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Hypoelliptic Bi-Invariant Laplacians on Infinite Dimensional Compact Groups

A. Bendikov and L. Saloff-Coste

Abstract. On a compact connected group G, consider the infinitesimal generator $-L$ of a central symmetric Gaussian convolution semigroup $(\mu_t)_{t>0}$. Using appropriate notions of distribution and smooth function spaces, we prove that L is hypoelliptic if and only if $(\mu_t)_{t>0}$ is absolutely continuous with respect to Haar measure and admits a continuous density $x \mapsto \mu_t(x)$, $t > 0$, such that $\lim_{t\to 0} t \log \mu_t(e) = 0$. In particular, this condition holds if and only if any Borel measure u which is solution of $Lu = 0$ in an open set Ω can be represented by a continuous function in Ω. Examples are discussed.

1 Introduction

Let G be a compact connected group equipped with its normalized Haar measure ν. Let $(\mu_t)_{t>0}$ be a symmetric Gaussian convolution semigroup of measures and $-L$ be its infinitesimal generator. The operator L can be interpreted as a second order differential operator on G and we call such operator L a sub-Laplacian on G (sub-Laplacians can be characterized in several different ways, see [12]). This paper is concerned with the hypoellipticity of L under the additional assumption that L is bi-invariant (this is equivalent to saying that $(\mu_t)_{t>0}$ is central). The notion of hypoellipticity must be carefully defined since several interpretations are possible in this setting. The aim of this paper and the companion paper [12] is to introduce and explore some of these possibilities. In finite dimension, there are (at least) two well established theories of hypoellipticity, namely, $\mathcal{C}^\infty$-hypoellipticity and analytic-hypoellipticity. In the present infinite dimensional setting, there are many possible choices to measure smoothness and it seems important from a practical viewpoint to allow for as much flexibility as possible, as one may encounter many different types of smooth functions.

Denote by $\mathcal{C}(G)$ the space of all continuous functions on G equipped with the uniform topology. Let $\mathcal{M}(G)$ be the space of all Borel signed measures on G, viewed as the dual of $\mathcal{C}(G)$ and equipped with the total variation norm $\|\mu\| = |\mu|(G)$, where $\mu = \mu^+ - \mu^-$ is the usual decomposition of μ in positive and negative parts and $|\mu| = \mu^+ + \mu^-$.

Recall that G is the projective limit of compact connected Lie groups [25, 27]. Denote by $\mathcal{B}(G)$ the set of all smooth cylindric functions on G, *i.e.,* all Bruhat test functions. These are smooth functions on Lie quotients of G, lifted to G. We equip

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$\mathcal{B}(G)$ with its natural topology [21, Définition 1] and let $\mathcal{B}'(G)$ be the topological dual of $\mathcal{B}(G)$, *i.e.,* the space of Bruhat distributions on G. Given an open set Ω, let $\mathcal{C}(\Omega)$ be the set of all continuous functions in Ω, and let $\mathcal{B}_0(\Omega)$ be the set of all smooth cylindric functions with support in Ω.

Let $\mathcal{A}$ be a vector space of elements of $\mathcal{B}'(G)$. The vector space $\mathcal{A}$ will come equipped with its own topology, not weaker than that of $\mathcal{B}'(G)$. Assume that $\mathcal{A}$ is stable by multiplication by elements of $\mathcal{B}(G)$. We then say that $\mathcal{A}$ is a *space of distributions* on G. One typical construction of a space of distributions is as the dual $\mathcal{T}'$ of a *test function space* where a test function space is a topological space $\mathcal{T}$ such that $\mathcal{B}(G) \subset \mathcal{T} \subset \mathcal{C}(G)$ and $\mathcal{B}(G)$ acts continuously on $\mathcal{T}$ by pointwise multiplication. The space $\mathcal{T}'$ is equipped with the strong dual topology, *i.e.,* the topology of uniform convergence on bounded sets, see [20, Chapitre III]. Of course, we have $\mathcal{M}(G) \subset \mathcal{T}' \subset \mathcal{B}'(G)$ and the topology of $\mathcal{T}'$ is not weaker than that of $\mathcal{B}'(G)$. Moreover, $\mathcal{B}(G)$ acts by multiplication on $\mathcal{T}'$. Thus $\mathcal{T}'$ is a space of distributions as defined above.

Definition 1.1 Let P be a left invariant differential operator on G of finite order (see definitions in Section 2.1). Let $\mathcal{A}$ be a fixed space of distributions. Let $\mathcal{S}$ be a space of continuous functions. We say that P is $\mathcal{A}$-$\mathcal{S}$-hypoelliptic if, for any $U \in \mathcal{A}$ and $F \in \mathcal{B}'(G)$ such that

$$PU = F \text{ in } \mathcal{B}'(G),$$

and for any open set Ω such that

$$\forall\, \phi \in \mathcal{B}_0(\Omega), \quad \phi F \in \mathcal{S},$$

we have

$$\forall\, \phi \in \mathcal{B}_0(\Omega), \quad \phi U \in \mathcal{S}.$$

The larger $\mathcal{A}$ is, the stronger $\mathcal{A}$-$\mathcal{S}$-hypoellipticity is for any fixed $\mathcal{S}$. Varying $\mathcal{S}$ leads to possibly different, *a priori* not comparable, notions. Let us illustrate this definition by some examples:

Example 1.2 Let G be a compact Lie group. Take $\mathcal{T} = \mathcal{S} = \mathcal{B}(G) = \mathcal{C}^\infty(G)$, $\mathcal{A} = \mathcal{T}'$. The definition above reduces to the usual notion of $\mathcal{C}^\infty$-hypoellipticity. In this case, if $P = \sum_1^k X_i^2$ for some left invariant vector fields X_i, then P is hypoelliptic if and only if $\{X_1, \dots, X_k\}$ generates the full Lie algebra of G (Hörmander's theorem [28]).

Example 1.3 Let $G = \mathbb{T}^\infty = (\mathbb{R}/2\pi\mathbb{Z})^\infty$. Take $\mathcal{T} = \mathcal{C}(G)$, $\mathcal{S} = \mathcal{C}^\infty$ where $\mathcal{C}^\infty$ is the space of continuous functions having continuous partial derivatives of all orders in the canonical directions of the fixed product structure of $\mathbb{T}^\infty$. In this case, $\mathcal{A} = \mathcal{T}' = \mathcal{M}(\mathbb{T}^\infty)$. In [3, §5.3], the corresponding hypoellipticity is studied for operators of the form $L = -\sum_i a_i \partial_i^2$. There it is shown that $\mathcal{M}$-$\mathcal{C}^\infty$-hypoellipticity holds if and only if $N(s) = \#\{i : a_i \le s\}$ satisfies $N(s) = o(s)$ as s tends to infinity.

Remarks

(i) Note that we are not assuming that $PU = F$ in $\mathcal{A}$ but only in $\mathcal{B}'(G)$. This allows us, for instance, to consider $\mathcal{M}(G)$-$\mathcal{C}(G)$-hypoellipticity although, of course, for a measure U, PU is not, in general, a measure. That is, we do not need the hypothesis that the distribution space $\mathcal{A}$ is stable by P.

(ii) The notion of hypoellipticity is a local property of the operator P. In the definition above, we have chosen to localize by multiplication by functions in $\mathcal{B}(G)$. This makes sense even if the space $\mathcal{S}$ is not stable by multiplication by such functions. However, if $\mathcal{S}$ is not stable by multiplication by $\mathcal{B}(G)$, it might be extremely difficult to prove hypoellipticity and also to apply the result in specific examples. We will see in Section 5 that it is sometimes possible and preferable to use a different localization procedure.

Definition 1.4 Let $(\mu_t)_{t>0}$ be a Gaussian semigroup on G.

