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Subsystems of transitive subshifts with linear complexity

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Abstract. We bound the number of distinct minimal subsystems of a given transitive subshift of linear complexity, continuing work of Ormes and Pavlov [On the complexity function for sequences which are not uniformly recurrent. *Dynamical Systems and Random Processes (Contemporary Mathematics, 736)*. American Mathematical Society, Providence, RI, 2019, pp. 125–137]. We also bound the number of generic measures such a subshift can support based on its complexity function. Our measure-theoretic bounds generalize those of Boshernitzan [A unique ergodicity of minimal symbolic flows with linear block growth. *J. Anal. Math.* **44**(1) (1984), 77–96] and are closely related to those of Cyr and Kra [Counting generic measures for a subshift of linear growth. *J. Eur. Math. Soc.* **21**(2) (2019), 355–380].

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1. *Introduction*

In this work, we study symbolically defined dynamical systems called subshifts. A subshift is defined by a finite set $\mathcal{A}$ (called an alphabet), the (left) shift action σ on $\mathcal{A}^{\mathbb{Z}}$, and a set $X \subset \mathcal{A}^{\mathbb{Z}}$ of sequences that is closed in the product topology and σ-invariant. For convenience, we will refer to a subshift (X, σ) only as X since the dynamics are always understood to come from σ. (See §2.1 for more details.)

Given a subshift X, let $c_X(n)$ denote the number of words of length n that appear in X, that is, the complexity function of X. Assuming that X is transitive (meaning that X is the closure of the σ-orbit of some $x \in X$) and $c_X(n)$ grows linearly, we ask: what is the interplay between $c_X(n)$ and the structure of the subdynamical systems of X? We study this question in both the topological and measure-theoretic categories.

In the topological category, we provide bounds on the number of distinct minimal subsystems that the transitive subshift X can have depending on how quickly $c_X(n)$ grows. By the Morse–Hedlund theorem (Theorem 3.2), if there exists $n \geq 1$ such that $c_X(n) \leq n$, then X is merely a finite set of periodic points. In [10], Ormes and Pavlov showed that if

$$\limsup_{n\to\infty}(c_X(n) - 1.5n) < \infty,$$

then X must itself be minimal. They proved this by contrapositive: were there to exist a proper minimal subsystem $M \subset X$, then by carefully counting the words that occur in points (which exist by transitivity) whose orbits enter and exit neighborhoods of M, they showed that $\limsup_{n\to\infty}(c_X(n) - 1.5n) = \infty$.

Moreover, they showed that this bound is *sharp* in the sense that there would exist non-minimal examples were the threshold $1.5n$ increased to $1.5n + g(n)$, where $g : \mathbb{N} \to \mathbb{N}$ is any non-decreasing, unbounded function.

For X such that $c_X(n)$ grows faster than $1.5n$, in §3 we establish the following.

THEOREM 1.1. *Let X be a transitive subshift which is the orbit closure of a recurrent point x, where X has $j \geq 2$ proper minimal subsystems, exactly i of which are infinite $(0 \leq i \leq j)$. For $k \in \mathbb{N}$, if either:*
(1) $\limsup_{n\to\infty}(c_X(n) - (k+1)n) < \infty$*; or*
(2) $\liminf_{n\to\infty}(c_X(n) - kn) < \infty$,
then $i + j < k$. Moreover, this bound is sharp.

The notion that the bound is sharp is the same as in [10]; namely, if either of the thresholds $(k+1)n$ or kn in (1) or (2) were increased by adding a non-decreasing, unbounded function $g : \mathbb{N} \to \mathbb{N}$, then there would exist X for which $i + j = k$.

In §4, we consider the case of a general (not necessarily recurrent) transitive subshift. We establish bounds on the growth rate of $c_X(n)$ in the special cases where X contains one or two minimal subsystems and then prove the following.

THEOREM 1.2. *Let X be a transitive (not necessarily recurrent) subshift where X has $j \geq 3$ minimal subsystems, exactly i of which are infinite $(0 \leq i \leq j)$. For $k \in \mathbb{N}$, if*

$$\liminf_{n\to\infty}(c_X(n) - kn) < \infty,$$

then $i + j < k$.

Of course, the condition

$$\limsup_{n\to\infty}(c_X(n) - kn) < \infty$$

also implies that $i + j < k$, and we show that this bound is sharp in the same sense as above.

Turning our attention to the measure-theoretic category, we consider the well-studied problem of bounding the number of ergodic measures that a given subshift can support.

For example, in [2], Boshernitzan showed that if X is minimal and

$$\liminf_{n\to\infty}(c_X(n) - Kn) = -\infty,$$

then X can support at most $K-1$ ergodic measures. He also showed, again assuming minimality, that X is uniquely ergodic provided that

$$\limsup_{n\to\infty}\frac{c_X(n)}{n} < 3.$$

In [3], Cyr and Kra, motivated by work of Katok [6] and Veech [13] on interval exchange transformations, extended Boshernitzan's work by considering arbitrary (not necessarily minimal) subshifts and non-atomic generic (not necessarily ergodic) measures (restated here as Theorem 5.1). In [4, 5], Damron and Fickenscher proved stronger bounds on the number of ergodic measures for transitive subshifts satisfying linear complexity plus additional technical conditions; these results apply, in particular, to codings of interval exchanges.

Because Cyr and Kra do not assume transitivity of the subshift X, they are not able to bound the number of non-atomic measures. As an example to illustrate this issue, suppose that X is known to have generic measures δ_{0^∞} and δ_{1^∞} (supported entirely on the sequences of all 0's and all 1's). With no other assumptions, it would be possible to simply have $X = \{0^\infty, 1^\infty\}$, in which case $c_X(n) = 2$ for every n. However, by merely assuming that there exists a point $x \in X$ that is not eventually periodic in both directions and whose orbit is dense (as we do in Theorem 1.4 below), one can show that $\liminf_{n\to\infty}(c_X(n) - 2n) > \infty$, as is done in [10, Lemma 3.3].

Our measure-theoretic results are as follows and are proved in §5.

THEOREM 1.3. *Let $X = \overline{\mathcal{O}(x)}$ be a transitive subshift where x is recurrent and aperiodic. If*

$$\limsup_{n\to\infty}\frac{c_X(n)}{n} < 3,$$

then X is uniquely ergodic.

THEOREM 1.4. *Let $X = \overline{\mathcal{O}(x)}$ be a transitive subshift where x is not eventually periodic in both directions. If*

$$\liminf_{n\to\infty}(c_X(n) - gn) = -\infty$$

for $g \in \mathbb{N}$, then X has at most $g-1$ generic measures.

Note that Theorem 1.4 does not imply Theorem 1.2. Indeed, if X contains g minimal subsystems, then there are at least g generic measures on X. Theorem 1.4 would then imply that

$$\liminf_{n\to\infty}(c_X(n) - gn) > -\infty,$$

whereas Theorem 1.2 gives the stronger conclusion

$$\liminf_{n\to\infty}(c_X(n) - gn) = \infty.$$

The methods we use to prove Theorems 1.3 and 1.4 take advantage of the given 'transitive point' $x \in X$. Specifically, drawing on techniques developed in [2], our proofs revolve around keeping track of so-called *right special* words that occur in x. (A word is *right special* if it can be followed by at least two different letters.) This differs from the approach in [3], where transitivity was not assumed.

While this paper focuses on two-sided subshifts (indexed by $\mathbb{Z}$), one could study similar questions for one-sided subshifts (indexed by $\mathbb{N}$). With some necessary modifications, and by treating two-sided subshifts as natural extensions of one-sided subshifts, we believe that it would be possible to prove some analogous versions of our results in the context of one-sided subshifts. But for brevity we restrict our attention to the two-sided context.

The paper is organized as follows. We first establish preliminary definitions in §2. In §3, we prove Theorem 1.1. As in [10], we give a proof by contrapositive by supposing that there exist proper minimal subsystems $M_1, \ldots, M_j$ and counting the words that occur in points whose orbits transition between neighborhoods of these subsystems. We handle the possibility that i of these j subsystems are infinite by first reducing to the $i = 0$ case with factor maps; the $i = 0$ case then completes the counting argument. To complete the proof of Theorem 1.1, we provide a family of examples that establish the sharpness of our bounds. In §4, we prove Theorem 1.2, which involves modifying arguments from §3 to more a general setting where systems are transitive but not necessarily recurrent. In §5, we transition to the measure-theoretic category and prove Theorems 1.3 and 1.4.

2. *Preliminaries*

2.1. *Subshifts.*

We recall some basic definitions; for more information, see [7, 8, 14].

A *full shift* is a pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$, where $\mathcal{A}$ is a finite alphabet, $\mathcal{A}^{\mathbb{Z}}$ has the product of the discrete topology on $\mathcal{A}$, and $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is the left shift defined by $\sigma(x)_i = x_{i+1}$ for each $x = (x_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$. A *subshift* is a pair (X, σ), where X is a closed and σ-invariant subset of some $\mathcal{A}^{\mathbb{Z}}$. To conserve notation, we will often refer to the subshift (X, σ) as simply X.

A subshift X is *transitive* if there exists $x \in X$ such that $X = \overline{\mathcal{O}(x)}$, the closure of the orbit $\mathcal{O}(x) = \{\sigma^n(x) : n \in \mathbb{Z}\}$. We call such a point $x \in X$ a *transitive point*. If $X = \overline{\mathcal{O}(x)}$ for every $x \in X$, then X is *minimal*. A transitive subshift X is *periodic* if it has a transitive point x which is periodic, meaning that there exists $p \in \mathbb{Z}$ such that $\sigma^p(x) = x$. Note that a transitive subshift X is periodic if and only if X has finite cardinality.

Given a subshift X, a *word of length* n *in* X is a block of symbols $w = w_1 w_2 \cdots w_n$ that occurs in some point $x \in X$, that is, $w = x_i x_{i+1} \cdots x_{i+n-1}$ for some $i \in \mathbb{Z}$. Let $\mathcal{L}_n(X)$ denote the set of all words of length n occurring in some point in X, and $\mathcal{L}(X) = \bigcup_{n=1}^{\infty} \mathcal{L}_n(X)$. The *complexity function* of X is the function $c_X(n) : \mathbb{N} \to \mathbb{N}$ that gives the cardinality of $\mathcal{L}_n(X)$. If X is transitive, then $X = \overline{\mathcal{O}(x)}$ for some $x \in X$, and in this case $c_X(n)$ is equal to the number of words of length n in x.

For a symbol $a \in \mathcal{A}$ and $n \geq 1$, the expression a^n denotes the word of length n formed by concatenating a with itself n times. Correspondingly, a^∞ denotes the infinite concatenation of a with itself. Depending on the situation, a^∞ may denote a bi-infinite sequence, a left-infinite sequence, or a right-infinite sequence. The choice of meaning should be clear from context.

