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SOME LOCAL-GLOBAL PRINCIPLES FOR FORMALLY REAL FIELDS

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1. Introduction. Let F be a formally real field, and let Δ be a *preordering* of F; that is, a subset of F satisfying $\Delta + \Delta = \Delta$, $\Delta\Delta = \Delta$, $F^2 \subseteq \Delta$. Denote by X_Δ the set of all orderings P of F satisfying $P \supseteq \Delta$. Thus $\Delta = \bigcap_{P \in X_\Delta} P$. This result is well known. It was first proved by Artin [**3**, Satz 1] in the case $\Delta = \sum F^2$.

For P an ordering of F, denote by F_P, the real closure of F at P. Denote by $W = W(F)$ the Witt ring of F [**14**], and by $M = M(F)$ the augmentation ideal of W, that is, the ideal of even dimensional forms. $W_\Delta = W_\Delta(F)$ will denote the ideal of W generated by all forms of the type $\langle 1, -s\rangle$, $s \in \Delta^\times$. (Thus, $W_\Delta = M$, if $\Delta = F$.)

THEOREM 1. *Suppose Δ is a (proper) preordering of F. Then the kernel of the natural ring homomorphism from $W(F)$ into $\prod_{P \in X_\Delta} W(F_P)$ is $W_\Delta(F)$.*

This theorem was originally proved by Pfister [**12**] in the case $\Delta = \sum F^2$. The general case follows, for example from the theory developed in [**4**]. For completeness, a proof is given here following the proof of Theorem 2.

Denote by k_*F the graded ring $\bigoplus_{i=0}^\infty k_iF$ as defined by Milnor in [**11**]. Let $k_{*\Delta}F$ denote the ideal of k_*F generated by the elements $l(s)$, $s \in \Delta^\times$. (Thus $k_{*\Delta}F = \bigoplus_{i=0}^\infty k_{i\Delta}F$ where $k_{i\Delta}F$ is generated by all elements of the form $l(a_1) \ldots l(a_i)$, $a_1 \in \Delta^\times$, $a_2 \ldots a_i \in F^\times$.)

In this paper we examine the following conjectures.

CONJECTURE 1. The kernel of the natural ring homomorphism from k_*F into $\prod_{P \in X_\Delta} k_*F_P$ is $k_{*\Delta}F$.

CONJECTURE 2. For each positive integer i, $M^i \cap W_\Delta = M^{i-1}W_\Delta$.

Conjecture 1 is the main conjecture. Its connection with Conjecture 2 is described in Corollaries 2 and 3, Section 2.

Both of these conjectures are shown to be true if either
(i) Δ satisfies the descending chain condition, or
(ii) Δ is 2-stable.

These results are proved in Sections 4 and 5 respectively. The major result is Theorem 7. Sections 2 and 3 are devoted to pointing out various consequences of Conjecture 1.

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Although the conjectures are stated for Δ an arbitrary (proper) preordering, there is special interest in the case $\Delta = \sum F^2$. At the present time we know about as much in this case as we do in the more general situation.

Conjecture 1 amounts to the statement that, for each non-negative integer i, $k_{i\Delta}F$ is the kernel of the natural mapping from k_iF into $\prod_{P\in X_\Delta} k_i\, F_P$. It is trivially true for $i = 0$ (since we assume $\Delta \neq F$). Also, the natural isomorphism $F^\times/F^{\times 2} \cong k_1F$ carries $\Delta^\times/F^{\times 2}$ onto $k_{1\Delta}F$. Thus, the Artin formula, $\bigcap_{P\in X_\Delta} P = \Delta$, implies the conjecture is valid for $i = 1$. The question is open for $i \geqq 2$.

The second conjecture is trivially true for $i = 1$. It is also true for $i = 2$. The proof is a simple modification of that given in [**7**, Theorem 2.8] for the case $\Delta = \sum F^2$. The question is open for $i \geqq 3$. However, for $\Delta = \sum F^2$, Elman and Lam have shown in [**9**, Theorem 3], that $M^2W_\Delta = 0$ implies $M^3 \cap W_\Delta = 0$. It follows that Conjecture 2 is true in case $\Delta = \sum F^2$, $M^2W_\Delta = 0$.