- We say that $(\mu_t)_{t>0}$ has property (AC) if, for all $t > 0$, μ_t is absolutely continuous with respect to the Haar measure ν on G.
- We say that $(\mu_t)_{t>0}$ has property (CK) if (AC) holds and, for all $t > 0$, μ_t admits a continuous density $x \mapsto \mu_t(x)$.
- We say that $(\mu_t)_{t>0}$ has property (CK$*$) if (CK) holds and

$$\lim_{t\to 0} t \log \mu_t(e) = 0.$$

In the companion paper [12], we consider a number of different possible choices for $\mathcal{A}$ and prove the following theorem.

Theorem 1.5 ([12, Theorem 1.3]) *Let G be a compact connected group. Let $-L$ be the infinitesimal generator of a symmetric Gaussian semigroup $(\mu_t)_{t>0}$. Let $\mathcal{S}$ be a space of continuous functions whose topology is not weaker than the uniform topology.*

(i) *If G is not a Lie group, L is not $\mathcal{B}'(G)$-$\mathcal{S}$-hypoelliptic.*

(ii) *If the operator L is $L^\infty(G)$-$\mathcal{S}$-hypoelliptic, then the group G is locally connected and metrizable.*

(iii) *The operator L is $L^\infty(G)$-$\mathcal{C}(G)$-hypoelliptic if and only if $(\mu_t)_{t>0}$ satisfies* (AC).

Assume in addition that $(\mu_t)_{t>0}$ is central, i.e., *L is bi-invariant.*

(iv) *Fix $1 \le p < +\infty$. If L is $L^p(G)$-$\mathcal{C}(G)$-hypoelliptic, then $(\mu_t)_{t>0}$ satisfies* (CK$*$).

The purpose of the present paper is to complete this picture in the case of bi-invariant Laplacians and prove that property (CK$*$) implies some rather strong form of hypoellipticity. More precisely, if (CK$*$) is satisfied, there exists a space of test functions $\mathcal{T}_L$ such that L is $\mathcal{T}'_L$-$\mathcal{S}$-hypoelliptic for several natural choices of smooth function spaces $\mathcal{S}$ including $\mathcal{T}_L$ itself and $\mathcal{C}(G)$. As we shall see, the space $\mathcal{T}_L$ has a very explicit description. Functions in $\mathcal{T}_L$ have infinitely many derivatives in certain directions and $\mathcal{T}'_L$ contains many distributions that are not Borel measures. Let us state a simple corollary of the main result of this paper and Theorem 1.5(iv) which does not require further notation.

Theorem 1.6 *Let G be a compact connected group. Let $-L$ be the infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t)_{t>0}$. Then L is $\mathcal{M}(G)$-$\mathcal{C}(G)$-hypoelliptic if and only if* (CK$*$) *is satisfied.*

Together with the companion paper [12], the present work shows that for the bi-invariant operators considered here, many reasonable types of hypoellipticity are equivalent and hold if and only if (CK$*$) is satisfied. That property (CK$*$) is an essential property in this setting is already apparent in previous works by the authors [3, 6–8]. For instance, it is proved in [8] that the sheaf of all continuous harmonic functions associated to L is a Brelot sheaf (*i.e.,* satisfies Harnack's inequality) if and only if the corresponding central symmetric Gaussian semigroup $(\mu_t)_{t>0}$ satisfies (CK$*$). At this point the reader may wonder whether there indeed exists any central symmetric Gaussian semigroup having property (CK$*$) in infinite dimension! The first such examples where constructed in [2, 16] on $\mathbb{T}^\infty$. By [8], any compact connected locally connected metrizable group carries a host of such Gaussian semigroups although this existence result is anything but obvious. On the other hand, if G is either not metrizable or not locally connected, condition (CK$*$) (or even the much weaker condition (AC)) is never satisfied, see [9, 26].

Several approaches to hypoellipticity are available for second order sub-elliptic differential operators in finite dimension. Most seem very difficult to adapt in the present setting. We will follow the line of reasoning developed by Kusuoka and Stroock [30, Section 8] for second order differential operators in $\mathbb{R}^n$. The tools used in Kusuoka–Stroock's approach are: (1) a reasonable theory of distributions and smooth functions spaces (readily available in finite dimension) and (2) Gaussian estimates for the time and space derivatives of the heat kernel.

In the present setting, both (1) and (2) need to be developed before Kusuoka–Stroock's approach can be brought to bear. Various spaces of smooth functions are introduced in [10, 12, 15]. The crucial spaces of distributions are introduced and studied in [15]. Gaussian estimates for the derivatives of the density $(t, x) \mapsto \mu_t(x)$ of $(\mu_t)_{t>0}$ are obtained in [10] under the hypothesis that $(\mu_t)_{t>0}$ is central and satisfies (CK$*$). In fact, the main purpose of [10, 15] is to develop the material needed for the proofs of the main results of the present paper, in particular Theorem 1.6. The authors are grateful to D. Stroock for asking whether hypoellipticity could be studied by the method of [30, §8] in the present infinite dimensional setting.

2 Background and Notation

2.1 Projective Structure

The following setup and notation will be in force throughout this article. Let G be a connected compact group with neutral element e. Assume further that G is locally connected and metrizable. By Theorem 1.5(ii), this hypothesis is no loss of generality when studying hypoellipticity. Such a group contains a decreasing family of compact normal subgroups $K_\alpha, \alpha \in \aleph$ such that $\bigcap_{\alpha\in\aleph} K_\alpha = \{e\}$ and, for each α, $G_\alpha = G/K_\alpha$ is a Lie group. As G is metrizable, $\aleph$ is either finite (if G is a Lie group) or countable. Consider the projection maps $\pi_\alpha \colon G \to G_\alpha$ and $\pi_{\alpha,\beta} \colon G_\beta \to G_\alpha$, $\beta \geq \alpha$. Then G is

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the projective limit of the projective system $(G_\alpha, \pi_{\alpha,\beta})_{\beta\geq\alpha}$.

For a compact Lie group N denote by $\mathcal{C}^\infty(N)$ the set of all smooth functions on N. For any compact connected group G, set

$$\mathcal{B}(G) = \left\{ f\colon G \to \mathbb{R} \;\middle|\; f = \phi \circ \pi_\alpha \text{ for some } \alpha \in \aleph \text{ and } \phi \in \mathcal{C}^\infty(G_\alpha) \right\}. \tag{2.1}$$

The space $\mathcal{B}(G)$ is the space of Bruhat test functions introduced in [21]. The space $\mathcal{B}(G)$ is the inductive limit of the topological vector spaces $\mathcal{B}(G_\alpha)$ [21, p. 46]. By [21, Lemme 1], $\mathcal{B}(G)$ is independent of the choice of the family $(K_\alpha)_{\alpha\in\aleph}$.

The Lie algebra $\mathfrak{G}$ of G is defined to be the projective limit of the Lie algebras $\mathfrak{G}_\alpha$ of the groups G_α equipped with the projection maps $d\pi_{\alpha,\beta}$. Following [18, 21], we define the notion of projective family and projective basis.

Definition 2.1 Given a descending family $(K_\alpha)_{\alpha\in\aleph}$ as above, we say that a family $(Y_i)_{i\in I}$ of elements of the projective Lie algebra $\mathfrak{G}$ is:

- a projective family (w.r.t. $(K_\alpha)_\aleph$) if for each $\alpha \in \aleph$ there is a finite subset $I_\alpha \subset I$ such that $d\pi_\alpha(Y_i) = 0$ if $i \notin I_\alpha$;
- a projective basis if for each $\alpha \in \aleph$ there is a finite subset $I_\alpha \subset I$ such that $d\pi_\alpha(Y_i) = 0$ if $i \notin I_\alpha$ and $(d\pi_\alpha(Y_i))_{i\in I_\alpha}$ is a basis of the Lie algebra $\mathfrak{G}_\alpha$.