A point $x \in X$ is *recurrent* if every word in x occurs at least twice (equivalently, infinitely often). If every word in x occurs infinitely often with uniformly bounded gaps between occurrences, then x is *uniformly recurrent*. By [1], a subshift X is minimal if and only if $X = \overline{\mathcal{O}(x)}$ for some uniformly recurrent $x \in X$. (See also [7, Theorem 2.19].)

If X and Y are subshifts, then for any continuous $f : X \to Y$ such that $f \circ \sigma = \sigma \circ f$, there exist $m, a \in \mathbb{N}$ such that for every $x \in X$, $f(x)_0$ is determined by the word $x_{-m} \cdots x_0 \cdots x_a$. In this case f is called an $(m + a + 1)$-*block map* with *memory* m and *anticipation* a. If such a map f is surjective, then it is a *factor map*. In this paper, when we define a factor map f on a subshift X, we always define $Y = f(X)$. In addition, any $(m + a + 1)$-block map has an obvious associated action on finite words as well (for any n-letter word w, $f(n)$ has length $n - m - a$); we use f to refer to this function also since usage should always be clear from context.

A word w in $\mathcal{L}(X)$ is *right-special* if there exist $a, b \in \mathcal{A}$ such that $wa, wb \in \mathcal{L}(X)$ with $a \neq b$. Similarly, w is *left-special* if there exist $a, b \in \mathcal{A}$ such that $aw, bw \in \mathcal{L}(X)$ with $a \neq b$. Let $RS_X(n)$ (or just $RS(n)$ if X is understood) denote the set of right-special words in X of length n.

Let x be an element of a subshift X. By the *omega-limit set of* x, we mean the set

$$\omega(x) = \bigcap_{N \geq 1} \overline{\{\sigma^n(x) : n \geq N\}}.$$

For any $x \in X$, the set $\omega(x)$ is a closed and shift-invariant subset of X, so is itself a subshift.

We say that a point x in a subshift X is *eventually periodic to the right* if there exist integers $p > 0$ and $N > 0$ such that for all $i > N$, $x_i = x_{i+p}$. Similarly, we say that x is *eventually periodic to the left* if there exist integers $p > 0$ and $N > 0$ such that for all $i < -N$, $x_i = x_{i-p}$.

2.2. *Sturmian subshifts.* There are several different approaches to defining Sturmian subshifts (see [12] for an introduction). We outline one such approach here.

For any irrational β, define the map $R_\beta : [0, 1) \to [0, 1)$ by $R_\beta(x) = x + \beta \bmod 1$. For any $x \in (0, 1)$, define the sequence $s(x) \in \{0, 1\}^{\mathbb{Z}}$ by

$$s_n(x) = \begin{cases} 1 & \text{if } R_\beta^n(x) \in [0, \beta), \\ 0 & \text{if } R_\beta^n(x) \in [\beta, 1). \end{cases}$$

The bi-infinite sequence $s(x)$ is called a *Sturmian sequence* for β. A subshift $X \subset \{0, 1\}^{\mathbb{Z}}$ is a *Sturmian subshift* if X can be obtained as the orbit closure of a Sturmian sequence.

For a Sturmian subshift X, a couple of properties that will be useful (e.g., in the proof of Theorem 4.1) are:

- for every $n \in \mathbb{N}$, there is exactly one right-special word of length n; and
- if $(1/n) < \beta < (1/(n + 1))$, then the number of zeros in any word of the form $1000 \cdots 01$ must be either n or $n - 1$.

2.3. *Bounds on the complexity function.* Here we introduce shorthand notation for bounds on the complexity function.

Definition 2.1. Given a function $f : \mathbb{N} \to \mathbb{R}$, write $c_X(n) \,\overline{\succ}\, f(n)$ if

$$\limsup_{n\to\infty}(c_X(n) - f(n)) = \infty.$$

Definition 2.2. Given a function $f : \mathbb{N} \to \mathbb{R}$, write $c_X(n) \succeq f(n)$ if

$$\liminf_{n\to\infty}(c_X(n) - f(n)) = \infty.$$

To prove Theorem 1.1, note that it is sufficient to prove, under the given hypotheses on X, that both $c_X(n) \,\overline{\succ}\, (j + i + 1)n$ and $c_X(n) \succeq (j + i)n$. This will be our approach in §3.2.

Similarly, to prove Theorem 1.2, in §4.2 we will show (under the given hypotheses) that $c_X(n) \succeq (j + i)n$.

A bound of the form $c_X(n) \,\overline{\succ}\, f(n)$ (or $c_X(n) \succeq f(n)$) which holds under some hypotheses on X is *sharp* if it fails when any non-decreasing unbounded $g : \mathbb{N} \to \mathbb{N}$ is added to $f(n)$, that is, if for any such g there exists a subshift X satisfying the relevant hypotheses for which $c_X(n) \,\overline{\succ}\, f(n) + g(n)$ (or $c_X(n) \succeq f(n) + g(n)$) is false.

3. *Transitive systems with a recurrent transitive point*

The main goal of this section is to prove Theorem 1.1, which assumes that X contains two or more minimal subsystems. We will do this in §3.2 by showing that both $c_X(n) \,\overline{\succ}\, (j + i + 1)n$ and $c_X(n) \succeq (j + i)n$. But first we recall some existing results that can be used to treat X containing a single minimal subsystem.

3.1. *Single minimal subsystem.* If X is itself minimal and not a periodic orbit, then the following two results establish that

$$c_X(n) \geq n + 1$$

for all $n \geq 1$ and that this bound cannot be improved.

THEOREM 3.1. [9] *If X is a Sturmian subshift, then*

$$c_X(n) = n + 1 \quad \textit{for all } n \geq 1.$$

THEOREM 3.2. [9] *Let X be any subshift. If there exists $n \geq 1$ such that $c_X(n) \leq n$, then X is a finite set of periodic points.*

Thus, the Sturmian subshifts are examples of the lowest-complexity subshifts that are not periodic (see [11] for more). If X is transitive and not minimal, then the following (sharp) bound was proved in [10].

THEOREM 3.3. [10] *Suppose that X is a transitive subshift with a recurrent transitive point x. If X is not minimal, then*

$$c_X(n) \,\overline{\succ}\, 1.5n$$

and this bound is sharp.

Combining the results of [10] and techniques of the proof of Theorem 1.1, we will be able to establish the following.

THEOREM 3.4. *Suppose that X is a transitive subshift with a recurrent transitive point x. If X properly contains an infinite minimal subsystem, then*

$$c_X(n) \succsim 2.5n$$

and this bound is sharp.

We postpone the proof of Theorem 3.4 until after the proof of Theorem 1.1.

3.2. *Multiple minimal subsystems.* For $j \geq 2$ and $0 \leq i \leq j$, consider the set of subshifts $S = S(j, i)$, where $X \in S(j, i)$ if and only if X is transitive with a recurrent transitive point, and X has j distinct minimal subsystems, exactly i of which are infinite. To prove Theorem 1.1 for $X \in S(j, i)$, we first reduce to the $i = 0$ case via factor maps.

3.2.1. *Reduction to the $i = 0$ case.* We define a factor map π that maps the j distinct minimal subsystems of X to j distinct points, which are fixed by σ. Lemma 3.5 will provide an inequality between the complexity sequences for X and $\pi(X)$, which will allow us to simply work with $\pi(X)$ moving forward.

Let $M_1, \ldots, M_j$ denote the minimal subsystems of X, where, without loss of generality, $M_1, \ldots, M_i$ are infinite and $M_{i+1}, \ldots, M_j$ are finite. Since the sets $M_1, \ldots, M_j$ are pairwise disjoint closed subsets of X, there is an $r \geq 1$ such that the sets $\mathcal{L}_r(M_1), \ldots, \mathcal{L}_r(M_j)$ are pairwise disjoint. Fix such a value of r; pick j distinct symbols $a_1, \ldots, a_j$ that do not occur in x and define an r-block map ϕ with domain X, memory zero, and anticipation $(r - 1)$ as follows:

$$\phi(y)_q = \begin{cases} a_p & \text{if } y_q \cdots y_{q+r-1} \in \mathcal{L}_r(M_p), \\ y_q & \text{otherwise.} \end{cases}$$

Now pick a symbol b that does not occur in $\phi(x)$ and define a 2-block map ψ with domain $\phi(X)$, memory zero, and anticipation 1 as follows:

$$\psi(z)_q = \begin{cases} b & \text{if } z_q z_{q+1} \in \{a_p s, s a_p\} \text{ for some } p \text{ and } s \neq a_p, \\ z_q & \text{otherwise.} \end{cases}$$

Note that post-composing ϕ with ψ has the effect of ensuring that words of the form a_p^n are always preceded and followed by the 'marker' symbol b. Define $\pi = \psi \circ \phi$. Then the minimal subsystems of $\pi(X)$ are simply the one-point sets $\pi(M_p) = \{a_p^\infty\}$. If $X \in S(j, i)$, it follows that $\pi(X) \in S(j, 0)$.

LEMMA 3.5. *If $X \in S(j, i)$ with $j \geq 2$ and π is the factor map defined above, then, for every $n > r$, $c_X(n) \geq c_{\pi(X)}(n - r) + in$.*

Proof. Note that for any q-block factor map from a subshift X onto a subshift Y, a word of length m in X determines a word of length $m - q + 1$ in Y. It follows that $c_X(m) \geq c_Y(m - q + 1)$.