It is worthwhile pointing out that a weak version of the two conjectures is known to hold in general, namely:

THEOREM 2. *Let Δ be any preordering of F.*

(i) *Let $f \in k_*F$. Then f is zero in k_*F_P for all $P \in X_\Delta$ if and only if $l(-1)^v \cdot f \in k_{*,\Delta}F$ holds for some integer $v \geqq 0$.*

(ii) *Let $f \in M^i \cap W_\Delta$. Then there exists an integer $v \geqq 0$ such that $2^v \times f \in M^{i+v-1}\, W_\Delta$.*

(Thus, if it were always possible to choose $v = 0$, we would have the conjectures.)

It should be noted that, in case $\Delta = \sum F^2$, each element of $k_{*\Delta}F$ is annihilated by some power of $l(-1)$. Thus the equation $l(-1)^v f \in k_{*\Delta}F$ can be replaced by $l(-1)^v f = 0$ in this case. Also note that, in this same case, (ii) is a triviality, since the elements of W_Δ are two-power torsion.

A proof of (i) in the case $\Delta = \sum F^2$ is found in [**1**]. However, for completeness, a full proof is presented here.

Proof of Theorem 2. For $a = (a_1, \ldots, a_n)$ a tuple of elements of $F^\times$, $\Delta(a) = \Delta(a_1, \ldots, a_n)$ will denote the preordering of F generated by $a_1, \ldots, a_n$ over Δ. Use the notations of [**13**, p. 42], i.e. $\psi_a = \langle a_1, \ldots, a_n\rangle$, $\pi_a = \langle\langle a_1, \ldots, a_n\rangle\rangle$. Let $\psi_a \in M^i$ and suppose its signatures satisfy $\operatorname{sgn}_P \psi_a = 0$ for all $P \in X_\Delta$ (respectively, $\operatorname{sgn}_P \psi_a \equiv 0 \bmod 2^{i+1}$ for all $P \in X_\Delta$). By [**13**, Lemma 2.1.6] we have

$$2^n \times \psi_a = \oplus_\epsilon\, \psi_\epsilon \otimes \pi_{\epsilon a} \quad \text{in } W,$$

the sum running through all n-tuples $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ of ± 1. There are two cases to consider:

(i) $\Delta(\epsilon a) \neq F$. In this case there exists $P \in X_\Delta$ satisfying $\operatorname{sgn}_P a_i = \operatorname{sgn}_P \epsilon_i$, $i = 1, \ldots, n$, so $\psi_\epsilon = 0$ in W (respectively, $\psi_\epsilon = 2^{i+1}v$ in W, v some integer). Thus $\psi_\epsilon \otimes \pi_{\epsilon a} = 0$ in W (respectively, $\psi_\epsilon \otimes \pi_{\epsilon a} \in M^{i+n+1}$).

(ii) $\Delta(\epsilon a) = F$. This implies that for a sufficiently large set of elements $s_1, \ldots, s_v \in \Delta^\times$, we have that $\pi_s \otimes \pi_{\epsilon a}$ is isotropic and hence zero in W.

Thus, by picking $s_1, \ldots, s_v$ to be a sufficiently large set of elements of $\Delta^\times$ [for example including n ones, and all elements required by the terms of type (ii)] we have $\pi_s \otimes \psi_a = 0$ in W (respectively, $\pi_s \otimes \psi_a \in M^{v+i+1}$).

Now write $\langle 1, s\rangle = 2 - \langle 1, -s\rangle$ and expand what we have just obtained to get $2^v \times \psi_a \in M^{i+v-1} W_\Delta$ (respectively, $2^v \times \psi_a \in M^{i+v-1} W_\Delta + M^{i+v+1}$). This proves the second statement.

To prove the first let $f \in k_i F$ be zero over F_P for all $P \in X_\Delta$, and let $s_i(f)$ denote the image of f in M^i/M^{i+1} under the natural map [**11**]. Say $s_i(f) = \psi_a + M^{i+1}$. Then ψ_a satisfies $\operatorname{sgn}_P \psi_a \equiv 0 \bmod 2^{i+1}$ for all $P \in X_\Delta$, so by our previous considerations, there is a non-negative integer v such that $2^v \times \psi_a \in M^{i+v-1} W_\Delta + M^{i+v+1}$. Thus we can write $l(-1)^v f = g + h$, $g \in k_{i+v,\Delta}F$, h in the kernel of the mapping $s_{i+v} : k_{i+v} F \to M^{i+v}/M^{i+v+1}$. Thus, by [**11**, p. 332] $l(-1)^t h = 0$ for some non-negative integer t, so $l(-1)^{v+t} f \in k_{*\Delta}F$.