By [18], $\mathfrak{G}$ does admit a projective basis. If $(Y_i)_{i\in I}$ is a projective basis, we can identify $\mathfrak{G}$ with $\mathbb{R}^I$ as topological vector space. For any $Z \in \mathfrak{G}$, there exists a unique $a = (a_i)_{i\in I}$ such that for any $\alpha \in \aleph$, $d\pi_\alpha(Z) = \sum_{i\in I} a_i d\pi_\alpha(Y_i)$ and convergence in $\mathfrak{G}$ is equivalent to convergence coordinate by coordinate. Since the group G is assumed to be metrizable, projective families have at most a countable number of elements. Given a projective family $Y = (Y_i)_I \subset \mathfrak{G}$, we set

$$Y^\ell = Y_{\ell_1} \cdots Y_{\ell_k}$$

for $\ell = (\ell_1, \dots, \ell_k) \in I^k$. By definition, in any given projective basis $Y = (Y_i)_I$, a homogeneous left invariant differential operator of degree k on G is a sum (possibly infinite)

$$P = \sum_{\ell\in I^k} a_\ell Y^\ell, \quad a_\ell \in \mathbb{C}.$$

One easily checks that this notion does not depend on Y. Such a P can be interpreted as a linear operator from $\mathcal{B}(G)$ to $\mathcal{B}(G)$. Indeed, if $f = \phi \circ \pi_\alpha \in \mathcal{B}(G)$, we have

$$Pf(x) = \sum_{\ell\in I^k} a_\ell Y^\ell f(x) = \sum_{(\ell_1,\ell_2,\dots,\ell_k)\in I_\alpha^k} a_\ell \left[d\pi_\alpha(Y_{\ell_1}) d\pi_\alpha(Y_{\ell_2}) \cdots d\pi_\alpha(Y_{\ell_k})\phi \right] (\pi_\alpha(x))$$

where the right-hand side is a finite sum since I_α is finite for each $\alpha \in \aleph$. The formal adjoint of P is the homogeneous left invariant differential operator P^* of the same degree k defined by

$$P^* = (-1)^k \sum_{\ell\in I^k} \overline{a_\ell} Y^{\check{\ell}}$$

where $\check{\ell} = (\ell_k, \dots, \ell_1)$ if $\ell = (\ell_1, \dots, \ell_k)$. A differential operator P of finite order k is a finite sum of homogeneous differential operators of degree at most k and its formal adjoint P^* is defined by linearity.

2.2 Gaussian Semigroups and Sums of Squares

Given a Borel measure μ, define $\check{\mu}$ by $\check{\mu}(A) = \mu(A^{-1})$ for any Borel set A. We say that μ is central if $\mu(a^{-1}Aa) = \mu(A)$ for any $a \in G$ and any Borel subset $A \subset G$. Recall that the convolution of two functions $u, v \in \mathcal{B}(G)$ is defined by

$$u * v(x) = \int_G u(y)v(y^{-1}x)\, d\nu(y) = \int_G u(xy^{-1})v(y)\, d\nu(y).$$

Accordingly, for $f \in \mathcal{B}(G)$, define the left and right convolution by a measure μ as

$$\mu * f(x) = \int_G f(y^{-1}x)\, d\mu(y), \quad f * \mu(x) = \int f(xy^{-1})\, d\mu(y).$$

If μ is central, *i.e.*, $\mu(a^{-1}Ba) = \mu(B)$ for any Borel set B and any $a \in G$, then $\mu * f = f * \mu$.

A convolution semigroup $(\mu_t)_{t>0}$ is a family of probability measures such that $\mu_t * \mu_s = \mu_{t+s}$ and $\mu_t \to \delta_e$ weakly as $t \to 0$. A convolution semigroup $(\mu_t)_{t>0}$ is symmetric if $\check{\mu}_t = \mu_t$ for all $t > 0$. It is central if μ_t is central for all $t > 0$. It is *Gaussian* if

$$t^{-1}\mu_t(G \setminus V) \to 0$$

as $t \to 0$ for any neighborhood V of the identity $e \in G$. For background, see [13, 14, 26].

Given a convolution semigroup $(\mu_t)_{t>0}$, define the associated Markov semigroup $(H_t)_{t>0}$ acting on continuous functions by

$$H_t f(x) = \int_G f(xy)\, d\mu_t(y). \tag{2.2}$$

Thus $H_t f = f * \check{\mu}_t$. As $(H_t)_{t>0}$ is a C_0-semigroup on $\mathcal{C}(G)$, it has an infinitesimal generator $-L$. By construction, L is left invariant and $H_t = e^{-tL}$. We call $-L$ the infinitesimal generator of $(\mu_t)_{t>0}$. This terminology may be a little misleading when $(\mu_t)_{t>0}$ is not symmetric. Indeed, $-L$ has a natural extension to L^2 (the infinitesimal generator of the semigroup $(H_t)_{t>0}$ extended to L^2). Abusing notation, call $-L^*$ the adjoint of this extension. Assume that for all $t > 0$, μ_t has a continuous density $x \mapsto \mu_t(x)$ with respect to Haar measure. Then this density belongs to the domain of L^* and satisfies $\partial_t \mu_t = -L^*\mu_t$, whereas

$$\partial_t \check{\mu}_t = -L\check{\mu}_t. \tag{2.3}$$

This clearly follows from (2.2).

Using a celebrated theorem of Hunt [29] and the projective structure, Heyer and Born [19, 26] obtained a general Lévy–Khintchin formula for L. In particular, in the case of a Gaussian semigroup, they give the following description of L as a differential operator. Given a (finite or) countable set I, let $\mathbb{R}^{(I)}$ be the set of all $z = (z_i) \in \mathbb{R}^I$ with finitely many non-zero entries.

Theorem 2.2 *Given a projective basis $(Y_i)_{i\in I}$, the infinitesimal generators of Gaussian convolution semigroups on G are exactly the second order left invariant differential operators*

$$L = -\sum_{i,j\in I} a_{i,j} Y_i Y_j + \sum_{i\in I} b_i Y_i$$

where $b = (b_i) \in \mathbb{R}^I$ and $A = (a_{i,j})_{I\times I}$ is a real symmetric non-negative matrix in the sense that $a_{i,j} = a_{j,i} \in \mathbb{R}$ and $\forall\, \xi \in \mathbb{R}^{(I)}$, $\sum a_{i,j}\xi_i\xi_j \geq 0$. Moreover, the Gaussian semigroup is symmetric if and only if $b = 0$.

This theorem describes the form of the infinitesimal generator of any Gaussian semigroup in a fixed projective basis. Based on this description, it is easy to show that if $-L$ is the infinitesimal generator of a symmetric Gaussian semigroup $(\mu_t)_{t>0}$, then there are many projective families $X = (X_i)_{i\in I}$ which are adapted to L in the sense that

$$\forall\, f \in \mathcal{B}(G), \quad Lf = -\sum_{i\in I} X_i^2 f.$$

See [7, 10, 11] for details. What such projective families have in common is that they span a certain Hilbert space $\mathcal{H}(L)$ contained in $\mathfrak{G}$ and canonically attached to L. More precisely, define the field operator Γ to be the symmetric bilinear form

$$\Gamma(f,g) = \frac{1}{2}\big(-L(fg) + fLg + gLf\big), \quad f,g \in \mathcal{B}(G). \tag{2.4}$$

Definition 2.3 Given the generator $-L$ of a symmetric Gaussian semigroup on G, let $\mathcal{H}(L)$ be the vector space

$$\mathcal{H}(L) = \{Z \in \mathfrak{G} : \exists\, c(Z),\ \forall\, f \in \mathcal{B}(G),\ \ |Zf(e)|^2 \leq c(Z)\Gamma(f,f)(e)\}$$

equipped with the norm

$$\|Z\|_L = \sup_{\substack{f\in\mathcal{B}(G)\\ \Gamma(f,f)(e)\leq 1}} \{|Zf(e)|\}.$$