Applying this to our situation, since ψ is a 2-block map, $c_{\phi(X)}(n) \geq c_{\pi(x)}(n-1)$. To complete the proof, it is enough to show that $c_X(n) \geq c_{\phi(X)}(n-r+1) + in$. For any n, let W_0 denote the set of words in $\phi(X)$ of the form a_p^{n-r+1}, where $n \geq r$ and $1 \leq p \leq i$, and let $W_1 = \mathcal{L}_{n-r+1}(\phi(X)) \setminus W_0$. Each word in W_1 has at least one ϕ-preimage in $\mathcal{L}_n(X)$, but each $a_p^{n-r+1} \in W_0$ has at least $n+1$ preimages in $\mathcal{L}_n(M_p)$, since any word in $\mathcal{L}_n(M_p)$ is a preimage of a_p^{n-r+1}, and $c_{M_p}(n) \geq n+1$ for all n, using Theorem 3.2 and the fact that M_p is infinite. □

LEMMA 3.6. *For each $p \in \{1, \ldots, j\}$ and $n \in \mathbb{N}$, a_p^n is both right- and left-special in $\pi(X)$.*

Proof. Recall that $\pi(M_p) = \{a_p^\infty\}$. Therefore, $a_p^\infty \in \pi(X)$, so a_p^n can be followed by a_p. Assume for a contradiction that a_p^n is not right-special, that is, it can only be followed by a_p. Since $\pi(X) \neq \{a_p^\infty\}$, there must exist $m \in \mathbb{Z}$ so that

$$\sigma^m(x) = \cdots y_{-3}y_{-2}y_{-1}.a_p^\infty,$$

where $y_{-1} \neq a_p$. Since x is recurrent, $y_{-k} \cdots y_{-k+n} = y_{-1}a_p^n$ for some $k \geq n+2$. Then $-k+n \leq -2$, so there is some smallest $q \geq 1$ such that $y_{-k+n+q} \neq a_p$. Then $y_{-k+q} \cdots y_{-k+n+q-1} = a_p^n$ and $y_{-k+n+q} \neq a_p$, implying that a_p^n is right-special, which is a contradiction. Our original assumption was then false, that is, a_p^n is right-special; a symmetric argument shows that a_p^n is left-special. □

To establish that $c_X(n) \not\preceq (j+i+1)n$ and $c_X(n) \succeq (j+i)n$, first observe that, by Lemma 3.5, it is enough to show only that these bounds hold for $\pi(X)$. Indeed, if we prove that $c_{\pi(X)}(n) \not\preceq (j+1)n$, then there is a strictly increasing sequence (n_k) such that $c_{\pi(X)}(n_k - r) > (j+1)(n_k - r) + k$ for all k. So, by Lemma 3.5,

$$\begin{aligned} c_X(n_k) &\geq c_{\pi(X)}(n_k - r) + in_k \\ &> (j+1)(n_k - r) + k + in_k \\ &= (j+i+1)n_k + k - r(j+1), \end{aligned}$$

which implies that $c_X(n) \not\preceq (j+i+1)n$. By a similar argument, $c_{\pi(X)}(n) \succeq jn$ implies that $c_X(n) \succeq (j+i)n$.

Because of this, to simplify notation we will replace $\pi(X)$ with X and make the following assumptions.

ASSUMPTIONS 3.7. *Throughout §3.2.2 below, we assume that:*

- *(A1): $X \in S(j, 0)$;*
- *(A2): the minimal subsystems of X are the one-point systems $M_1 = \{a_1^\infty\}, \ldots, M_j = \{a_j^\infty\}$; and*
- *(A3): for any word in X of the form $a_p s$ or $s a_p$, $s = b$ or $s = a_p$.*

3.2.2. *Proof that $c_X(n) \not\preceq (j+i+1)n$ and $c_X(n) \succeq (j+i)n$ when $i = 0$.* Under the hypotheses of Theorem 1.1 (and under Assumptions 3.7), there is a transitive point x for X that is recurrent, and that therefore cannot be both eventually periodic to the left and eventually periodic to the right. Without loss of generality, assume that x is not eventually periodic to the right. Our proof will involve bounding from below the number of right-special words in X of various lengths (were x eventually periodic to the right, we would instead count left-special words). The following elementary lemma will be used to verify that $c_X(n) \overline{\succ} (j+1)n$.

LEMMA 3.8. *Under* Assumptions 3.7, *for each* $n \geq 2$, $c_X(n) \geq c_X(n-1) + \#RS(n-1)$. *Therefore, for* $m < n$, $c_X(n) \geq c_X(m) + \sum_{\ell=m}^{n-1} \#RS(\ell)$.

Proof. For every n, each word of length $n-1$ can be extended to at least one word of length n, while each right-special word of length $n-1$ can be extended to at least two words of length n. This yields $c_X(n) \geq c_X(n-1) + \#RS(n-1)$. Applying this recursively, we obtain the inequality $c_X(n) \geq c_X(m) + \sum_{\ell=m}^{n-1} \#RS(\ell)$ for $m < n$. □

We begin by considering the set of right-special words provided by Lemma 3.6, which we will call B:

$$B := \{a_p^n : 1 \leq p \leq j \text{ and } n \in \mathbb{N}\}.$$

By just considering the elements of B, we see that $\#RS(n) \geq j$ for all n.

By combining the inequality $\#RS(n) \geq j$ with Lemma 3.8, we can show that the bound $c_X(n) \overline{\succ} (j+1)n$ would imply that $c_X(n) \succeq jn$. Indeed, note that $c_X(n) \overline{\succ} (j+1)n$ implies that there exists an increasing sequence of integers (n_k) such that $c_X(n_k) \geq (j+1)n_k$. Thus, for $n > n_k$,

$$\begin{aligned} c_X(n) &\geq c_X(n_k) + \sum_{\ell=n_k}^{n-1} \#RS(n) \\ &\geq (j+1)n_k + j(n-n_k) \\ &\geq jn + n_k, \end{aligned}$$

which gives $c_X(n) \succeq jn$.

Thus, we will be done if we establish the bound $c_X(n) \overline{\succ} (j+1)n$. We divide the proof into cases, beginning with the simplest case.

Case (i). For all sufficiently large n, $\#RS(n) \geq j+1$ and there is a strictly increasing sequence (n_k) where $\#RS(n_k) \geq j+2$ for all k.

The assumptions imply that, for $n > n_k$,

$$\sum_{\ell=1}^{n} \#RS(\ell) \geq (j+1)n + k.$$

This implies that $c_X(n) \overline{\succ} (j+1)n$, which completes the proof of Theorem 1.1 in Case (i).

We now assume that the hypotheses of Case (i) do not hold. That is, precisely one of the following two conditions holds.

Case (ii). There is a strictly increasing sequence (n_k) such that $\#RS(n_k) = j$, that is, $RS(n_k) \subset B$ for all k;

Case (iii). For all sufficiently large n, $\#RS(n) = j+1$.

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LEMMA 3.9. *Under* Assumptions 3.7, *suppose that either Case (ii) or (iii) holds. Then there exists a strictly increasing sequence (n_k) such that for each k, after re-indexing the minimal subsystems $M_1, \ldots, M_j$ if necessary, there exist words $w_{12}^{(k)}, w_{23}^{(k)}, \ldots, w_{j1}^{(k)}$, each of which begins and ends with the symbol b, such that the transitive point $x \in X$ has one of the following forms:*

$$x = \cdots\ a_1^{\geq n_k}\ w_{12}^{(k)}\ a_2^{\geq n_k}\ w_{23}^{(k)}\ \cdots\ a_j^{\geq n_k}\ w_{j1}^{(k)}\ a_1^{\geq n_k}\ w_{12}^{(k)}\ \cdots \text{ or}$$

$$x = a_1^{\infty}\ w_{12}^{(k)}\ a_2^{\geq n_k}\ w_{23}^{(k)}\ \cdots\ a_j^{\geq n_k}\ w_{j1}^{(k)}\ a_1^{\geq n_k}\ w_{12}^{(k)}\ \cdots,$$

where each $a_p^{\geq n_k}$ represents a word of the form a_p^n for some $n \geq n_k$.

Proof. Recall that x is a transitive point that is not eventually periodic to the right. For all $p \in \{1, \ldots, j\}$ and all $n \geq 1$, a_p^n occurs in x. Since x is not eventually periodic to the right, $a_p^n b$ must occur as well. Since x is recurrent, $a_p^n b$ occurs infinitely many times in x. It follows that $b a_p^n$ also occurs infinitely many times in x.

In Case (ii), we are given a strictly increasing sequence (n_k) such that $\#RS(n_k) = j$ for all k. Fix $k \geq 1$ and $p \in \{1, \ldots, j\}$. Define $w_1 = b$. Since $\#RS(n_k) = j$, $a_p^{n_k - 1} w_1 \notin RS(n_k)$, so there is only one symbol, call it w_2, that can appear after $a_p^{n_k-1} w_1$. Similarly, there is only one symbol that can appear after $a_p^{n_k-2} w_1 w_2$. Continuing in this way, each successive symbol w_i is forced (at least) until some $w_i \cdots w_{i+n_k-1} \in RS(n_k)$, that is, $w_i \cdots w_{i+n_k-1} = a_r^{n_k}$ for some r. There must exist some smallest i for which this is true, since the omega-limit set $\omega(x)$ must contain one of the minimal subsystems $M_1, \ldots, M_j$. Set $w(p) = w_1 w_2 \cdots w_{i-1}$. Note that $w(p)$ is the only word that begins with b that can follow $a_p^{n_k}$ in x, and that $a_r^{n_k}$ is the only word that can follow $w(p)$. Set $f(p) = r$.

Since $1 \leq p \leq j$ was arbitrary, we obtain a function $f : \{1, \ldots, j\} \to \{1, \ldots, j\}$. Since $b a_p^{n_k}$ appears in x for each p, each $p \in \{1, \ldots, j\}$ is equal to $f(q)$ for some $q \in \{1, \ldots, j\}$, that is, f is a bijection and thereby a composition of cyclic permutations. If f were the composition of two or more cyclic permutations, then x would not contain all $a_p^{n_k}$ for $1 \leq p \leq j$, in contradiction to transitivity of x. Therefore, f must cyclically permute the elements of $\{1, \ldots, j\}$. Re-indexing if necessary, we may assume that $f(p) = p + 1$ for $p < j$ and $f(j) = 1$. For $p \in \{1, \ldots, j\}$, set $w_{pf(p)}^{(k)} = w(p)$. After re-indexing again in the case where x is eventually periodic to the left, it follows that x has one of the two prescribed forms.

Now assume the hypothesis of Case (iii). By the assumptions on x, there must exist a strictly increasing sequence (n_k) such that $b a_1^{n_k} b$ occurs in x for all k. Because $b a_1^{n_k+1}$ also occurs in x, we know that $b a_1^{n_k}$ is right-special for every k. By deleting the first few terms of the sequence (n_k) if necessary, this together with the assumption of Case (iii) implies that

$$RS(n_k + 1) = \{b a_1^{n_k}, a_1^{n_k+1}, \ldots, a_j^{n_k+1}\}$$

for every k. The word $a_1^{n_k} b \notin RS(n_k + 1)$, so, as in Case (ii), the word $a_1^{n_k} b$ forces a transition word $w(1)$ (whose first symbol is b). But note that the only symbols that can follow $b a_1^{n_k}$ are also a_1 or b, which similarly implies that the word $b a_1^{n_k} b$ forces the same transition word $w(1)$. The rest of the argument from Case (ii) carries through with n_k replaced by $n_k + 1$. We obtain the same possible forms of x as in Case (ii). $\square$

We proceed by assuming the conclusion of Lemma 3.9. For each $p \in \{1, \ldots, j\}$ and $k \in \mathbb{N}$, define

$$\ell_{k,p} := \min\{\ell \geq n_k : ba_p^\ell b \in \mathcal{L}(X)\}$$

and set $\ell_{0,p} := 0$. Consider the following set of words:

$$E(k) := \{a_1^{\ell_{k,1}} w_{12}^{(k)} a_2^{\ell_{k,2}}, \ldots, a_j^{\ell_{k,j}} w_{j1}^{(k)} a_1^{\ell_{k,1}}\}.$$

It follows from Lemma 3.9 that, if $u \in E(k)$, then any suffix of u is right-special. Some of these suffixes, namely, the *constant* suffixes $a_2^{\ell_{k,2}}, \ldots, a_1^{\ell_{k,1}}$, are in B. But many of these suffixes are not constant and therefore not in B.