Using the proof of Theorem 2 given above, there is now a very easy proof of Theorem 1.

Proof of Theorem 1. Suppose ψ_a is zero over F_P for all $P \in X_\Delta$. Then by the proof of Theorem 2, $\pi_s \otimes \psi_a = 0$ in W_F for some v-tuple $s = (s_1, \ldots, s_v)$ of elements of $\Delta^\times$. By the distributive property, this yields $a_1 \pi_s \oplus \ldots \oplus a_n \pi_s = 0$ so there exist $t_1, \ldots, t_n$ represented by π_s (and hence in Δ) not all zero, such that $a_1 t_1 + \ldots + a_n t_n = 0$. Let $a' = (a_1', \ldots, a_n')$ be defined by $a_i' = a_i$, if $t_i = 0$; $a_i' = a_i t_i$, if $t_i \neq 0$. Then $\psi_{a'}$ is isotropic and $\psi_{a'} - \psi_a \in W_\Delta$. Also, since $t_i \pi_s \cong \pi_s$ if $t_i \neq 0$, we have $\pi_s \otimes \psi_{a'} = \pi_s \otimes \psi_a = 0$ in W. Thus, by induction on the dimension, we have $\psi_{a'}$ (and hence ψ_a) is in W_Δ.

It is not clear how this proof could be modified to yield a proof of the conjectures.

2. Some consequences of Conjecture 1. Denote by gr $W(F)$ the graded ring $\bigoplus_{i=0}^{\infty} M^i/M^{i+1}$, and by $\operatorname{gr}_\Delta W(F)$ the ideal of gr $W(F)$ generated by the elements $\langle 1, -s\rangle + M^2$, $s \in \Delta^\times$.

THEOREM 3. *If Conjecture* 1 *holds for* Δ, *then the kernel of the ring homomorphism from* gr $W(F)$ *into* $\prod_{P \in X_\Delta}$ gr $W(F_P)$ *is* $\operatorname{gr}_\Delta W(F)$.

Proof. The natural ring homomorphism $s : k_*F \to \operatorname{gr} W(F)$ is surjective, carries $k_{*\Delta}F$ onto $\operatorname{gr}_\Delta W(F)$, and is an isomorphism if F is real closed.

Note that if the assumption that Conjecture 1 holds is dropped, then one still has the following weaker result: $f \in \operatorname{gr} W(F)$ is zero in gr $W(F_P)$ for all $P \in X_\Delta$ if and only if $(2 + M^2)^v \cdot f \in \operatorname{gr}_\Delta W(F)$ holds for some integer $v \geqq 0$. This is implicit in the proof of Theorem 2.

COROLLARY 1. *Assume Conjecture* 1 *holds for* Δ. *Let* k *be a positive integer, and let* $f \in W$. *Then* $\operatorname{sgn}_P f \equiv 0 \bmod 2^k$ *holds for all* $P \in X_\Delta$ *if and only if* $f \in W_\Delta + M^k$.

Proof. This is immediate from the previous theorem, using induction on k.

If $f \in M^i \cap W_\Delta$, then $\bar{f} \in M^i/M^{i+1}$ is zero locally for all $P \in X_\Delta$. It follows by Theorem 3 that if Conjecture 1 holds for Δ, then $f \in M^{i-1} W_\Delta + M^{i+1}$. Thus $M^i \cap W_\Delta = M^{i-1} W_\Delta + M^{i+1} \cap W_\Delta$. Repeating this process we get:

COROLLARY 2. *Suppose Conjecture* 1 *holds for* Δ. *Then for all integers* $i, k \geqq 1$, $M^i \cap W_\Delta = M^{i-1} W_\Delta + M^{i+k} \cap W_\Delta$,

This is "almost" Conjecture 2. In particular we have the following:

COROLLARY 3. *Suppose that Conjecture* 1 *holds for* Δ, *and that* $M^k \cap W_\Delta = 0$ *holds for some positive integer* k. *Then Conjecture* 2 *holds for* Δ.