It is proved in [10, 15] that the space $\mathcal{H}(L)$ equipped with the norm $\|Z\|_L$ is a Hilbert space. In fact, let $X = (X_i)_{i\in I}$ be a projective family extracted from a projective basis in $\mathfrak{G}$ such that $L = -\sum_{i\in I} X_i^2$. Then $\Gamma(f,g) = \sum_I (X_i f)(X_i g)$ and X is an orthonormal basis of $\mathcal{H}(L)$, that is,

$$\mathcal{H}(L) = \Big\{ Z = \sum_I \zeta_i X_i : \sum_I |\zeta_i|^2 < \infty \Big\} \quad \text{and} \quad \Big\| \sum_I \zeta_i X_i \Big\|_L^2 = \sum \zeta_i^2.$$

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2.3 The Spaces $\mathcal{C}_X^k$, $\mathcal{D}_X^{2k}$, $\mathcal{S}_X^k$ and $\mathcal{T}_X^k$

Any left invariant vector field $Z \in \mathfrak{G}$ generates a one parameter group $t \mapsto e^{tZ}$ in G. By definition, a function $f\colon G \mapsto \mathbb{R}$ has a derivative at x in the direction of Z if

$$Zf(x) = \lim_{t\to 0} \frac{f(xe^{tZ}) - f(x)}{t} = \frac{d}{dt} f(xe^{tZ})\Big|_{t=0}$$

exists. Fix a projective family $X = (X_i)_{i\in I}$ of $\mathfrak{G}$ and let $L = -\sum_I X_i^2$. Fix $k, n \in \mathbb{N} = \{0, 1, 2, \dots\}$. Consider the set

$$\Lambda(k,n) = \Big\{ \lambda = (\lambda_0, \lambda_2, \dots, \lambda_k) : \lambda_i \in \mathbb{N}, \sum \lambda_i = n \Big\}$$

of all possible ways to put n indistinguishable balls in a rack of $k+1$ boxes (these are called compositions of n into $k+1$ parts). For $f \in \mathcal{B}(G)$, $\ell \in I^k$, $\lambda \in \Lambda(k,n)$, set

$$P^{\ell,\lambda} f = P_X^{\ell,\lambda} f = L^{\lambda_0} X_{\ell_1} L^{\lambda_1} X_{\ell_2} \dots L^{\lambda_{k-1}} X_{\ell_k} L^{\lambda_k} f \tag{2.5}$$

For any integer k and any $\ell \in I^k$, consider the following seminorms on $\mathcal{B}(G)$:

$$N_X^\ell(f) = \|X^\ell f\|_\infty, \tag{2.6}$$

$$D_X^{2k}(f) = \sup_{n\le k} \|L^n f\|_\infty, \tag{2.7}$$

$$S_X^k(f) = \sup_{m\le k} \Big\| \Big(\sum_{\ell\in I^m} |X^\ell f|^2 \Big)^{1/2} \Big\|_\infty, \tag{2.8}$$

$$M_X^k(f) = \sup_{\substack{(n,m)\in\mathbb{N}\\ n+2m\le k}} \sup_{\lambda\in\Lambda(n,m)} \Big\| \Big(\sum_{\ell\in I^n} |P_X^{\ell,n} f|^2 \Big)^{1/2} \Big\|_\infty. \tag{2.9}$$

Definition 2.4 Let X be a projective family.

(i) For each $k = 0, 1, 2, \dots$, let $\mathcal{C}_X^k$ be the completion of $\mathcal{B}(G)$ for the system of seminorms N^ℓ, $\ell \in I^k$.

(ii) For each $k = 0, 1, 2, \dots$, let $\mathcal{D}_X^{2k}$, $\mathcal{S}_X^k$ and $\mathcal{T}_X^k$ be, respectively, the completion of $\mathcal{B}(G)$ for the norm D_X^{2k}, S_X^k and M_X^k.

(iii) Set $\mathcal{C}_X^\infty = \bigcap_{k\in\mathbb{N}} \mathcal{C}_X^k$, $\mathcal{D}_X^\infty = \bigcap_{k\in\mathbb{N}} \mathcal{D}_X^k$, $\mathcal{S}_X^\infty = \bigcap_{k\in\mathbb{N}} \mathcal{S}_X^k$, $\mathcal{T}_X^\infty = \bigcap_{k\in\mathbb{N}} \mathcal{T}_X^k$, and equip each of these spaces with its natural system of seminorms and the corresponding topology.

Remarks

(i) Obviously, $D_X^{2k}(f)$ depends only on X through $L = -\sum_I X_i^2$. It is proved in [10, 15] that the same is true for $S_X^k(f)$ and $M_X^k(f)$. Hence we will also use the notation

$$D_L^{2k}(f) = D_X^{2k}(f), \quad S_L^k(f) = S_X^k(f), \quad M_L^k(f) = M_X^k(f)$$

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and, for $k = 0, 1, 2, \dots, \infty$, $\mathcal{D}_L^{2k} = \mathcal{D}_X^{2k}$, $\mathcal{S}_L^k = \mathcal{S}_X^k$, $\mathcal{T}_L^k = \mathcal{T}_X^k$. To simplify notation, we also set

$$\mathcal{T}_L = \mathcal{T}_L^\infty. \tag{2.10}$$

(ii) If we view $-L$ as the $\mathcal{C}(G)$-infinitesimal generator of the semigroup $(H_t)_{t>0}$ then $\mathcal{D}_L^{2k}$ is the domain of the closed operator L^k.

(iii) The spaces $\mathcal{C}_X^k$, $\mathcal{S}_X^k$, and $\mathcal{T}_X^k$, can also be described as spaces of continuous functions with continuous derivatives of order k with respect to the family X. Each of these spaces is an algebra for pointwise multiplication on which, for any Borel measure μ, the convolution $f \mapsto \mu * f$ is a bounded operator. Note that if $E \subset \mathcal{C}(G)$ is such that, for any projective basis Y, $E \subset \mathcal{C}_Y^\infty$ then $E \subset \mathcal{B}(G)$. For these facts and further properties see [10, 12, 15, 21].

We will need to linearize the seminorms involved in the definition of the spaces $\mathcal{S}_L^k$ and $\mathcal{T}_L^k$. For any fixed $k, n, \lambda \in \Lambda(k, n)$ and $a = (a_\ell) \in \ell^2(I^k)$, set

$$Q_a^k = \sum_{\ell \in I^k} a_\ell X^\ell, \quad Q_a^{k,\lambda} = \sum_{\ell \in I^k} a_\ell P^{\ell,\lambda}. \tag{2.11}$$

The following Lemma is proved in [15].

Lemma 2.5 *For any $f \in \mathcal{S}_L^k$, any $n \le k$ and any $a = (a_\ell) \in \ell^2(I^n)$ with $\sum |a_\ell|^2 \le 1$, we have $Q_a^n f \in S_L^{k-n}$ and $S_L^{k-n}(Q_a^n f) \le S_L^k(f)$. Moreover,*

$$S_L^k(f) = \sup_{n \le k} \sup_{\substack{a \in \ell^2(I^n) \\ \sum a_\ell^2 \le 1}} \{\|Q_a^n f\|_\infty\}.$$

Similarly, for any $f \in \mathcal{T}_L^k$, any n, m, λ with $n + 2m \le k$ and $\lambda \in \Lambda(n, m)$, we have

$$Q_a^{n,\lambda} f \in \mathcal{T}_L^{k-n-2m} \quad \textit{and} \quad M_L^{k-n-2m}(Q_a^{n,\lambda} f) \le M_L^k(f).$$

Moreover,

$$M_L^k(f) = \sup_{\substack{(n,m)\in\mathbb{N} \\ n+2m \le k}} \sup_{\lambda \in \Lambda(n,m)} \sup_{\substack{a \in \ell^2(I^n) \\ \sum a_\ell^2 \le 1}} \{\|Q_a^{n,\lambda} f\|_\infty\}.$$

