\frac{(-1)^{\ell} 3^{2\alpha 3^k + 3^{\alpha} \alpha m + 3^{\alpha} m + 3^k - 1} m^{3^{\alpha} m + 2 \cdot 3^k}}{\bullet} \right).$$

Thus, $D_{\alpha,m,k}(z) \in M_{\ell}(\Gamma_0(N), \chi)$ where ℓ, N and χ are as above. By Theorem 2.5, the Fourier coefficients of $D_{\alpha,m,k}(z)$ are almost all divisible by $r = 3^k$. From (4.2), this holds for $\overline{a}_{3^{\alpha} m}(n)$ also. This completes the proof of Theorem 1.3.

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

5. Proof of Theorem 1.4

First we recall the following result of Ono and Taguchi [13] on the nilpotency of Hecke operators.

THEOREM 5.1 [13, Theorem 1.3(3)]. *Let n be a nonnegative integer and k be a positive integer. Let χ be a quadratic Dirichlet character of conductor $9 \cdot 2^a$. Then there is an integer $c \geq 0$ such that for every $f(z) \in M_k(\Gamma_0(9 \cdot 2^a), \chi) \cap \mathbb{Z}[[q]]$ and every $t \geq 1$,*

$$f(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_{c+t}} \equiv 0 \pmod{2^t}$$

whenever the primes $p_1, \ldots, p_{c+t}$ are coprime to 6.

We apply this theorem to the modular form $B_{1,1,k}(z)$ to prove Theorem 1.4. Putting $\alpha = 1$ and $m = 1$ in (3.3), we find that

$$B_{1,1,k}(z) \equiv \sum_{n=0}^{\infty} \overline{a}_3(n) q^{24n} \pmod{2^{k+1}},$$

which yields

$$B_{1,1,k}(z) := \sum_{n=0}^{\infty} F_k(n) q^n \equiv \sum_{n=0}^{\infty} \overline{a}_3\Big(\frac{n}{24}\Big) q^n \pmod{2^{k+1}}. \tag{5.1}$$

Now, $B_{1,1,k}(z) \in M_{2^{k-1}+1}(\Gamma_0(9 \cdot 2^6), \chi_3)$ for $k \geq 1$ where χ_3 is the associated character (which is χ_1 evaluated at $\alpha = 1$ and $m = 1$). In view of Theorem 5.1, there is an integer $c \geq 0$ such that for any $d \geq 1$,

$$B_{1,1,k}(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_{c+d}} \equiv 0 \pmod{2^d}$$

whenever $p_1, \ldots, p_{c+d}$ are coprime to 6. It follows from the definition of Hecke operators that if $p_1, \ldots, p_{c+d}$ are distinct primes and if n is coprime to $p_1 \cdots p_{c+d}$, then

$$F_k(p_1 \cdots p_{c+d} \cdot n) \equiv 0 \pmod{2^d}. \tag{5.2}$$

Combining (5.1) and (5.2) completes the proof of the theorem.

6. Concluding remarks

Theorems 1.2 and 1.3 of this paper and [7, Theorem 1.8] give the arithmetic densities of $\overline{a}_t(n)$ for odd t, but similar techniques cannot be used to obtain the arithmetic density of $\overline{a}_t(n)$ when t is even. It would be interesting to study the arithmetic density of $\overline{a}_t(n)$ for even values of t.

Computational evidence suggests that there are Ramanujan type congruences for $\overline{a}_t(n)$ modulo powers of 2, 3 and other primes ≥ 5 for various t that are not covered by the results of [2, 7]. It would be of interest to find new congruences for $\overline{a}_t(n)$.

https://doi.org/10.1017/S000497272400042X Published online by Cambridge University Press

Asymptotic formulae for partition functions and other related functions have been widely studied in the literature. For instance, the asymptotic formulae for $p(n)$ and $c_t(n)$ were obtained by Hardy and Ramanujan [9] and Anderson [1], respectively. It will be desirable to find an asymptotic formula for $\overline{a}_t(n)$.

Some relations connecting $\overline{a}_t(n)$ and $c_t(n)$ have been discussed in [2]. A combinatorial treatment for $\overline{a}_t(n)$ might reveal more interesting partition theoretic connections.

Acknowledgements

The author is extremely grateful to his PhD supervisor, Professor Nayandeep Deka Baruah, for his guidance and encouragement. The author is indebted to Professor Rupam Barman for many helpful comments and suggestions. The author thanks the Council of Scientific and Industrial Research, Government of India, for supporting the project. The author would like to thank the anonymous referee for carefully reading the paper.

References

[1] J. Anderson, 'An asymptotic formula for the t-core partition function and a conjecture of Stanton', *J. Number Theory* **128** (2008), 2591–2615.

[2] S. Bandyopadhyay and N. D. Baruah, 'Arithmetic identities for some analogs of the 5-core partition function', *J. Integer Seq.* **27** (2024), Article no. 24.4.5.

[3] B. C. Berndt, *Ramanujan's Notebooks, Part III* (Springer-Verlag, New York, 1991).

[4] H. Cho, B. Kim, H. Nam and J. Sohn, 'A survey on t-core partitions', *Hardy–Ramanujan J.* **44** (2022), 81–101.

[5] T. Cotron, A. Michaelsen, E. Stamm and W. Zhu, 'Lacunary eta-quotients modulo powers of primes', *Ramanujan J.* **53** (2020), 269–284.

[6] F. Garvan, D. Kim and D. Stanton, 'Cranks and t-cores', *Invent. Math.* **101** (1990), 1–17.

[7] D. S. Gireesh, C. Ray and C. Shivashankar, 'A new analogue of t-core partitions', *Acta Arith.* **199** (2021), 33–53.

[8] A. Granville and K. Ono, 'Defect zero p-blocks for finite simple groups', *Trans. Amer. Math. Soc.* **348** (1996), 331–347.

[9] G. H. Hardy and S. Ramanujan, 'Asymptotic formulae in combinatory analysis', *Proc. Lond. Math. Soc. (3)* **17** (1918), 75–115.

[10] A. Jindal and N. K. Meher, 'Arithmetic density and congruences of t-core partitions', *Results Math.* **79** (2024), Article no. 4, 23 pages.

[11] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms* (Springer-Verlag, New York, 1991).

[12] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, CBMS Regional Conference Series in Mathematics, 102 (American Mathematical Society, Providence, RI, 2004).

[13] K. Ono and Y. Taguchi, '2-adic properties of certain modular forms and their applications to arithmetic functions', *Int. J. Number Theory* **1** (2005), 75–101.

[14] S. Radu and J. A. Sellers, 'Congruence properties modulo 5 and 7 for the pod function', *Int. J. Number Theory* **7** (2011), 2249–2259.

[15] J.-P. Serre, 'Divisibilité de certaines fonctions arithmétiques', *Sém. Delange-Pisot-Poitou, Théor. Nombres* **16** (1974), 1–28.

[16] J.-P. Serre, ‘Valeurs propres des opérateurs de Hecke modulo ℓ’, *Astérisque* **24** (1975), 109–117.
[17] J. Tate, ‘The non-existence of certain Galois extensions of $\mathbb{Q}$ unramified outside 2’, in: *Arithmetic Geometry*, Contemporary Mathematics, 174 (eds. N. Childress and J. W. Jones) (American Mathematical Society, Providence, RI, 1994), 153–156.

PRANJAL TALUKDAR, Department of Mathematical Sciences,
Tezpur University, Napaam 784028, Assam, India
e-mail: pranjaltalukdar113@gmail.com