For example, let $E_1(k)$ denote the set of suffixes of $a_1^{\ell_{k,1}} w_{12}^{(k)} a_2^{\ell_{k,2}}$ that are longer than $\ell_{k,2}$ but no longer than $\ell_{k,2} + \ell_{k,1}$. Then each $u \in E_1(k)$ is right-special and not in B.

In general, for $1 \leq p \leq j$, let $E_p(k)$ denote the set of suffixes of $a_p^{\ell_{k,p}} w_{pq}^{(k)} a_q^{\ell_{k,q}}$ that are longer than $\ell_{k,q}$ but no longer than $\ell_{k,q} + \ell_{k,p}$, where $q = p + 1$ if $2 \leq p < j$ and $q = 1$ if $p = j$. Note that $\#E_p(k) = \ell_{k,p}$ and therefore

$$\sum_{p=1}^{j} \#E_p(k) = \ell_{k,1} + \cdots + \ell_{k,j}. \tag{3.1}$$

Also, each $E_p(k) \cap B = \varnothing$ and, for $r \neq p$, $E_p(k) \cap E_r(k) = \varnothing$.

PROPOSITION 3.10. *Under* Assumptions 3.7, *for each* $k \in \mathbb{N}$,

$$c_X(\ell_{k,1} + \cdots + \ell_{k,j}) \geq (j + 1)(\ell_{k,1} + \cdots + \ell_{k,j}) + (\ell_{k-1,1} + \cdots + \ell_{k-1,j}).$$

Proof. To simplify notation, let $L_k = \ell_{k,1} + \cdots + \ell_{k,j}$. We proceed by induction, where, since $L_0 = 0$, the base case is the assertion that $c_X(L_1) \geq (j + 1)L_1$. This assertion follows from Lemma 3.8 together with the following observations:

- for $1 \leq \ell \leq L_1 - 1$, there are j right-special words of length ℓ in B;
- there are L_1 (distinct) right-special words in $\bigcup_{p=1}^{j} E_p(1)$, none of which are in B.

Now assume that $c_X(L_{k-1}) \geq (j + 1)L_{k-1} + L_{k-2}$ and observe that, by Lemma 3.8 together with equation (3.1),

$$\begin{aligned} c_X(L_k) &\geq c_X(L_{k-1}) + \sum_{\ell=L_{k-1}}^{L_k - 1} \#RS(\ell) \\ &\geq [(j + 1)L_{k-1} + L_{k-2}] + [j(L_k - L_{k-1}) + L_k] \\ &\geq (j + 1)L_k + L_{k-1}. \end{aligned}$$

$\square$

Since $(\ell_{k-1,1} + \cdots + \ell_{k-1,j}) \to \infty$ as $k \to \infty$, it follows from Proposition 3.10 that $c_X(n) \not\preceq (j + 1)n$, which by our earlier discussion completes the proof of Theorem 1.1 in Cases (ii) and (iii).

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3.2.3. *Sharpness of Theorem 1.1 when $i = 0$.* Here we define a family of examples that demonstrates the sharpness of the bounds $c_X(n) \overline{\succ} (j + i + 1)n$ and $c_X(n) \succeq (j + i)n$. We will consider the $i = 0$ cases first, and then show how to modify the argument for $i > 0$. Let $g : \mathbb{N} \to \mathbb{N}$ be a given non-decreasing unbounded function.

Define a sequence $\omega = \omega_1\omega_2\omega_3 \cdots$ via the rule that $\omega_{2^m k + 2^{m-1}} = m$ for $m \geq 1$ and $k \geq 0$, so that $\omega = 1213121412131215 \cdots$. Then define a doubly-infinite sequence

$$x = j^\infty.\,(1^{n_1^1}2^{n_1^2} \cdots j^{n_1^j})(1^{n_2^1}2^{n_2^2} \cdots j^{n_2^j})(1^{n_1^1}2^{n_1^2} \cdots j^{n_1^j})(1^{n_3^1}2^{n_3^2} \cdots j^{n_3^j}) \cdots$$

for a doubly-indexed sequence n_k^p satisfying

$$n_1^1 << n_1^2 << \cdots << n_1^j << n_2^1 << n_2^2 \cdots << n_2^j << \cdots .$$

(The pattern of n_k^p within x is as follows: the superscript p is always the same as the letter being repeated, and the subscript k comes from ω, in that it is $\omega_1 = 1$ for the first j exponents, then $\omega_2 = 2$ for the next j, then $\omega_3 = 1$ for the next j, and so on.) Note that while ω is fixed, different sequences (n_k^p) give rise to different sequences x. In other words, x represents a family of examples parametrized by (n_k^p).

Regardless of choice of (n_k^p), the transitive subshift $X = \overline{\mathcal{O}(x)}$ has j minimal subsystems: $M_1 = \{1^\infty\}, \ldots, M_j = \{j^\infty\}$. Also, we can enumerate the right-special words in X as follows.

All words in the previously defined set B are again right-special, and any right-special word not in B must end with a word from the set

$$C = \{12^{n_k^2}, 23^{n_k^3}, \ldots, j1^{n_k^1} : k \in \mathbb{N}\}.$$

For $1 \leq p \leq j - 1$ and for any given k, the word

$$p^{n_k^p}(p + 1)^{n_k^{p+1}}$$

is *maximally right-special* in the sense that:

- any suffix of $p^{n_k^p}(p + 1)^{n_k^{p+1}}$ is right-special; and
- for any symbol s, the word $sp^{n_k^p}(p + 1)^{n_k^{p+1}}$ fails to be right-special.

Let $D = D(k)$ denote the set of words that are not in B and are suffixes of such a maximally right-special word $p^{n_k^p}(p + 1)^{n_k^{p+1}}$. Each n in an interval of the form $(n_k^{p+1}, n_k^{p+1} + n_k^p]$ corresponds to the length of a word in D. Moreover, we can ensure that these intervals are disjoint by requiring that

$$n_k^3 > n_k^2 + n_k^1\,, \quad n_k^4 > n_k^3 + n_k^2\,, \ldots,\ n_k^j > n_k^{j-1} + n_k^{j-2}.$$

Now consider right-special words ending in $j1^{n_k^1}$. In ω, the left-most occurrence of k is ω_{2^k-1}; any occurrence of a symbol $m \geq k$ in ω is directly preceded by the word $\omega_1 \cdots \omega_{2^{k-1}-1}$; and any occurrence of the word $\omega_1 \cdots \omega_{2^{k-1}-1}$ is directly followed by a symbol $m \geq k$. Moreover, any occurrence of the word $k\omega_1 \cdots \omega_{2^{k-1}-1}$ must be followed by a symbol $m > k$; and any occurrence of the word $m\omega_1 \cdots \omega_{2^{k-1}-1}$ for $m > k$ must be followed by k.

It follows from these observations about ω that, in x, there is a unique word $u = u(k)$ of length

$$L = L(k) = \sum_{i=1}^{j} (n^i_{\omega_1} + n^i_{\omega_2} + \cdots + n^i_{\omega_{2^{k-1}-1}}),$$

namely,

$$u := x_0 \cdots x_{L-1} = \underbrace{(1^{n^1_1} \cdots j^{n^j_1})(1^{n^1_2} \cdots j^{n^j_2}) \cdots (1^{n^1_1} \cdots j^{n^j_1})}_{2^k - 1 \text{ parenthetical blocks}}$$

that directly precedes any occurrence of $1^{n^1_k}$. It follows that $j^{n^j_k} u 1^{n^1_k}$ is maximally right-special.

Let $F = F(k)$ denote the set of words that are not in B and are suffixes of $j^{n^j_k} u 1^{n^1_k}$, and note that $F \cap D = \varnothing$. Then, for each $n \in (n^1_k, n^1_k + n^j_k + L(k)]$, there is exactly one word of length n in F and that word is right-special.

We have established the following.

PROPOSITION 3.11. *The set of right-special words in X is*

$$\bigcup_{n=1}^{\infty} RS(n) = B \cup \bigcup_{k=1}^{\infty} (D(k) \cup F(k)).$$

Moreover,

$$\#RS(n) \le \begin{cases} j & \text{if } n^j_k + n^1_k + L(k) < n \le n^1_{k+1} \text{ for some } k, \\ j+2 & \text{if } n^p_k < n \le n^p_k + n^{p-1}_k \text{ for some } k \text{ and } 2 \le p \le j, \\ j+1 & \text{otherwise.} \end{cases}$$

Set $n^j_0 = 0$. Since g is non-decreasing and unbounded, we can choose the sequence $n^1_1 < n^2_1 < \cdots < n^j_1 < n^1_2 < \cdots$ to grow fast enough so that for each $k \in \mathbb{N}$ and $p \in \{1, \ldots, j\}$, $g(n^p_k)$ is larger than the sum of all n^q_ℓ smaller than n^p_k. More specifically, choose (n^p_k) so that for all $k \ge 1$ and $p \in \{1, \ldots, j\}$,

$$g(n^p_k) > \sum_{\ell=1}^{k-1} \sum_{q=1}^{j} n^q_\ell + \sum_{q=1}^{p-1} n^q_k.$$

Note that the right-hand side above provides an upper bound on the number of $n \in [1, n^{p+1}_k)$ with $\#RS(n) = j + 2$. Lemma 3.8 then implies that for $n \in [n^p_k, n^{p+1}_k)$,

$$c_X(n) \le (j+1)n + g(n^p_k) \le (j+1)n + g(n).$$

Similarly, if $n \in [n^j_{k-1}, n^1_k)$ for some $k \ge 1$, then

$$c_X(n) \le (j+1)n + g(n^j_{k-1}) \le (j+1)n + g(n).$$

We have shown that $c_X(n) \le (j+1)n + g(n)$ for all n. Since g was arbitrary, this shows that the bound $c_X(n) \gtrsim (j+1)n$ is sharp.