2.4 Bruhat Distributions

Let us denote by $\mathcal{B}'(G)$ the (strong) topological dual of $\mathcal{B}(G)$. This is the space of Bruhat distributions on G introduced in [21] and we refer to [21] for details. Recall the following definitions. Let $U \in \mathcal{B}'(G)$. The distribution $\check{U}$ is defined by

$$\forall \phi \in \mathcal{B}(G), \quad \check{U}(\phi) = U(\check{\phi}),$$

where $\check{\phi}(x) = \phi(x^{-1})$. Convolutions of a function $f \in \mathcal{B}(G)$ and a distribution U in $\mathcal{B}'(G)$ are defined by

$$[f * U](\phi) = U(\check{f} * \phi), \quad [U * f](\phi) = U(\phi * \check{f}), \quad \phi \in \mathcal{B}(G).$$

The distributions $f * U$ and $U * f$ are in fact functions in $\mathcal{B}(G)$ and we have

$$U * f(z) = U(\mathcal{L}_{z^{-1}}\check{f}) = U(y \to f(y^{-1}z)),$$
$$f * U(z) = U(\mathcal{R}_{z^{-1}}\check{f}) = U(y \to f(zy^{-1}))$$

where $\mathcal{L}_a, \mathcal{R}_a$ are the left and right translation operators defined by

$$\mathcal{L}_a \colon f \mapsto \mathcal{L}_a f, \quad \mathcal{L}_a f(x) = f(ax), \quad a \in G \tag{2.12}$$
$$\mathcal{R}_a : f \mapsto \mathcal{R}_a f, \quad \mathcal{R}_a f(x) = f(xa), \quad a \in G. \tag{2.13}$$

These formulas allow us to define the convolution of two distributions $U, V \in \mathcal{B}'(G)$ by setting

$$\forall \phi \in \mathcal{B}(G), \quad U * V(\phi) = V(\check{U} * \phi) = U(\phi * \check{V}).$$

For any left invariant vector field Z, $ZU \in \mathcal{B}'(G)$ is defined by $ZU(\phi) = -U(Z\phi)$, $\phi \in \mathcal{B}(G)$. This definition extends to any left invariant finite order differential operator P by setting (see §2.1)

$$\forall \phi \in \mathcal{B}(G), \quad PU(\phi) = U(P^*\phi).$$

Given a distribution $U \in \mathcal{B}(G)$ and an open set Ω, we say that U restricted to Ω can be identified with a continuous function if there is a continuous function u defined in Ω such that

$$\forall \phi \in \mathcal{B}_0(\Omega), \quad U(\phi) = \int u\phi \, d\nu.$$

The next lemma tells us how to recognize that a distribution coincides with a smooth function in an open set Ω. We omit the proof, which is standard.

Lemma 2.6 *Let X be a projective family. Fix an open set Ω and an integer k. Let U be a distribution in $\mathcal{B}'(G)$.*

(i) *Assume that for each integer $m \le k$ and each $\ell \in I^m$, the distribution $X^\ell U$ restricted to Ω can be identified with a continuous function u_ℓ. Then for any $\phi \in \mathcal{B}_0(\Omega)$, the distribution ϕU can be identified with a function in $\mathcal{C}^k_X$.*
If, in addition, for each $m \le k$, $\sum_{\ell \in I^m} |u_\ell|^2$ is a continuous function in Ω, then for any $\phi \in \mathcal{B}_0(\Omega)$, the distribution ϕU can be identified with a function in $\mathcal{S}^k_X$.

(ii) *Assume that for each pair of integers m, n with $m + 2n \le k$, each $\ell \in I^m$ and each $\lambda \in \Lambda(m, n)$, the distribution $P^{\ell,\lambda}U$ restricted to Ω can be identified with a continuous function $u_{\ell,\lambda}$ and that, for all m, n, $m + 2n \le k$, $\sum_{\ell \in I^m} |u_{\ell,\lambda}|^2$ is continuous on Ω. Then, for any $\phi \in \mathcal{B}_0(\Omega)$, the distribution ϕU can be identified with a function in $\mathcal{T}^k_X$.*

2.5 The Space of Distributions $\mathcal{T}'_L$

Fix a symmetric Gaussian semigroup with infinitesimal generator $-L$. Consider the topological vector space of smooth functions $\mathcal{T}_L$ introduced in Definition 2.4. It is a Fréchet space [20, II.26]. Denote by $\mathcal{T}'_L$ the strong topological dual of $\mathcal{T}_L$ [20, III.14]. Elements in $\mathcal{T}'_L$ are linear functionals U on $\mathcal{T}_L$ such that there exist an integer $m = M(U)$ and a real $C = C(U) > 0$ such that

$$\forall\, \phi \in \mathcal{T}_L, \quad |U(\phi)| \leq C M_L^m(\phi).$$

The topology of $\mathcal{T}'_L$ is defined by the family of seminorms

$$p_B(U) = \sup_{\phi \in B} |U(\phi)|$$

where B runs over all bounded sets in $\mathcal{T}_L$ (recall that $B \subset \mathcal{T}_L$ is bounded if for any integer k, $\sup_{\phi \in B} M^k(\phi) < \infty$). The space $\mathcal{T}'_L$ is complete [20, III.24]. As $\mathcal{T}_L$ is an algebra, contains $\mathcal{B}(G)$ and has a weaker topology, it follows that $\mathcal{T}'_L$ is contained in $\mathcal{B}'(G)$ and that $\mathcal{B}(G)$ acts on $\mathcal{T}'_L$ by pointwise multiplication. We will need some basic results concerning convolution on $\mathcal{T}'_L$. It turns out that there are essential differences between the general case and the case when $L = -\sum_I X_i^2$ is bi-invariant. We refer the reader to [15] for details. The following proposition gathers the results needed in the sequel.

Proposition 2.7 *Assume that L is bi-invariant.*

(i) *Let $U \in \mathcal{T}'_L$ and $f \in \mathcal{T}_L$. Then $U * f \in \mathcal{T}_L$ and*

$$X^\ell L^n[U * f] = U * [X^\ell L^n f]. \tag{2.14}$$

Moreover, there exists a constant C_U and an integer $m = m(U)$ such that

$$\forall\, k, \quad M_L^k(U * f) \leq C_U M_L^{m+k}(f). \tag{2.15}$$

(ii) *Let ϕ_ϵ, $\epsilon > 0$ be a family of functions in $\mathcal{T}_L$ such that $\phi_\epsilon \to \delta_e$ as ϵ tends to zero. Then for any $U \in \mathcal{T}'_L$, the distribution $U^\epsilon = U * \phi_\epsilon$ is represented by a function in $\mathcal{T}_L$, and*

$$\forall\, \psi \in \mathcal{T}_L, \quad \lim_{\epsilon \to 0} U^\epsilon(\psi) = U(\psi).$$

3 $\mathcal{T}'_L$-$\mathcal{C}(G)$-Hypoellipticity

This section contains our main results concerning the hypoellipticity of the infinitesimal generators of central Gaussian semigroups. For brevity we will use the following notation. Let U be a distribution in $\mathcal{B}'(G)$. Let Ω be an open set. We say that U belongs to $\mathcal{C}(\Omega)$ if there exists a continuous function u on Ω such that, for all $\phi \in \mathcal{B}_0(\Omega)$, $U(\phi) = \int u\phi\, d\nu$. In this case, we set $\|U\|_{\Omega,\infty} = \|u\|_{\Omega,\infty}$ where $\|u\|_{\Omega,\infty} = \sup_\Omega\{|u|\}$. Given a projective family $X = (X_i)_I$ and an open set Ω, we set (see (2.9))

$$M_X^k(\Omega, f) = \sup_{\substack{(n,m)\in\mathbb{N} \\ n+2m\leq k}} \sup_{\lambda\in\Lambda(n,m)} \sup_{x\in\Omega} \Big(\sum_{\ell\in I^n} |P_X^{\ell,n} f|^2\Big)^{1/2}.$$

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3.1 Smoothing

In this section we present basic but critical technical computations that will be used to prove hypoellipticity results. We focus our attention on the bi-invariant infinitesimal generator $-L$ of symmetric central Gaussian semigroup $(\mu_t^L)_{t>0}$. Let also $Z \in \mathfrak{G}$. Assume that for each $k = 0, 1, 2, \dots$ there exist a constant $A(Z, k) \geq 1$ and an integer $\kappa(Z, k)$ such that

$$(3.1) \qquad \forall f \in \mathcal{B}(G), \quad M_L^k(Zf) \leq A(Z, k) M_L^{\kappa(Z,k)}(f).$$

Set

$$(3.2) \qquad P = L + Z.$$

Note that Z satisfies (3.1) if and only if $Z\mathcal{T}_L \subset \mathcal{T}_L$ (see also Lemma 4.3). Our aim is to study the hypoellipticity of P. To this end we assume throughout this section that the symmetric central Gaussian semigroup $(\mu_t^L)_{t>0}$ satisfies the property (CK), that is, μ_t^L is absolutely continuous with respect to Haar measure and admits a continuous density for all $t > 0$. Under this condition, it is proved in [10, Theorem 4.2] that $\mu_t^L \in \mathcal{T}_L$, for all $t > 0$. We will need the following observation.