Note that the assumption $M^k \cap W_\Delta = 0$ implies $2^{k-1} \langle 1, -s\rangle = 0$ in W for all $s \in \Delta^\times$; i.e. that $\Delta = \sum F^2$. Thus Corollary 3 is really a result only about $\Delta = \sum F^2$.

Denote by G_F the Galois group of $\bar{F}/F$; $\bar{F}$ being the algebraic closure of F. Denote by $H^*(G_F, \mathbf{Z}/2\mathbf{Z})$ the graded ring of cohomology groups $H^*(G_F, \mathbf{Z}/2\mathbf{Z}) = \bigoplus_{i=0}^{\infty} H^i(G_F, \mathbf{Z}/2\mathbf{Z})$, and by $h : k_*F \to H^*(G_F, \mathbf{Z}/2\mathbf{Z})$ the canonical ring homomorphism. Thus $h(k_*F)$ is the subring of $H^*(G_F, \mathbf{Z}/2\mathbf{Z})$ generated by the elements $\delta(a)$, $a \in F^\times$.

THEOREM 4. *Assume Conjecture* 1 *holds for* Δ. *Then the kernel of the natural ring homomorphism from* $h(k_*F)$ *into* $\prod_{P \in X_\Delta} H^*(G_{F_P}, \mathbf{Z}/2\mathbf{Z})$ *is the ideal of* $h(k_*F)$ *generated by the elements* $\delta(s)$, $s \in \Delta^\times$.

Proof. $h : k_*F \to H^*(G_F, \mathbf{Z}/2\mathbf{Z})$ is an isomorphism if F is real closed.

Since $h(k_2F)$ can be identified with the subgroup of the Brauer group B_F generated by the quaternion algebras $(a, b/F)$, $a, b \in F^\times$, we have the following result:

COROLLARY. *Suppose that Conjecture* 1 *holds for* Δ. *Suppose* $x \in B_F$ *is a product of quaternion algebras, and that* x *splits over all real closures* F_P, $P \in X_\Delta$. *Then* x *is of the form*

$$\prod_{i=1}^{n} \left(\frac{a_i, s_i}{F}\right),$$

with $a_i \in F^\times$, $s_i \in \Delta^\times$, $i = 1, \ldots, n$.

3. The injectivity of the homomorphisms s, h. It has been conjectured in [**11**] that the mappings $s : k_*F \to \operatorname{gr} W(F)$, $h : k_*F \to H^*(F_F, \mathbf{Z}/2\mathbf{Z})$ are injective. In case Conjecture 1 holds for $\Delta = \sum F^2$, one can get some partial results in this direction.

The first result holds without the assumption of Conjecture 1, and is an immediate consequence of [**2**] and [**6**, Theorem 3.2].

THEOREM 5. *Let i be any positive integer, and let $\Delta = \sum F^2$. Then the following statements are equivalent:*
(i) $k_{i\Delta}F = 0$
(ii) $M^{i-1} W_\Delta = 0$
(iii) $M^{i-1} W_\Delta = M^i W_\Delta$.

Define the *v-invariant* of F to be the greatest integer (or ∞ if no such integer exists) satisfying $k_{v,\Delta}F \neq 0$, (with $\Delta = \sum F^2$). In view of the previous theorem, this invariant can be characterized as the greatest integer for which there is a v-fold Pfister form over F which is both anisotropic and torsion.

If u denotes the generalized u-invariant of F as defined in [**8**], then it is clear that $2^v \leqq u$. In fact $2^v = u$, if $u \leqq 4$.

For the rest of this section, assume that Conjecture 1 *holds for* $\Delta = \sum F^2$.

THEOREM 6. $s_i : k_iF \to M^i/M^{i+1}$ *and* $h_i : k_iF \to H^i(G_F, \mathbf{Z}/2\mathbf{Z})$ *are injective if and only if they are injective on* $k_{i\Delta}F$. ($\Delta = \sum F^2$).

Proof. This is clear, since s_i, h_i are locally isomorphisms.