To see that the bound $c_X(n) \succeq jn$ is sharp, we consider the complexity along the subsequence (n_k^1). If we choose (n_k^p) to grow fast enough, then, for all $k \geq 1$,

$$g(n_{k+1}^1) > (j+2)(n_k^j + n_k^1 + L(k)).$$

Then Lemma 3.8 and Proposition 3.11 imply that

$$\begin{aligned} c_X(n_{k+1}^1) &\leq c_X(n_k^j + n_k^1 + L(k)) + j(n_{k+1}^1 - (n_k^j + n_k^1 + L(k))) \\ &\leq (j+2)(n_k^j + n_k^1 + L(k)) + jn_{k+1}^1 - j(n_k^j + n_k^1 + L(k)) \\ &\leq g(n_{k+1}^1) + jn_{k+1}^1 - j(n_k^j + n_k^1 + L(k)). \end{aligned}$$

Since $c_X(n) \leq jn + g(n)$ along the sequence n_k^1 and g was arbitrary, the bound $c_X(n) \succeq jn$ is sharp.

3.2.4. *Sharpness of Theorem 1.1 when $i > 0$.* We now wish to show that the sharp examples constructed in §3.2.3 can be extended to the $i > 0$ case. For this, first consider a system $X \in S(j, 0)$ constructed using the form from §3.2.3, that is, X is the orbit closure of a sequence of the form

$$x = j^\infty.(1^{n_1^1} 2^{n_1^2} \cdots j^{n_1^j})(1^{n_2^1} 2^{n_2^2} \cdots j^{n_2^j})(1^{n_1^1} 2^{n_1^2} \cdots j^{n_1^j})(1^{n_3^1} 2^{n_3^2} \cdots j^{n_3^j}) \cdots .$$

Now let $1 \leq i \leq j$, and let $S_{j-i+1}, \ldots, S_j$ be arbitrary Sturmian subshifts with alphabets disjoint from each other and from $\{1, \ldots, j-i\}$. Our goal is to replace constant strings of symbols from $\{j-i+1, \ldots, j\}$ with sequences chosen from $S_{j-i+1}, \ldots, S_j$ to create $X' \in S(j, i)$ in such a way that the complexity is increased by exactly in. We begin with an elementary observation about minimal subshifts, which applies in particular to $S_{j-i+1}, \ldots, S_j$.

LEMMA 3.12. *Given a minimal subshift S and a word $w \in \mathcal{L}(S)$, there exist arbitrarily long words v with the property that $wvw \in \mathcal{L}(S)$.*

Proof. By minimality, every point in $x \in S$ contains w infinitely many times. Therefore, we can find two instances of w in x occurring at indices that are arbitrarily far apart. □

To stitch $S_{j-i+1}, \ldots, S_j$ into X, we need to impose a further assumption on the sequences $(n_m^i)_{m \in \mathbb{N}}$. By Lemma 3.12, we can recursively define $n_1^1, n_1^2, \ldots, n_1^j, n_2^1, n_2^2, \ldots, n_2^j, n_3^1, \ldots$ in such a way that, associated to each $p \in \{j-i+1, \ldots, j\}$ and $k \in \mathbb{N}$, there is a word $w_k^p \in \mathcal{L}_{n_k^p}(S_p)$, and every such w_k^p is both a prefix and a suffix of w_{k+1}^p. The proof of sharpness only required rapid growth of the sequence (n_k^p), and Lemma 3.12 ensures that we may recursively choose n_k^p with arbitrarily rapid growth such that the words w_k^p have the desired conditions. Since each w_k^j is a suffix of w_{k+1}^j, the sequence w_k^j has a left-infinite limit (as $k \to \infty$), which we denote by w_∞^j.

Now define

$$x' = w_\infty^j.\big(1^{n_1^1} \cdots (j-i)^{n_1^{j-i}} w_1^{j-i+1} \cdots w_1^j\big)\big(1^{n_2^1} \cdots (j-i)^{n_2^{j-i}} w_2^{j-i+1} \cdots w_2^j\big) \cdots ,$$

the sequence obtained by replacing each $p^{n_k^p}$ in x by w_k^p, and replacing j^∞ by w_∞^j. Then each S_p for $p \in \{j-i+1, \ldots, j\}$ is an infinite minimal subsystem of $X' := \overline{\mathcal{O}(x')}$, and each $\{p^\infty\}$ for $p \in \{1, \ldots, j-i\}$ is a finite minimal subsystem of X'. It is not hard to check that X' contains no other minimal subsystems and so $X' \in S(j, i)$. It remains to show that $c_{X'}(n) = c_X(n) + in$.

For this, consider the following 1-block factor map ϕ applied to X'. Since the alphabets of the S_p are disjoint, we may map any letter in the alphabet of S_p to p and leave other letters (for $p \in \{1, \ldots, j-1\}$) unchanged. The map ϕ induces a surjection from $\mathcal{L}_n(X')$ to $\mathcal{L}_n(X)$ for all n. We claim that every word in $\mathcal{L}(X)$ which is not constant has only a single ϕ-preimage. To see this, consider $w \in \mathcal{L}(X)$ containing multiple letters. Without loss of generality, we can extend w on the left and right so that w contains some $a^{n_m^a}$ as a prefix and $b^{n_m^b}$ as a suffix; if the extension has only one preimage, then of course w did as well. Then, by construction of x', the only subword of x' mapping to w under ϕ is obtained by replacing every maximal subword of the form $p^{n_k^p}$ in x by w_k^p for $p \in \{j-i+1, \ldots, j\}$. (The only possible ambiguity comes from $a^{n_m^a}$ and $b^{n_m^b}$, but recall that w_k^p is a prefix and a suffix of $w_{k'}^p$ for all $k' > k$, and so even if $a^{n_m^a}$ and/or $b^{n_m^b}$ were portions of longer runs of a's or b's in x, the corresponding word in x' still contains w_m^a and/or w_m^b at those locations if a and/or b are in $\{j-i+1, \ldots, j\}$.)

On the other hand, for $k \in \{j-i+1, \ldots, j\}$, any constant word of the form k^n has every word in $\mathcal{L}_n(S_k)$ as a preimage, since $S_k \subset X'$, and all words in $\mathcal{L}_n(S_k)$ map to k^n under ϕ. Since $c_{S_k}(n) = n+1$ for all n, this means that all such words have $n+1$ preimages under ϕ. Combining this yields $c_{X'}(n) = 1 \cdot (c_X(n) - i) + (n+1)i = c_X(n) + in$ for all n.

Now the proof from §3.2.3 provides examples of $X \in S(j, 0)$ demonstrating sharpness of the bounds

$$c_X(n) \not\succ (j+1)n \quad \text{and} \quad c_X(n) \succeq jn$$

in the $i = 0$ case. The procedure above yields, for any $0 \le i \le j$, $X' \in S(j, i)$ with $c_{X'}(n) = c_X(n) + in$, and so such X' demonstrate the sharpness of the more general bounds $c_X(n) \not\succ (j+i+1)n$ and $c_X(n) \succeq (j+i)n$.

3.3. *Proof of Theorem 3.4.* We sketch the proof of Theorem 3.4 here. Suppose that X is a transitive subshift with a recurrent transitive point x such that X properly contains an infinite minimal subshift M. Because $M \neq X$, there is an $r \ge 1$ such that $\mathcal{L}_r(M) \neq \mathcal{L}_r(X)$. Define a factor map π on X such that

$$\pi(z)_k = \begin{cases} 0 & \text{if } z_k \cdots z_{k+r-1} \in \mathcal{L}_r(M), \\ 1 & \text{otherwise.} \end{cases}$$

The image $\pi(x)$ is a recurrent transitive point for $\pi(X)$, and the subshift $\pi(X)$ contains a unique minimal subshift $\pi(M) = \{0^\infty\}$. Since $\pi(X) \neq \{0^\infty\}$, Theorem 3.3 gives $c_{\pi(X)}(n) \not\succ 1.5n$. Using an estimate as in Lemma 3.5 with $i = 1$ yields $c_X(n) \not\succ 2.5n$.

To see that the bound $c_X(n) \not\succ 2.5n$ is sharp, let $g : \mathbb{N} \to \mathbb{N}$ be a given non-decreasing unbounded function. In [10], the bound $c_{\pi(X)}(n) \not\succ 1.5n$ is shown to be sharp using

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examples of the form $\pi(X) = \overline{\mathcal{O}(y)}$, where

$$y = 0^{\infty} \,.\, 1\, 0^{n_1}\, 1\, 0^{n_2}\, 1\, 0^{n_1}\, 1\, 0^{n_3}\, 1\, 0^{n_1}\, 1\, 0^{n_2}\, 1\, 0^{n_1}\, 1\, 0^{n_4}\, 1 \cdots .$$

Here the sequence of subscripts on the exponents n_i is the sequence $\omega = 12131214\cdots$ used in §3.2.3; by choosing $n_1 << n_2 << n_3 << \cdots$ to grow sufficiently fast, it is shown in [10] that the inequality $c_{\pi(X)}(n) < 1.5n + g(n)$ can be achieved for all sufficiently large n. Now simply modify these examples using the technique (from §3.2.4) of filling in blocks 0^{n_i} with Sturmian words to obtain X that properly contains an infinite minimal subshift and satisfies $c_X(n) < 2.5n + g(n)$.

4. *General transitive systems*

Our main theorem in the recurrent case, Theorem 1.1, provides bounds in terms of i and j for subshifts in the sets $S(j, i)$. Our main theorem in the general transitive case, Theorem 1.2, offers similar bounds in terms of sets that we will refer to as $T(j, i)$. Here $T(j, i) \supset S(j, i)$ is the set of all transitive subshifts with $j \geq 1$ minimal subsystems, exactly i of which are infinite where $0 \leq i \leq j$.

4.1. *Two or fewer subsystems.*

We first tackle the cases where $j \leq 2$. When $j = 1$, the results of Theorems 3.1 and 3.2 yield the minimal complexity sequence of $c_X(n) = n + 1$ for X not periodic.

When $j = 2$, the orbit closure of the sequence

$$x = \ldots 0000.11111 \ldots$$

produces a transitive system in $T(2, 0)$ satisfying $c_X(n) = n + 1$ for all n.

THEOREM 4.1. *Let $X \in T(2, i)$, where $i > 0$. Then*

$$\liminf_{n\to\infty}(c_X(n) - (i + 1)n) > -\infty.$$

Moreover, this bound is optimal in that the $-\infty$ cannot be replaced by any integer.