Lemma 3.1 *Let $(\mu_t^L)_{t>0}$ be a symmetric central Gaussian semigroup with infinitesimal generator $-L$. Let Z be a left invariant vector field. Let $(\mu_t)_{t>0}$ be the Gaussian semigroup such that the semigroup $H_t f = f * \check{\mu}_t$ has infinitesimal generator $-P = -(L+Z)$.*

(i) *The measure μ_t^L has a continuous density if and only if μ_t does, and if they exist, the two continuous densities have the same uniform norm.*
(ii) *Assume (3.1) and that $(\mu_t^L)_{t>0}$ satisfies (CK). Let $x \mapsto \mu_t(x)$ be the continuous density of the measure μ_t. Then $\partial_t^n \mu_t, \partial_t^n \check{\mu}_t, \in \mathcal{T}_L$ for any integer $n = 0, 1, 2, \dots$.*

Proof Since L is bi-invariant it commutes with Z. Hence, for any Borel set B,

$$(3.3) \qquad \mu_t(B) = \mu_t^L(Bz_t) = \mu_t^L(z_t B), \quad \check{\mu}_t = \mu_t^L(Bz_t^{-1})$$

where $z_t = \exp(tZ)$ is the one parameter subgroup generated by Z. The first assertion follows. Moreover, if the two semigroups have property (CK), it follows from [10, Theorem 4.2] and (3.3) that the densities are in $\mathcal{T}_L$ and satisfy $M_L^k(\mu_t) = M_L^k(\mu_t^L)$. Moreover $\partial_t^n \mu_t = (-L + Z)^n \mu_t$. By (3.1) this shows that $\partial_t^n \mu_t \in \mathcal{T}_L$. ■

Lemma 3.2 *Let L, Z, P be as above with Z satisfying (3.1). Let $U \in \mathcal{T}_L'$. Let $A(U) \geq 1$ and the integer $\kappa(U)$ be such that*

$$(3.4) \qquad \forall \phi \in \mathcal{T}_L, \quad |U(\phi)| \leq A(U) M^{\kappa(U)}(\phi).$$

Then for any $\eta \in \mathcal{B}(G)$ there exists C_η such that for all $\phi \in \mathcal{T}_L$

$$\max\left\{ \left| [\eta P U](\phi) \right|, \left| [P \eta U](\phi) \right| \right\} \leq C_\eta A(U, Z) M^{\kappa(U,Z)}(\phi)$$

with

$$(3.5) \quad A(U, Z) = A(U) A(Z, \kappa(U)) \quad \textit{and} \quad \kappa(U, Z) = \max\{\kappa(Z, \kappa(U)), 2 + \kappa(U)\}.$$

Proof We have $[\eta PU](\phi) = U((L-Z)\eta\phi)$ and

$$M_L^k\big((L-Z)(\eta\phi)\big) \leq \big[M_L^k(L(\eta\phi)) + M_L^k(Z(\eta\phi))\big].$$

Hence, by (3.1) and the fact that $\mathcal{T}_L^s$ is an algebra for any fixed integer s,

$$M_L^k((L-Z)(\eta\phi)) \leq [M_L^{2+k}(\eta\phi) + A(Z)M_L^{\kappa(Z,k)}(\eta\phi)] \leq CM_L^{m(Z,k)}(\eta)M_L^{m(Z,k)}(\phi)$$

with $m(Z,k) = \max\{\kappa(Z,k), 2+k\}$. It follows that

$$\big|[\eta PU](\phi)\big| \leq C_\eta A(U)A(Z,\kappa(U))M_L^{m(Z,\kappa(U))}(\phi).$$

For $[P\eta U](\phi) = U(\eta(L+Z)\phi)$, a similar argument gives the desired result.

Consider two distributions $U \in \mathcal{T}_L'$, $F \in \mathcal{B}'(G)$ such that

$$PU = F \ \text{ in } \mathcal{B}'(G).$$

By construction and (3.1), $P\mathcal{T}_L \subset \mathcal{T}_L$. Hence $P\mathcal{T}_L' \subset \mathcal{T}_L'$. It follows that

$$F \in \mathcal{T}_L' \quad \text{and} \quad PU = F \ \text{ in } \mathcal{T}_L'.$$

Let Ω be an open set such that $F \in \mathcal{C}(\Omega)$. Fix an open set Ω_0 such that $\overline{\Omega_0} \subset \Omega$. Fix a function $\eta_0 \in \mathcal{B}_0(\Omega)$ such that $\eta_0 \equiv 1$ on a neighborhood of Ω_0 and set

$$\widetilde{U} = \eta_0 U, \quad \widetilde{F} = \eta_0 F, \quad \widetilde{V} = P\widetilde{U} - \widetilde{F}. \tag{3.6}$$

Observe that $\widetilde{U} \in \mathcal{T}_L'$, $\widetilde{F} \in \mathcal{C}(G) \subset \mathcal{T}_L'$, hence $\widetilde{V} \in \mathcal{T}_L'$. Moreover, $\widetilde{V}$ is supported in $\Omega \setminus \overline{\Omega_0}$. Under the standing assumption that $(\mu_t^L)_{t>0}$ satisfies (CK), Lemma 3.1 shows that the density $x \mapsto \check{\mu}_t(x)$ of the measure $\check{\mu}_t$ belongs to $\mathcal{T}_L$. Hence, by Proposition 2.7, we can consider

$$\widetilde{U}^t = \widetilde{U} * \check{\mu}_t, \quad \widetilde{F}^t = \widetilde{F} * \check{\mu}_t, \quad \widetilde{V}^t = \widetilde{V} * \check{\mu}_t \tag{3.7}$$

which are all in $\mathcal{T}_L$. ■

Lemma 3.3 *Let $L, Z, P, (\mu_t^L)_{t>0}, (\mu_t)_{t>0}$ be as above with $(\mu_t^L)_{t>0}$ satisfying* (CK) *and Z satisfying* (3.1). *Let W be in $\mathcal{T}_L'$ and set $W^t = W * \check{\mu}_t$. Then, for all $t > 0$,*

$$W^t, PW_t \in \mathcal{T}_L \quad \textit{and} \quad \partial_t W^t = -PW^t.$$

Proof Note that for any $w \in \mathcal{B}(G)$ we have

$$P(w * \check{\mu}_t) = w * (P\check{\mu}_t) = (Pw) * \check{\mu}_t$$

because P and right convolution by $\check{\mu}_t$ commute. By Lemma 3.1, $\check{\mu}_t$ and $P\check{\mu}_t$ are in $\mathcal{T}_L$. By Proposition 2.7, W^t, PW^t belong to $\mathcal{T}_L$ and it follows that the double equality above holds true if we replace w by a distribution W in $\mathcal{T}'_L$. For any $0 < s < t < +\infty$, we now have

$$\begin{aligned} W^t - W^s &= W * (\check{\mu}_t - \check{\mu}_s) = W * \int_s^t \check{\mu}'_\tau \, d\tau \\ &= -W * \int_s^t P\check{\mu}_\tau \, d\tau = -\int_s^t W * P\check{\mu}_\tau \, d\tau \\ &= -\int_s^t (PW) * \check{\mu}_\tau \, d\tau = \int_s^t (-PW)^\tau \, d\tau. \end{aligned}$$