COROLLARY 1. *If* $i > v$, *then* s_i, h_i *are injective.*

Proof. This follows from Theorems 5 and 6.

COROLLARY 2. *Suppose* $v \leqq 2$. *Then* s, h *are injective.*

Proof. Since s_0, s_1, s_2, h_0, and h_1 are injective, and since s_i, h_i are injective for $i \geqq 3$ by the previous corollary, all that is required is to show that h_2 is injective. The mapping $c = h_2 \circ s_2^{-1} : M^2/M^3 \to H^2(G_F, \mathbf{Z}/2\mathbf{Z})$ is the Clifford mapping. By [**9**, Theorem 3], this mapping is injective on $(MW_\Delta + M^3)/M^3$. Thus h_2 is injective on $k_{2,\Delta}F$ and hence (by Corollary 1) on k_2F also.

4. The descending chain condition.

For Δ a preordering of F, and a_i, $i \in I$ elements of $F^\times$, $\Delta(a_i | i \in I)$ will denote the preordering of F generated by a_i, $i \in I$ over Δ.

We will say that Δ satisfies the *descending chain condition* (abbreviated D.C.C.) if every descending chain $\Delta_1 \supseteq \Delta_2 \supseteq \Delta_3 \supseteq \ldots$ of preorderings, each of which is finitely generated over Δ, terminates. This is equivalent to the condition that every non-empty set consisting of preorderings of F which are finitely generated over Δ has minimal elements.

Examples. (i) If the group index $(F^\times : \Delta^\times)$ is finite, then Δ satisfies D.C.C. Such preorderings are obtained by taking the intersection of a finite set of orderings of F.

(ii) Let $F = \bigcup_{n=1}^\infty \mathbf{R}((x_1)) \ldots ((x_n))$. Then F is superpythagorian, so the proper preorderings of F are just the sets of the form $\Delta = \Delta^\times \cup \{0\}$ where $\Delta^\times$

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is a *subgroup* of $F^\times$ which does not contain -1 [**7**, Theorem 4.3]. Thus all preorderings of F satisfy D.C.C. This provides an example of a field F and a preordering $\Delta \subseteq F$ satisfying D.C.C. for which the index $(F^\times : \Delta^\times)$ is infinite. Other such examples can be found where F is not superpythagorian.†

(iii) Let F be an algebraic number field (possibly infinite dimensional over $\mathbf{Q}$). Every preordering finitely generated over $\Delta = \sum F^2$ is of the form $\Delta(a)$ for some $a \in F^\times$. Suppose Δ satisfies D.C.C. Let P be any ordering of F. Pick $\Delta(a)$ minimal such that $a \notin P$. Then, among all preorderings in P finitely generated over Δ, $\Delta(-a)$ is maximal. It follows that $\Delta(-a) = P$. Thus X_Δ is discrete. Since X_Δ is also compact, this implies X_Δ is finite. Thus, if we pick an algebraic number field F in such a way that it has infinitely many orderings (for example, pick $F = \mathbf{Q}(\sqrt{p}|p$ a prime integer) then $\Delta = \sum F^2$ *does not* satisfy D.C.C. (However Conjectures 1 and 2 still hold for such a field since it is a direct limit of fields for which the conjectures hold!)

THEOREM 7. *Suppose that Δ_0 is a preordering of F satisfying D.C.C. Then Conjectures* 1 *and* 2 *hold for all preorderings Δ of F satisfying $\Delta_0 \subseteq \Delta$.*

Proof. (1) Let $i > 1$, and let $f \in k_iF$ be such that it is zero in k_iF_P for all $P \in X_\Delta$. We wish to show that $f \in k_{i,\Delta}F$. Using Theorem 2, for example, we may assume that Δ is finitely generated over Δ_0, and hence that Δ itself satisfies D.C.C. There exists a preordering Δ_1 finitely generated over Δ such that $f \in k_{i,\Delta_1}F$. (For example, take $\Delta_1 = \Delta(-1) = F$.) By D.C.C. we may suppose that Δ_1 is chosen minimal such that this is so. Suppose Δ_2 is any preordering over Δ (but not necessarily finitely generated over Δ) such that $f \in k_{i\Delta_2}F$, $\Delta_2 \subseteq \Delta_1$, $\Delta_2 \neq \Delta_1$. Then evidently there exists Δ_3 finitely generated over Δ such that $\Delta_3 \subseteq \Delta_2$, $f \in k_{i,\Delta_3}F$. This contradicts the minimality of Δ_1. Thus Δ_1 is minimal among *all* preorderings Δ_2 over Δ satisfying $f \in k_{i,\Delta_2}F$.