Proof. Suppose that X contains two minimal subsystems M_1 and M_2. Then there is an $r > 0$ such that $\mathcal{L}_r(M_1) \cap \mathcal{L}_r(M_2) = \varnothing$. Define a factor map π on X such that

$$\pi(z)_k = \begin{cases} i & \text{if } z_k \cdots z_{k+r-1} \in \mathcal{L}_r(M_i) \text{ for } i = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $y = \pi(x)$. Then, for all $n \geq 1$, the words 1^n and 2^n occur in y. This means that y is not periodic, so, by the Morse–Hedlund theorem, $c_Y(n) \geq n + 1$ for all n. Since $i > 0$, we may assume without loss of generality that M_1 is infinite and so again by Morse–Hedlund, $c_{M_1}(n) \geq n + 1$ for all n. Since π is an r-block map, for $n \geq r$, the word 1^{n-r+1} has at least $c_{M_1}(n) \geq n + 1$ π-preimages, so we obtain

$$c_X(n) \geq c_{\pi(X)}(n - r + 1) + n > 2n - r + 1.$$

If M_2 is also infinite, then both 1^{n-r+1} and 2^{n-r+1} have at least $n+1$ preimages and all of those preimages are distinct. Therefore,

$$c_X(n) \geq c_{\pi(X)}(n-r+1) + 2n > 3n - r + 1.$$

We also claim that the $-\infty$ in Theorem 4.1 cannot be replaced by any integer. We will use the properties of Sturmian subshifts described in §2.2.

We first treat the $i = 1$ case: choose any $N > 3$ and define a Sturmian subshift $Z \subset \{0, 1\}^{\mathbb{Z}}$ created by β, where $(1/(N+1)) < \beta < (1/N)$. Then $10^k1 \in \mathcal{L}(Z)$ if and only if $k = N-1$ or N. Consider a right-infinite word s in Z, let

$$x = 0^{\infty}.s,$$

and let $X = \overline{\mathcal{O}(x)}$. Then, for $1 \leq n < N$, there is only one right-special word in $\mathcal{L}_n(X)$, namely, 0^n. Therefore, $c_X(N) = N+1$. Sturmian systems contain exactly one right-special word of length n for every $n \geq 1$. Therefore, when $n \geq N$, there are two right-special words in X: 0^n and a different one that is a subword of s. Lemma 3.8 then implies that $c_X(n) = 2n - N + 1$ for $n \geq N$, so $\liminf c_X(n) - 2n = -N + 1$. Since N could be arbitrarily large, there is no uniform lower bound in the $i = 1$ case of Theorem 4.1.

For the $i = 2$ case, consider two Sturmian subshifts $Z_1, Z_2 \subset \{0, 1\}^{\mathbb{Z}}$ created by distinct $\beta_1, \beta_2 \in (1/(N+1), 1/N)$ for any $N > 3$. Consider a left-infinite word $r \in Z_1$ ending in 1 and a right-infinite word $s \in Z_2$ beginning with 1. Let

$$x = r.s$$

and let $X = \overline{\mathcal{O}(x)}$.

There are three types of words in x: subwords of r, subwords of s, and words that contain the word 11. For $n \geq 2$, there are exactly $n-1$ subwords of x that contain 11. For $1 \leq n \leq N$, $\mathcal{L}_n(Z_1) = \mathcal{L}_n(Z_2)$; both of these equal the set of words of length n in $\{0, 1\}^n$ that contain at most one 1. Therefore, $c_X(n) = (n-1) + (n+1) = 2n$ for $1 \leq n \leq N$.

For $n > N$, there are at most three right-special words: a single word w_1 in $\mathcal{L}_n(Z_1)$ that can be extended in two ways in $\mathcal{L}(Z_1)$, a single word w_2 in $\mathcal{L}_n(Z_2)$ that can be extended in two ways in $\mathcal{L}(Z_2)$, and the n-letter suffix w_3 of r. Therefore, by Lemma 3.8, $c_X(n) = 3n - N$ for $n > N$ and so $\liminf c_X(n) - 3n = -N$, implying that there is no uniform lower bound in the $i = 2$ case of Theorem 4.1. □

4.2. *Three or more subsystems.* We now proceed with the proof of Theorem 1.2, which gives the bound $c_X(n) \succeq (j+i)n$ for $X \in T(j, i)$ with $j \geq 3$. For such an X, let $x \in X$ be a transitive point. Consider the same r-block factor map π as constructed in §3.2.1. Then $\pi(X) \in T(j, 0)$ for $j \geq 3$. Set $y = \pi(x)$ and note that y is a transitive point for $\pi(X)$. We also note that y cannot be eventually periodic in both directions, or else $\pi(X)$ would have at most two minimal subsystems, namely, the alpha- and omega-limit sets of y:

$$\alpha(y) = \bigcap_{N \geq 1} \overline{\{\sigma^{-n}(x) : n \geq N\}}$$

and

$$\omega(y) = \bigcap_{N \geq 1} \overline{\{\sigma^n(x) : n \geq N\}}.$$

We then assume without loss of generality that y is not eventually periodic to the right. (If y were eventually periodic to the right, then below we would consider left-special, as opposed to right-special, words and arrive at the same conclusion.)

For each $p \in \{1, \ldots, j\}$ and $n \in \mathbb{N}$, we claim that a_p^n is right-special. Indeed, a_p^{n+1} must occur in y for all p since $\{a_p^\infty\}$ is a minimal subsystem of $\pi(X)$. Since y is not eventually periodic to the right, the word $a_p^n b$ must also occur in y.

Therefore, $c_{\pi(X)}(n) \geq jn$ for all $n \geq 1$. By Lemma 3.8, we may establish that $c_{\pi(X)}(n) \succeq jn$ by finding an infinite set of n for which $\#RS_{\pi(X)}(n) \geq j+1$. Fix p so that $\{a_p^\infty\}$ is in the omega-limit set $\omega(y)$. Then there exist infinitely many n for which $ba_p^n b$ occurs in y. Recall that ba_p^{n+1} occurs in y for all $n \geq 1$, so $ba_p^n \in RS_{\pi(x)}(n+1)$, that is, $\#RS_{\pi(x)}(n+1) \geq j+1$, for infinitely many n.

We then have $c_{\pi(X)}(n) \succeq jn$. Finally, as in Lemma 3.5, for every p with $\pi^{-1}\{a_p^\infty\}$ infinite, the number of π-preimages of a_p^{n-r+1} is at least n, establishing that

$$c_X(n) \succeq (j+i)n.$$

Remark 4.2. The proof above also works if $X \in S(2, i)$ has a transitive point x such that $\pi(x)$ is not both eventually periodic to the right and eventually periodic to the left.

THEOREM 4.3. *Let $X \in T(j, i)$, where $j \geq 3$. Then the bound*

$$c_X(n) \overline{\succ} (j+i)n$$

holds and is sharp.

Proof. The bound itself follows immediately from Theorem 1.2. To see that the bound is sharp, let $g : \mathbb{N} \to \mathbb{N}$ be any non-decreasing unbounded function. Consider a point of the form

$$x = 0^\infty.1^{n_1}2^{n_2} \cdots (j-1)^{n_{j-1}}1^{n_j}2^{n_{j+1}} \cdots (j-1)^{n_{2j-2}} \cdots, \tag{4.1}$$

where $n_1 << n_2 << n_3 << \cdots$, and set $X = \overline{\mathcal{O}(x)}$.

For any $n \geq 1$, $0^n, 1^n, \ldots, (j-1)^n$ are all right-special in X. No word in X containing 01 is right-special, nor is any word that contains three distinct symbols. The only other right-special words are of the form

$$p^m(p+1)^{n_k} \quad \text{where } k \equiv p + 1 \bmod (j-1), 1 \leq m \leq n_{k-1}$$

or

$$(j-1)^m 1^{n_k} \quad \text{where } k \equiv 1 \bmod (j-1), 1 \leq m \leq n_{k-1}.$$

In other words, we have $\#RS(n) = j+1$ only for $n \in (n_k, n_k + n_{k-1}]$. Therefore, if $n \in [n_k, n_{k+1})$, the number of right-special words of length less than n which are not of the form a^n is at most $\sum_{i=1}^{k-1} n_i$. If the sequence (n_k) grows sufficiently fast, then $g(n_k) > \sum_{i=1}^{k-1} n_i$ and then Lemma 3.8 implies that

$$c_X(n) < jn + g(n)$$

for all n.

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For the $T(j, i)$ bound where $j \geq 3$ and $i > 0$, consider the family of examples obtained by replacing the blocks $0^\infty, 1^{n_1}, 2^{n_2}, \ldots, (i-1)^{n_{i-1}}, 1^{n_j}, 2^{n_{j+1}}, \ldots, (i-1)^{n_{j+i-2}}, \ldots$ with blocks from Sturmian sequences as in §3.2.4. Then the estimate in Lemma 3.5 gives the sharpness of the bound. □

5. *Generic measures*

We say that a point x in a subshift X is *generic* for a measure μ if the measures $\nu_n(x) := (1/n)\sum_{i=0}^{n-1} \delta_{\sigma^i x}$ converge to μ in the weak topology, that is, if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\sigma^i(x)) = \int f \, d\mu$$

for all continuous $f : X \to \mathbb{R}$. The pointwise ergodic theorem implies that whenever μ is ergodic, μ-almost every $x \in X$ is generic for μ. We say that a measure μ is a *generic measure* on X if there exists $x \in X$ that is generic for μ. All ergodic measures are clearly generic by the pointwise ergodic theorem, but generic measures are not necessarily ergodic; for instance, it is easily checked that

$$x = 0^\infty.011000111100000111111\ldots$$

is generic and not ergodic for $\mu = (\delta_{0^\infty} + \delta_{1^\infty})/2$.

Our goal is to provide bounds on the number of generic measures a transitive subshift can support. One of the more general results in this vein is the following theorem of Cyr and Kra, which does not assume transitivity of the subshift, but also does not control the number of atomic measures supported on periodic subshifts.

THEOREM 5.1. [3] *Suppose that X is a subshift and there exists $k \geq 3$ such that*

$$\limsup_{n\to\infty} \frac{c_X(n)}{n} < k.$$

If X has a generic measure μ and there is a generic point $z \in X$ for μ such that the subshift $Z = \overline{\mathcal{O}(z)}$ is not uniquely ergodic, then X has at most $k-2$ distinct, non-atomic, generic measures.

Combining the Cyr–Kra result above with others in this paper, we obtain the same conclusion under a different hypothesis.

THEOREM 5.2. *Suppose that X is a transitive subshift which is the orbit closure of a recurrent point and there exists $k \geq 3$ such that*

$$\limsup_{n\to\infty} \frac{c_X(n)}{n} < k.$$

Then X has at most $k-2$ distinct, non-atomic, generic measures.

Proof. Let X be a transitive subshift with a recurrent transitive point. Fix a generic measure μ for X and let z be a generic point for μ. If $\overline{\mathcal{O}(z)}$ is not uniquely ergodic, then we are done by Theorem 5.1. Thus, we may assume that $Z = \overline{\mathcal{O}(z)}$ is uniquely ergodic.

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Since Z is uniquely ergodic, it follows that every point $x \in Z$ is generic for μ (see, for instance, [14]).