Hence $\partial_t W^t$ exists and $\partial_t W^t = (-PW)^t$. As $(-PW)^t = -PW^t$ this finishes the proof. ■

Fix $x_0 \in \Omega_0$. We want to show that $\widetilde{U} \in \mathcal{C}(\Omega_1)$ for some neighborhood $\Omega_1 \subset \overline{\Omega}_1 \subset \Omega_0$ of x_0. For this, it suffices to bound

$$\sup_{0<t\leq 1} \|\partial_t \widetilde{U}^t\|_{\Omega_1,\infty}.$$

Indeed, such a bound implies that $\widetilde{U}$ is the uniform limit of the continuous functions $\widetilde{U}^t$ in Ω_1 as t tends to zero and that

$$\|\widetilde{U}\|_{\Omega_1,\infty} \leq \|\widetilde{U}^1\|_{\Omega_1,\infty} + \sup_{0<t\leq 1} \|\partial_t \widetilde{U}^t\|_{\Omega_1,\infty}. \tag{3.8}$$

By Lemma 3.3,

$$\begin{aligned} \partial_t \widetilde{U}^t &= -P\widetilde{U}^t = (-P\widetilde{U}) * \check{\mu}_t = -(\widetilde{V} + \widetilde{F}) * \check{\mu}_t \\ &= -\widetilde{V}^t - \widetilde{F}^t. \end{aligned}$$

Thus it suffices to bound

$$\|\widetilde{F}^t\|_{\Omega_1,\infty} \quad \text{and} \quad \|\widetilde{V}^t\|_{\Omega_1,\infty}.$$

To handle $\widetilde{F}^t$ we do not need any additional localization provided by Ω_1.

Lemma 3.4 *We have $\|\widetilde{F}^t\|_\infty \leq \|\widetilde{F}\|_\infty < +\infty$.*

Proof By hypothesis $\widetilde{F} = \eta_0 F \in \mathcal{C}(G)$, and $\widetilde{F}^t = \widetilde{F} * \mu_t$. Hence,

$$\|\widetilde{F}^t\|_\infty \leq \|\widetilde{F}\|_\infty < +\infty.$$ ■

We are now left with the task of bounding $\widetilde{V}^t$. For this, we will need the additional localization in a small neighborhood Ω_1 of x_0. We will be able to control $\|\widetilde{V}^t\|_{\Omega_1,\infty}$ independently of the continuity of F in Ω. By construction, $\widetilde{V} \in \mathcal{T}'_L$ is supported in

$\Omega \setminus \overline{\Omega_0}$. Let Θ_0 be an open neighborhood of $G \setminus \Omega_0$ such that $x_0 \notin \overline{\Theta_0}$. Obviously we can choose the neighborhood Ω_1 of x_0 so that

$$x_0 \in \Omega_1 \subset \Omega_0, \quad e \notin \overline{\Theta_0^{-1}\Omega_1} \tag{3.9}$$

where e is the identity element. To see that such a choice is indeed possible, write $\Theta_0 = x_0\theta_0$ where θ_0 is an open set whose closure does not contain the identity e and write $\Omega_1 = x_0\omega_1$ where ω_1 is an open neighborhood of e. Then condition (3.9) amounts to $e \notin \overline{\theta_0^{-1}\omega_1}$, and this can obviously be arranged by choosing ω_1 small enough. From now on we fix Ω_1 so that (3.9) holds true.

Lemma 3.5 *Let Θ_0 be an open neighborhood of $G \setminus \Omega_0$ whose closure does not contain x_0. Let Ω_1 be an open neighborhood of x_0 such that* (3.9) *holds true and set $\Theta = \Theta_0^{-1}\Omega_1$. Then*

$$\|\widetilde{V}^t\|_{\Omega_1,\infty} \leq CAM_L^{\kappa}(\Theta, \mu_t)$$

where, referring to (3.1), (3.4) *and* (3.5), *$A = A(U, Z)$, $\kappa = \kappa(U, Z)$ and $C = C(\Omega_0, \Omega_1)$.*

Proof Since $\widetilde{V} \in \mathcal{T}_L$, we have

$$\widetilde{V}^t(x) = \widetilde{V} * \check{\mu}_t(x) = \widetilde{V}(\mathcal{L}_{x^{-1}}\mu_t)$$

and, since $\widetilde{V}$ is supported in $\Omega \setminus \overline{\Omega_0}$, we can write

$$\widetilde{V}^t(x) = \widetilde{V}(\eta_1 \mathcal{L}_{x^{-1}}\mu_t)$$

where $\eta_1 \in \mathcal{B}_0(\Theta_0)$ and $\eta_1 = 1$ in a neighborhood of $\overline{\Omega \setminus \overline{\Omega_0}}$. Since $\widetilde{V} = P\widetilde{U} - \widetilde{F} = P(\eta_0 U) - \eta_0 PU$, Lemma 3.2 gives

$$\forall\, \phi \in \mathcal{T}_L, \quad |\widetilde{V}(\phi)| \leq C_{\eta_0} AM^{\kappa}(\phi).$$

Thus

$$|\widetilde{V}^t(x)| \leq CAM_L^{\kappa}(\eta_1 \mathcal{L}_{x^{-1}}\mu_t)$$

If we assume $x \in \Omega_1$, then

$$y \in \Theta_0 \Longrightarrow y^{-1}x \in \Theta_0^{-1}\Omega_1 = \Theta$$

and it follows that there exists $C_2 = C_2(\eta_0, \eta_1)$ such that

$$M_L^{\kappa}(\eta_1 \mathcal{L}_{x^{-1}}\mu_t) \leq C_2 M_L^{\kappa}(\Theta, \mu_t).$$

This finishes the proof of the lemma. ■

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3.2 The Condition (CK$*$)

Recall from [12] that given a symmetric central Gaussian semigroup $(\mu_t)_{t>0}$ with generator $-L$, property (CK$*$) is a necessary condition for the $\mathcal{T}'_L$-$\mathcal{C}(G)$-hypoellipticity of L. We are now ready to prove that (CK$*$) is also a sufficient condition. The next result, taken from [10], captures the role of this condition. It follows from Gaussian estimates [10, Corollary 4.8] for space and time derivatives of the density $x \mapsto \mu_t(x)$.

Theorem 3.6 ([10, Corollary 4.9]) *Assume that $(\mu_t)_{t>0}$ is a symmetric central Gaussian semigroup satisfying condition* (CK$*$). *Then for any compact set K with $e \notin K$, any integer k, and any $\sigma > 0$, there exists a constant C (which depends on $(\mu_t)_{t>0}$, K, k and σ) such that*

$$\sup_{t\in(0,1)} \{t^{-\sigma} M_L^k(K, \mu_t)\} \le C.$$

In order to treat the operators of the form $P = L+Z$ where Z is a left invariant vector field satisfying (3.1) as in Section 3.1, we need the following corollary.