If $\Delta_1 = \Delta$ we are finished. Otherwise there exists $P \in X_\Delta$, $P \notin X_{\Delta_1}$. Let $\Delta_2 = \Delta_1 \cap P$. Then

$$\Delta_1^\times/\Delta_2^\times = \Delta_1^\times/\Delta_1^\times \cap P^\times \cong \Delta_1^\times P^\times/P^\times = F^\times/P^\times,$$

so the group index $(\Delta_1^\times : \Delta_2^\times)$ is two, and if a is a group generator of $\Delta_1^\times$ over $\Delta_2^\times$, then $\Delta_1 = \Delta_2 \cup \Delta_2\, a$. If we show $f \in k_{i,\Delta_2}F$, we will have the desired contradiction.

From $\Delta_1 = \Delta_2 \cup \Delta_2\, a$ it follows that

$$k_{*\Delta_1}F = k_{*\Delta_2}F + l(a)k_*F.$$

Thus $f = f_1 + l(a)g$, with $f_1 \in k_{i\Delta_2}F$, $g \in k_{i-1}F$. Evidently, we have $l(-1)g = 0$ in k_iF_P for all $P \in X_{\Delta_2(-a)}$. But $l(-1)$ is not a divisor of zero in k_*F_P, so $g = 0$ locally for all such P. It follows by induction on i that $g \in k_{i-1,\Delta_2(-a)}F$.

†Thomas C. Craven (University of Hawaii, Honolulu) has recently classified the Witt ring structure of pythagorian fields with only a finite number of places into the reals (preprint, *Characterizing reduced Witt rings of fields*). It can be shown that all such fields satisfy D.C.C.

However, if $s \in \Delta_2(-a)$, then $s = \alpha - \beta a$, $\alpha, \beta \in \Delta_2$. It follows that $l(a)\, l(s) \equiv 0$ modulo $k_{*\Delta_2}F$. (For if $\alpha, \beta \neq 0$, then $s/\alpha + \beta a/\alpha = 1$, and $l(a)\, l(s) \equiv l(\beta a/\alpha)\, l(s/\alpha) = 0 \bmod k_{*\Delta_2}F$. The proof is even simpler if either α or β is zero.) It follows that $l(a)\, k_{*\Delta_2(-a)}F \subseteq k_{*\Delta_2}F$, so $f = f_1 + l(a)g \in k_{*\Delta_2}F$.

(2) The proof of the second conjecture is completely similar. We may assume $i \geqq 3$. Let $f \in M^i \cap W_\Delta$. We wish to show $f \in M^{i-1} W_\Delta$. We may assume Δ satisfies D.C.C. Using D.C.C. pick $\Delta_1 | \Delta$ minimal such that $f \in M^{i-1} W_{\Delta_1}$. If $\Delta_1 \neq \Delta$, then define Δ_2 as above. If $s \in \Delta_2$, then $\langle 1, -sa\rangle = \langle 1, -s\rangle + s\langle 1, -a\rangle$. It follows that

$$W_{\Delta_1} = W_{\Delta_2} + \langle 1, -a\rangle W, \quad \text{and} \quad M^{i-1}W_{\Delta_1} = M^{i-1}W_{\Delta_2} + \langle 1, -a\rangle M^{i-1}.$$

Proceeding as in (1) this yields $f \in M^{i-1}W_{\Delta_2}$, contradicting the minimality of Δ_1.

Remark. In this proof one would like to replace D.C.C. by Zorn's Lemma but it is not clear how this can be done.

5. Stability. The preorder Δ is said to be *k-stable* if every preorder which is finitely generated over Δ is of the form $\Delta(a_1, \ldots, a_k)$ for some $a_1, \ldots, a_k \in F^\times$. Several conditions are known which are equivalent to k-stability [**5**]. For example, Δ is k-stable if and only if the image of M^k in Cont $(X_\Delta, \mathbf{Z})$ is Cont $(X_\Delta, 2^k\mathbf{Z})$.