Now assume for a contradiction that X has $k-1 \geq 2$ generic measures $\mu_1, \ldots, \mu_{k-1}$, with respective generic points $z_1, \ldots, z_{k-1}$. Then, by the preceding paragraph, we obtain $k-1$ disjoint subsystems $Z_1, \ldots, Z_{k-1}$ such that for all $x \in Z_i$, x is generic for μ_i. Each Z_i contains a distinct minimal subsystem. Therefore, by Theorem 1.1,

$$\limsup_{n\to\infty}(c_X(n) - kn) = \infty,$$

which is a contradiction. □

In general, there does not appear to be a simple way to combine our results with those of [3] in order to use an upper bound on $\limsup_{n\to\infty} c_X(n)/n$ to bound the number of generic (not necessarily non-atomic) measures in the transitive case. However, with some effort, we are able to show that in the case where $k = 3$, that is, when X is transitive with a recurrent transitive point and

$$\limsup_{n\to\infty} \frac{c_X(n)}{n} < 3,$$

then X is uniquely ergodic; this is our Theorem 1.3.

5.1. *Proof of Theorem 1.3.* If X is minimal, then this is Theorem 1.5 from [2]. If X is not minimal, then it cannot be the case that X contains two or more minimal subsystems, since our Theorem 1.1 would imply that $c_X(n) \succsim 3n$, which would contradict $\limsup_{n\to\infty}(c_X(n)/n) < 3$.

So, X contains a unique minimal subsystem M_1. By Theorem 1.5 from [2], there is a unique (ergodic) measure μ supported on M_1. Consider the factor map π (as defined in §3.2.1); then μ pushes forward under π to $\delta_{a_1^\infty}$ in $\pi(X)$ and it is the only measure which does so. So, if we are able to prove that $\pi(X)$ is uniquely ergodic, then its unique measure is $\delta_{a_1^\infty}$, which implies that μ is the unique measure on X.

Since the hypotheses of Theorem 1.3 are preserved under application of a factor map, we can assume without loss of generality that X has a unique minimal subsystem $\{a_1^\infty\}$. We can further reduce (by applying a 1-block factor map sending a_1 to 0 and all other letters to 1) to the case where $X \subseteq \{0, 1\}^{\mathbb{Z}}$ and that X has unique minimal subsystem $\{0^\infty\}$. Toward a contradiction, suppose that such an X has an ergodic $\mu \neq \delta_{0^\infty}$.

LEMMA 5.3. *Let $(n_k) \subseteq \mathbb{N}$ be a strictly increasing sequence and x a generic point for μ. Then, for all sufficiently large k, $x_{[0,\infty)}$ has the form*

$$x_{[0,\infty)} = w_0^{(k)}\, 0^{\geq n_k}\, w_1^{(k)}\, 0^{\geq n_k}\, w_2^{(k)}\, \cdots,$$

where every $w_i^{(k)}$ begins and ends with 1 and does not contain 0^{n_k}. Moreover, $(|w_0^{(k)}|/n_k) \to \infty$.

Proof. We first note that since x is generic for $\mu \neq \delta_{0^\infty}$, $x_{[0,\infty)}$ contains infinitely many 1's. Moreover, since 0^∞ is the only minimal subsystem of X, the omega-limit set of x

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must contain 0^∞, that is, $x_{[0,\infty)}$ contains 0^n for arbitrarily large n. Therefore, $x_{[0,\infty)}$ has the claimed form and it remains only to show that $(|w_0^{(k)}|/n_k) \to \infty$.

By genericity,

$$\begin{aligned}\mu([0]) &= \int \chi_{[0]}\, d\mu \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{[0]}(\sigma^i(x)) \\ &= \lim_{N\to\infty} \frac{\#\text{ zeros in } x_0\cdots x_{N-1}}{N}.\end{aligned}$$

Observe that $\mu([0]) < 1$ since we are assuming that $\mu \neq \delta_{0^\infty}$. It follows that $|w_0^{(k)}| \to \infty$. Therefore, if we let $r_k = |w_0^{(k)}|$ and z_k denote the number of zeros in $w_0^{(k)}$, then

$$\lim_{k\to\infty} \frac{z_k}{r_k} = \mu([0]) \quad\text{and}\quad \lim_{k\to\infty} \frac{z_k+n_k}{r_k+n_k} = \mu([0]).$$

Through some algebraic manipulation,

$$0 = \lim_{k\to\infty} \left(\frac{z_k}{r_k} - \frac{z_k+n_k}{r_k+n_k}\right) = \lim_{k\to\infty} \left(\left(\frac{n_k}{n_k+r_k}\right)\left(\frac{z_k-r_k}{r_k}\right)\right).$$

Now

$$\lim_{k\to\infty} \left(\frac{z_k-r_k}{r_k}\right) = \lim_{k\to\infty} \left(\frac{z_k}{r_k} - 1\right) = \mu([0]) - 1 \neq 0,$$

so $\lim_{k\to\infty}(n_k/(n_k+r_k)) = 0$, which implies that $(r_k/n_k) \to \infty$. □

LEMMA 5.4. *If x is a sequence which is not eventually periodic to the right, $n \in \mathbb{N}$, and w is a subword of x with length at least $c_X(n) + n$, then w contains an n-letter right-special word.*

Proof. Assume that x and w are as in the lemma. By assumption, w contains more than $c_X(n)$ subwords of length n and therefore one is repeated, call it u. If no n-letter subword of w is right-special, then for every n-letter subword of w, there is only one choice of a letter which may follow it in x. However, this would mean that the portion of x between the two occurrences of u would have to repeat indefinitely to the right, a contradiction to the assumption that x is not eventually periodic to the right. Therefore, w contains an n-letter right-special word. □

LEMMA 5.5. *There does not exist a strictly increasing sequence (n_k) such that $\#RS(n_k) = 1$ for all k.*

Proof. If such a sequence (n_k) exists, then, for every k, the only right-special word of length n_k is 0^{n_k}. Let x be generic for μ. We note that x cannot be eventually periodic to the right: if that periodic point were 0^∞, then by genericity $\mu = \delta_{0^\infty}$; and if it were not 0^∞, then X would contain a minimal subsystem other than $\{0^\infty\}$.

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Consider $w_0^{(k)}$ as defined in Lemma 5.3. We note that $w_0^{(k)}$ cannot contain a right-special word of length n_k, since by definition it would not be 0^{n_k}, which is a contradiction. Therefore, by Lemma 5.4, $|w_0^{(k)}| < c_X(n_k) + n_k$.

Next we observe that there exists $K \in \mathbb{N}$ such that $c_X(n) < Kn$ for all n. To see this, note that, since $\limsup_{n\to\infty}(c_X(n)/n) < 3$, there exists $N \in \mathbb{N}$ such that $c_X(n) < 3n$ for all $n \geq N$. And, for $1 \leq n \leq N-1$, we can simply let $K(n) \in \mathbb{N}$ be any value such that $K(n) > c_X(n)$. Then just define

$$K := \max\{3, K(1), \ldots, K(N-1)\}.$$

Then $|w_0^{(k)}| < c_X(n_k) + n_k < (K+1)n_k$ for all k, which contradicts the conclusion from Lemma 5.3 that $(|w_0^{(k)}|/n_k) \to \infty$. □

Following [2], we define a strictly increasing sequence (n_k) to be *logarithmically syndetic* if there exists M such that $(n_{k+1}/n_k) < M$ for all k.

LEMMA 5.6. *There is a logarithmically syndetic sequence (n_k) such that $\#RS(n_k) \leq 2$ for all k. Moreover, if $RS(n_k) = \{0^{n_k}, b_k\}$ for some k and $b_k \neq 0^{n_k}$, then b_k cannot be followed by three or more symbols.*

Proof. Let $\varepsilon = 3 - \limsup_{n\to\infty}(c_X(n)/n)$. Then there exists N such that $(c_X(n)/n) < 3 - \varepsilon/2$ for all $n \geq N$. We claim that any interval of the form $[i, (6/\varepsilon)\cdot i]$ where $i \geq N$ must contain a value of n such that $c(n+1) - c(n) \leq 2$. The statement of Lemma 5.6 then follows immediately by Lemma 3.8.

To prove the claim, let $i \geq N$ and suppose that $c_X(n+1) - c_X(n) \geq 3$ for all $n \in [i, j]$, where $j = (6/\varepsilon)\cdot i$. Then

$$\begin{aligned} c_X(j) &\geq c_X(i) + 3(j-i) \\ &> 3(j-i) \\ &= j(3-\varepsilon/2). \end{aligned}$$

But $(c_X(j)/j) < 3 - \varepsilon/2$ since $j \geq N$. □

Let (n_k) be a fixed logarithmically syndetic sequence as given by Lemma 5.6 and let $M > 1$ be a constant such that $(n_{k+1}/n_k) < M$. By Lemma 5.5, by deleting a finite number of terms in (n_k) if necessary, we can assume that $\#RS(n_k) = 2$ for all k. Let b_k denote the only right-special word of length n_k other than 0^{n_k}. By passing to a subsequence and increasing M to $10M$ if necessary, we can assume that

$$10 < \frac{n_{k+1}}{n_k} < M \quad \text{for all } k. \tag{5.1}$$

Since $(|w_0^{(k)}|/n_k) \to \infty$ and $\limsup_{k\to\infty}(c_X(n_k)/n_k) \leq \limsup_{n\to\infty}(c_X(n)/n) < 3$, we see that $|w_0^{(k)}| > c_X(n_k) + n_k$ for sufficiently large k. By Lemma 5.4, $w_0^{(k)}$ contains a right-special word of length n_k, which cannot be 0^{n_k}, so $w_0^{(k)}$ contains at least one occurrence of b_k.

Define a_k to be the shortest (possibly empty) word such that

$$w_0^{(k)} = a_k b_k r_k \tag{5.2}$$

for some word r_k and define d_k to be the shortest word such that

$$w_0^{(k)} = \ell_k b_k d_k \tag{5.3}$$

for some word ℓ_k. Therefore, we can write

$$w_0^{(k)} = a_k b_k u_k d_k$$

for some (possibly empty) word u_k. Observe that $|a_k|, |d_k| < c_X(n_k) + n_k < 4n_k$ for all sufficiently large k, since otherwise, by Lemma 5.4, a_k or d_k would contain b_k, contradicting minimality of length in their definitions. By definition, b_k is a suffix of $a_k b_k u_k$, but, since $(|w_0^{(k)}|/n_k) \to \infty$, for sufficiently large k we can assume that

$$\begin{aligned} |w_0^{(k)}| &> 13n_k \\ &= 4n_k + n_k + 4n_k + 4n_k \\ &> |a_k| + |b_k| + |d_k| + c_X(n_k) + n_k, \end{aligned}$$

which implies that $|u_k| > c_X(n_k) + n_k$. Therefore, by Lemma 5.4, u_k contains at least one copy of b_k.