In this section, some general theory is developed, and as a corollary of this it follows that Conjectures 1 and 2 hold if Δ is 2-stable.

We will say that the quadratic form $f = \langle b_1, \ldots, b_n\rangle$ over F *represents* $x \in F$ *modulo* W_Δ, if there exist elements $s_1, \ldots, s_n \in \Delta$ satisfying $x = s_1b_1 + \ldots + s_nb_n$. There is a well developed theory of forms modulo W_Δ, which may be found in [**4**] or (more generally) in [**10**]. A basic result is the following:

LEMMA 1. *A form $f = \langle a_1, \ldots, a_n\rangle$ represents $x \in F^\times$ modulo W_Δ if and only if there exist $x_2, \ldots, x_n \in F^\times$ such that $f \equiv \langle x, x_2, \ldots, x_n\rangle$* mod W_Δ.

LEMMA 2. *Let $f = l(-a_1) \ldots l(-a_n)$, let $f_1 = \langle\langle a_1, \ldots, a_n\rangle\rangle$, and let f_1' denote the form derived from f_1 via $f_1 \cong f_1' \oplus \langle 1\rangle$. Suppose f_1' represents b modulo W_Δ, $b \neq 0$. Then there exist $b_2, \ldots, b_n \in F^\times$ such that $f_1 \equiv \langle\langle b, b_2, \ldots, b_n\rangle\rangle$ modulo $M^{n-1} W_\Delta$ and $f \equiv l(-b)\, l(-b_2) \ldots l(-b_n)$ modulo $k_{*\Delta}F$.*

Proof. The proof is only given for f. The result concerning f_1 is obtained in an analogous way. To simplify notations in the proof, congruences will denote congruences modulo $k_{*\Delta}F$.

The proof is by induction on n. If $n = 1$, then $b = sa_1$, $s \in \Delta^\times$, so $l(-b) = l(-a_1) + l(s) \equiv l(-a_1)$. Assume $n > 1$. Let $g = l(-a_2) \ldots l(-a_n)$, $g_1 = \langle\langle a_2, \ldots, a_n\rangle\rangle$. Then $f_1 \cong g_1 \oplus a_1g_1$, so $f_1' \cong g_1' \oplus a_1g_1$. Thus $b = c + a_1d$. with c and d represented (modulo W_Δ) by g_1' and g_1 respectively. Also $d = s + d'$, $s \in \Delta$, d' represented by g_1' modulo Δ. Thus by induction, g decomposes as $g \equiv l(-d') \ldots$, so $l(d)g \equiv l(d)\, l(-d') \ldots$. Now $l(d)\, l(-d') \equiv$

$l(d/s)\, l(-d'/s) \equiv 0$, since $s \in \Delta$, and $(d/s) - (d'/s) = 1$. (If either $s = 0$ or $d' = 0$, the proof is even simpler.) Thus $l(-a_1)g - l(-a_1 d)g = -l(d)g \equiv 0$. Also, by induction g decomposes as $g \equiv l(-c) \ldots$, so $f = l(-a_1)g \equiv l(-a_1 d)g \equiv l(-a_1 d)\, l(-c) \ldots$. But $b = c + a_1 d$, so $l(-a_1 d)\, l(-c) = l(-b)\, l(-a_1 dc)$.

Note. The idea in the proof of this lemma is not new. It may be found in [**13**, Lemma 2.4.4), for example.

One should compare the next theorem to the "Main Theorem" in [**6**].

THEOREM 8. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in F^*$. Then the following statements are equivalent:*

(i) $\Delta(a_1, \ldots, a_n) = \Delta(b_1, \ldots, b_n)$

(ii) $\langle\langle a_1, \ldots, a_n\rangle\rangle \equiv \langle\langle b_1, \ldots, b_n\rangle\rangle \bmod W_\Delta$

(iii) $\langle\langle a_1, \ldots, a_n\rangle\rangle \equiv \langle\langle b_1, \ldots, b_n\rangle\rangle \bmod M^{n-1}\, W_\Delta$

(iv) $l(-a_1) \ldots l(-a_n) \equiv l(-b_1) \ldots l(-b_n) \bmod k_{*\Delta}F$.

Proof. It is clear that (i) $\Leftrightarrow$ (ii) and that (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii). For example, just compare signatures of $\langle\langle a_1, \ldots, a_n\rangle\rangle$ and $\langle\langle b_1, \ldots, b_n\rangle\rangle$ at all $P \in X_\Delta$. A proof is given here that (ii) $\Rightarrow$ (iv). The proof that (ii) $\Rightarrow$ (iii) is similar and will not be included.

The proof is by induction on n, the result being clear if $n = 1$. By the lemmas we have $c_2, \ldots, c_n \in F^\times$ such that $l(-a_1) \ldots l(-a_n) \equiv l(-b_1)\, l(-c_2) \ldots l(-c_n)$ modulo $k_{*\Delta}F$. Thus $\Delta(b_1, c_2, \ldots, c_n) = \Delta(a_1, \ldots, a_n) = \Delta(b_1, \ldots, b_n)$ so by induction $l(-c_2) \ldots l(-c_n) \equiv l(-b_2) \ldots l(-b_n)$ modulo $k_{*\Delta(b_1)}F$. Now if $s \in \Delta(b_1)^\times$, $s = \alpha + \beta b_1$, $\alpha, \beta \in \Delta$, so

$$l(-b_1)l(s) \equiv l\left(\frac{-\beta b_1}{\alpha}\right) l\left(\frac{s}{\alpha}\right) \equiv 0 \mod k_{*\Delta}F.$$

It follows that $l(-b_1)\, l(-c_2) \ldots l(-c_n) \equiv l(-b_1) \ldots l(-b_n)$ modulo $k_{*\Delta}F$ so by transitivity of $\equiv$, we have the required result.

COROLLARY. *If Δ is 2-stable, then Conjectures 1 and 2 hold.*

Proof. First look at the second conjecture. We may assume $i > 2$. For any $a_1, \ldots, a_i \in F^\times$ we have, by 2-stability, elements $a, b \in F^\times$ such that $\Delta(a_1, \ldots, a_i) = \Delta(a, b, 1, \ldots, 1)$ (with $i - 2$ ones). It follows by the previous theorem that $\langle\langle a_1, \ldots, a_i\rangle\rangle \equiv 2^{i-2} \langle\langle a, b\rangle\rangle$ modulo $M^{i-1}\, W_\Delta$. Thus $M^i = 2^{i-2} M^2 + M^{i-1} W_\Delta$, so

$$M^i \cap W_\Delta = 2^{i-2}(M^2 \cap W_\Delta) + M^{i-1}\, W_\Delta = 2^{i-2}\, M\, W_\Delta + M^{i-1}\, W_\Delta = \mathrm{M}^{i-1}\, W_\Delta.$$

Now look at Conjecture 1. If $i > 2$, then by the previous theorem $k_iF = l(-1)^{i-2}\, k_2F + k_{i\Delta}F$. Thus we are reduced to the case $i = 2$ (since $l(-1)$is not a divisor of zero locally). But $s_2 : k_2F \cong M^2/M^3$. Assume $f \in k_2F$ is zero in k_2F_P for all $P \in X_\Delta$. Write $s_2(f) = g + M^3, g \in M^2$, Then $\text{sgn}_P\, g \equiv 0 \bmod 8$ for all $P \in X_\Delta$, so by the alternate characterization of k-stability mentioned, $g = 2h + h_1$ with $h \in M^2$, $h' \in W_\Delta$. Thus $h_1 \in M^2 \cap W_\Delta = MW_\Delta$, and $s_2(f) = h_1 + M^3$. Since the isomorphism s_2 carries $k_{2\Delta}F$ onto $(MW_\Delta + M^3)/M^3$ this completes the proof.

Note. Certain of the consequences of Conjecture 1 are valid if Δ is k-stable, $k \geqq 3$. For example, Corollary 1 of Theorem 3 holds if Δ is k-stable. In particular if Δ is 3-stable, then this corollary holds for all $k \geqq 1$ (since it holds trivially for $k = 1, 2$). Also, the Corollary of Theorem 4 holds if Δ is 3-stable.

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