Remark 5.7. Let $p > 0$ be an arbitrary positive integer. As above, for sufficiently large k, we have

$$\begin{aligned} |w_0^{(k)}| &> (13 + 4p)n_k \\ &> |a_k| + |b_k| + |d_k| + p(c_X(n_k) + n_k). \end{aligned}$$

Applying Lemma 5.4, we see that for sufficiently large k, u_k contains at least p copies of b_k.

Observe that b_k is a suffix of $a_k b_k u_k$, and b_k is the only word in $RS(n_k)$ that appears in $w_0^{(k)}$. Also, a_k and d_k are the *shortest* words that satisfy (5.2) and (5.3) and (by Lemma 5.6) b_k can only be followed by two distinct symbols. Therefore, $w_0^{(k)}$ must have the form

$$w_0^{(k)} = a_k b_k (e_k b_k)^{m_k} d_k, \tag{5.4}$$

where e_k is the shortest (possibly empty) word such that $w_0^{(k)} = a_k b_k e_k b_k q_k$ for some word q_k.

In the form (5.4), observe (by Lemma 5.4) that

$$|e_k| < c_X(n_k) + n_k < 4n_k \tag{5.5}$$

for all sufficiently large k since e_k does not contain any right-special word of length n_k. Also, $m_k \to \infty$ by Remark 5.7. Therefore, by deleting a finite number of terms from the beginning of (n_k) if necessary, we can assume that

$$m_k > M^2 + 1 \quad \text{for all } k \geq 1. \tag{5.6}$$

Observe that the first symbols of e_k and d_k must be different: if they were the same, then, since b_k is the only word in $RS(n_k)$ that appears in $w_0^{(k)}$, we would have $e_k = d_k$ and $x_{[0,\infty)} = a_k b_k (e_k b_k)^\infty$. As noted, for example, in the proof of Lemma 5.5, this cannot happen because x cannot be eventually periodic to the right.

Now observe that all suffixes of $(e_k b_k)^{m_k-1}$ of length at least n_k are right-special: this is due to the fact that any such suffix can be followed by either e_k or d_k and, as we just noted, the first symbols of e_k and d_k must be different. By construction, none of these suffixes have the form 0^n. Therefore, $\#RS(n) \geq 2$ for each $n \in (n_k, (m_k - 1) \cdot |e_k b_k|]$.

LEMMA 5.8. *For all k, both b_{k+1} and b_{k+2} are suffixes of $(e_k b_k)^{m_k-1}$.*

Proof. Using (5.1) and (5.6), observe that

$$\begin{aligned}|b_{k+1}| &= n_{k+1}\\ &< M \cdot n_k\\ &< (m_k - 1) \cdot n_k\\ &\leq (m_k - 1) \cdot |e_k b_k|.\end{aligned}$$

Therefore, $|b_{k+1}| \in (n_k, (m_k - 1) \cdot |e_k b_k|)$. And, b_{k+1} is the only right-special word of length n_{k+1} other than $0^{n_{k+1}}$, which implies that b_{k+1} is a suffix of $(e_k b_k)^{m_k-1}$.

Next, observe that by (5.1) and (5.6),

$$\begin{aligned}|b_{k+2}| &= n_{k+2}\\ &< M \cdot n_{k+1}\\ &< M^2 \cdot n_k\\ &< (m_k - 1) \cdot n_k\\ &\leq (m_k - 1) \cdot |e_k b_k|.\end{aligned}$$

Therefore, b_{k+2} is also a suffix of $(e_k b_k)^{m_k-1}$. □

LEMMA 5.9. *For all k, $e_k b_k e_k b_k$ is a suffix of b_{k+1}.*

Proof. By Lemma 5.8, b_{k+1} is a suffix of $(e_k b_k)^{m_k-1}$, so it is enough to show that $|b_{k+1}| \geq |e_k b_k e_k b_k|$. To see this, observe that, using (5.1) and (5.5),

$$\begin{aligned}|b_{k+1}| &= n_{k+1}\\ &> 10 n_k\\ &= 4n_k + n_k + 4n_k + n_k\\ &> |e_k| + |b_k| + |e_k| + |b_k|\\ &= |e_k b_k e_k b_k|.\end{aligned}$$

□

PROPOSITION 5.10. *For all k, $e_{k+1} b_{k+1} e_{k+1} b_{k+1}$ is a suffix of $(e_k b_k)^{m_k-1}$.*

Proof. If we apply Lemma 5.9 to $k + 1$ instead of k, we see that $e_{k+1} b_{k+1} e_{k+1} b_{k+1}$ is a suffix of b_{k+2}, which, by Lemma 5.8, is a suffix of $(e_k b_k)^{m_k-1}$. □

Finally, to arrive at a contradiction, we will use the words $e_k b_k e_k b_k$ to construct a minimal subsystem of X other than $\{0^\infty\}$. To do this, first define L_k to be the length of the longest block of zeros occurring in $e_k b_k e_k b_k$. Then L_k is also the length of the longest block of zeros in $(e_k b_k)^{m_k - 1}$ (assuming that $m_k \geq 3$, which is trivial since $m_k \to \infty$).

PROPOSITION 5.11. *(L_k) is a constant sequence.*

Proof. By Proposition 5.10, $e_{k+1} b_{k+1} e_{k+1} b_{k+1}$ is a suffix of $(e_k b_k)^{m_k - 1}$, so $L_{k+1} \leq L_k$. On the other hand, by Lemma 5.9, $e_k b_k e_k b_k$ is a suffix of b_{k+1}, so $L_k \leq L_{k+1}$. □

For each k, define $y^{(k)}$ to be any point in X of the form

$$y^{(k)} = \ell_k \; e_k \; b_k \; . \; e_k \; b_k \; r_k,$$

where ℓ_k and r_k are left- and right-infinite sequences. By compactness, a subsequence of $(y^{(k)})$ converges to a point $y \in X$. But by construction (and applying Proposition 5.11), $\overline{\mathcal{O}(y)}$ is a subsystem of X that is disjoint from $\{0^\infty\}$. Therefore, $\overline{\mathcal{O}(y)}$ contains a minimal subsystem other than $\{0^\infty\}$, which is a contradiction, completing the proof of Theorem 1.3. □

Remark 5.12. The assumption of recurrence was only used above to establish that X has a single minimal subsystem. Our results in this paper show that the only transitive X with more than one minimal subsystem and $\limsup_{n\to\infty}(c_X(n)/n) < 3$ are of the following types.

- X has two periodic minimal subsystems (e.g., the orbit closure of $x = \ldots 000.1111 \ldots$ or $x = 1^\infty.0^{n_1}10^{n_2}10^{n_3}1 \ldots$).
- $\pi(X)$ is eventually periodic in both directions and X has one infinite subsystem (e.g., $x = \ldots 000.s \ldots$, where s is a one-sided sequence from a Sturmian system).

We now conclude with a proof of Theorem 1.4, which establishes an upper bound of $g - 1$ on the number of generic measures when $X = \overline{\mathcal{O}(x)}$ for some x that is not eventually periodic in both directions and

$$\liminf(c_X(n) - gn) = -\infty.$$

5.2. *Proof of Theorem 1.4.* Assume that X is as above and x is not eventually periodic to the right. Since $\liminf(c_X(n) - gn) = -\infty$, Lemma 3.8 implies that the number of n-letter right-special words is strictly less than g for infinitely many n. Therefore, there exist $C \leq g - 1$ and a strictly increasing sequence (n_k) such that there are exactly C right-special words of length n_k; call them $b_k^{(i)}$ for $1 \leq i \leq C$. We may further assume that

$$c_X(n_k) < 2gn_k$$

for every k by considering values of n where $c_X(n) - gn$ is smaller than all previous values $c_X(i) - gi$ for $i < n$; see the discussion following Theorem 2.2 in [2] for details. Therefore, by Lemma 5.4, every word of length $(2g + 1)n_k$ contains at least one word $b_k^{(i)}$.

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For each k and i, choose a point $x_k^{(i)}$ with the word $b_k^{(i)}$ appearing in coordinates 0 through $n_k - 1$, and define the measure $\nu_k^{(i)} := \nu_{n_k}^{x_k^{(i)}}$ as discussed at the start of §5, that is,

$$\nu_k^{(i)} = \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \delta_{\sigma^j x_k^{(i)}}.$$

By compactness, we can pass to a subsequence and assume without loss of generality that for all i, $(\nu_k^{(i)})_k$ converges to a limit $\nu^{(i)}$. We claim that every generic measure is some $\nu^{(i)}$. For this, assume that μ is an arbitrary generic measure and let $x \in X$ be generic for μ.

For each k, the word $x_0 \cdots x_{(2g+1)n_k - 1}$ contains some $b_k^{(i_k)}$ by Lemma 5.4 and, by again passing to a subsequence, we can assume that i_k is always equal to some fixed i. For each k, define $m_k \le 2gn_k$ so that $x_{m_k} \cdots x_{m_k + n_k - 1} = b_k^{(i)}$.

Since x is generic for μ, both $\nu_{m_k}^x$ and $\nu_{m_k+n_k-1}^x$ converge to μ in the weak topology. This topology is induced by the metric

$$d(\mu, \nu) := \sum_{n \in \mathbb{N}} 2^{-n} |\mu([w_n]) - \nu([w_n])|,$$

where $\{w_n\}$ is an arbitrary enumeration of the set of finite words on $\{0, 1\}$. Then, for any $\epsilon > 0$, we may choose K so that for all $k > K$, $d(\nu_{m_k}^x, \mu), d(\nu_{m_k+n_k}^x, \mu) < \epsilon$. This implies that

$$d(m_k \nu_{m_k}^x, m_k \mu) < m_k \epsilon \quad \text{and} \quad d((m_k + n_k)\nu_{m_k+n_k}^x, (m_k + n_k)\mu) < (m_k + n_k)\epsilon,$$

which together imply that

$$d((m_k + n_k)\nu_{m_k+n_k}^x - m_k \nu_{m_k}^x, n_k \mu) < (2m_k + n_k)\epsilon.$$

Therefore,

$$d(\nu_{n_k}^{\sigma^{m_k} x}, \mu) < \frac{2m_k + n_k}{n_k} \epsilon \le (4g + 1)\epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\nu_{n_k}^{\sigma^{m_k} x} \to \mu$. Recall that $\sigma^{m_k} x$ begins with $b_k^{(i)}$ and so $\nu_{n_k}^{\sigma^{m_k} x} \to \nu^{(i)}$ by definition of $\nu^{(i)}$. Therefore, $\mu = \nu^{(i)}$ and, since μ was an arbitrary generic measure, every generic measure is one of the $\nu^{(i)}$. Since there are $C \le g - 1$ such measures, the proof is complete. □

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