Corollary 3.7 *Let $-L$ be the bi-invariant infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t^L)_{t>0}$. Let Z be a left invariant vector field satisfying* (3.1). *Let $(\mu_t)_{t>0}$ be the Gaussian semigroup with infinitesimal generator $-P = -(L+Z)$. Assume that $(\mu_t^L)_{t>0}$ satisfies* (CK$*$). *Then, for any compact set K with $e \notin K$, any integer k, and any $\sigma > 0$, there exists a constant C (which depends on $(\mu_t)_{t>0}$, K, k and σ) such that*

$$\sup_{t\in(0,1)} \{t^{-\sigma} M_L^k(K, \mu_t)\} \le C.$$

Proof By Lemma 3.1, we know that $\mu_t \in \mathcal{T}_L$. In fact, using the same notation as in the proof of Lemma 3.1, we have $\mu_t(x) = \mu_t^L(xz_t) = \mu_t^L(z_t x)$ where $z_t = \exp(tZ)$. By definition $t \mapsto z_t$ is a continuous map. Thus, for any fixed compact set K and any open set Ω such that $K \subset \Omega$ and $e \notin \overline{\Omega}$, there exists $t_0 > 0$ such that $Kz_t \subset \Omega$ for all $t \in (0, t_0)$. Hence

$$\sup_{t\in(0,t_0)} \{t^{-\sigma} M_L^k(K, \mu_t)\} \le \sup_{t\in(0,t_0)} \{t^{-\sigma} M_L^k(\overline{\Omega}, \mu_t^L)\} \le C.$$

As $M_L^k(\mu_t)$ is decreasing in t, this proves the desired result. ∎

The following theorem describes our main hypoellipticity result in a form that will be used to obtain further information in the next few sections.

Theorem 3.8 *Let $-L$ be the bi-invariant infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t^L)_{t>0}$. Let Z be a left invariant vector field satisfying* (3.1). *Let $(\mu_t)_{t>0}$ be the Gaussian semigroup with infinitesimal generator $-P = -(L+Z)$. Assume that $(\mu_t^L)_{t>0}$ satisfies* (CK$*$). *Consider two distributions $U \in \mathcal{T}'_L$, $F \in \mathcal{B}'(G)$ such that*

$$PU = F \quad \text{in } \mathcal{B}'(G). \tag{3.10}$$

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Let Ω be an open set in G such that $F \in \mathcal{C}(\Omega)$. Then $U \in \mathcal{C}(\Omega)$ and for any open sets $\Omega_1 \subset \Omega_0 \subset \Omega$ with $\overline{\Omega_1} \subset \Omega_0$ and $\overline{\Omega_0} \subset \Omega$, we have

$$\|U\|_{\Omega_1,\infty} \leq C(\Omega_0,\Omega_1)(\|F\|_{\Omega_0,\infty} + A).$$

where, referring to (3.1), (3.4) *and* (3.5), $A = A(U, Z)$. *In particular P is $\mathcal{T}'_L$-$\mathcal{C}(G)$-hypoelliptic. Moreover, setting $U^t = U * \check{\mu}_t$,*

$$\forall\, t \in (0,1), \quad \|U - U^t\|_{\Omega_1,\infty} \leq C(\Omega_0,\Omega_1)t(\|F\|_{\Omega_0,\infty} + A).$$

Proof Referring to the setting and notation of Section 3.1, Corollary 3.7 and Lemma 3.5 give

$$\sup_{t\in(0,1)} \|\widetilde{V}^t\|_{\Omega_1,\infty} \leq C_1 A$$

where A is as above and $C_1 = C_1(\Omega_0,\Omega_1)$. This inequality together with Lemma 3.4, (3.8), and the smoothing argument of Section 3.1 yields

$$\forall\, t \in (0,1), \quad \|\widetilde{U} - \widetilde{U}^t\|_{\Omega_1,\infty} \leq C(\Omega_0,\Omega_1)\, t\, (\|F\|_{\Omega_0,\infty} + A).$$

This shows that U is continuous in Ω_1, being the uniform limit of the continuous functions $\widetilde{U}^t$ as t tends to zero. Together with (3.4), it also proves the first inequality in the theorem. To obtain the last statement, we need to estimate

$$\|U^t - \widetilde{U}^t\|_{\Omega_1,\infty}.$$

But $U^t - \widetilde{U}^t = [(1-\eta_0)U] * \check{\mu}_t$ and $(1-\eta_0)U$ is supported in $G \setminus \Omega_0$. Thus, for all $t \in (0,1)$, Theorem 3.6 yields

$$\begin{aligned}\big|\, [(1-\eta_0)U] * \check{\mu}_t(x)\big| &= |U[(1-\eta_0)\mathcal{L}_{x^{-1}}\mu_t]| \\ &\leq AM^{\kappa}_L((1-\eta_0)\mathcal{L}_{x^{-1}}\mu_t) \leq CAt\end{aligned}$$

Where C depends on $\Omega_0, \Omega_1, (\mu_t)_{t>0}$ and κ. Hence, we obtain $\|U^t - \widetilde{U}^t\|_{\Omega_1,\infty} \leq C(\Omega_0,\Omega_1)At$, as desired. ■

Taking $Z = 0$ in the above theorem yields the following result which contains the "if" part of Theorem 1.6 stated in the introduction.

Corollary 3.9 *Let $-L$ be the bi-invariant infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t)_{t>0}$. Assume that $(\mu_t)_{t>0}$ satisfies* (CK$*$). *Then L is $\mathcal{T}'_L$-$\mathcal{C}(G)$-hypoelliptic.*

4 $\mathcal{T}'_L$-$\mathcal{C}^k_Y$-Hypoellipticity

We are now ready to state and prove some of the main results of this paper concerning $\mathcal{T}'_L$-$\mathcal{S}$-hypoellipticity. The main question we need to answer is: for which spaces $\mathcal{S}$ of smooth functions can we prove hypoellipticity? This section is concerned with the case when $\mathcal{S} = \mathcal{C}^k_Y$ where Y is a fixed projective family.

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4.1 Bi-invariant Laplacians

The next proposition is simple but very useful. It shows that when a differential operator P is bi-invariant and the space of distributions $\mathcal{A}$ is stable under the action of any vector field Y_i of a projective family $Y = (Y_i)_I$, $\mathcal{A}$-$\mathcal{C}(G)$-hypoellipticity implies $\mathcal{A}$-$\mathcal{C}^k_Y$-hypoellipticity for each $k = 1, 2, \ldots, \infty$. Thus, in the present setting, and for bi-invariant operators, $\mathcal{A}$-$\mathcal{C}(G)$-hypoellipticity is really the main property one should study. This idea will be developed further in Section 5.

Proposition 4.1 *Let P be a bi-invariant differential operator of finite order on a connected compact group G. Fix a space of distributions $\mathcal{A}$ and a projective family $Y = (Y_i)_I$. Assume that for any distribution $U \in \mathcal{A}$ and any $i \in I$, Y_iU is in $\mathcal{A}$. Then $\mathcal{A}$-$\mathcal{C}(G)$-hypoellipticity implies $\mathcal{A}$-$\mathcal{C}^k_Y$-hypoellipticity, for all $k = 1, 2, \ldots, \infty$.*

Proof Let $U \in \mathcal{A}$, $F \in \mathcal{B}'(G)$ be such that $PU = F$ and $\phi F \in \mathcal{C}^k_Y$ for all $\phi \in \mathcal{B}_0(\Omega)$. Then for any integer $m \leq k$ and any $\ell \in I^m$, $Y^\ell U$ satisfies

$$PY^\ell U = Y^\ell PU = Y^\ell F$$

because P is bi-invariant. By hypothesis, the distribution $Y^\ell U$ belongs to $\mathcal{A}$ and for any $\phi \in \mathcal{B}_0(\Omega)$, $\phi Y^\ell F$ is continuous. Thus $\mathcal{A}$-$\mathcal{C}(G)$-hypoellipticity implies that $Y^\ell U$ coincides with a continuous function in Ω. This proves that ϕU belongs to $\mathcal{C}^k_Y$ for any $\phi \in \mathcal{B}_0(\Omega)$, as desired. ■

Theorem 4.2 *Assume that $(\mu_t)_{t>0}$ is a symmetric central Gaussian semigroup satisfying* (CK$*$). *Let $-L$ denote its infinitesimal generator. Let $Y = (Y_i)_{i\in I}$ be a projective family and assume that*

$$\forall\, i \in I, \quad Y_i\mathcal{T}_L \subset \mathcal{T}_L. \tag{4.1}$$

Then the operator L is $\mathcal{T}'_L$-$\mathcal{C}^k_Y$-hypoelliptic for each $k = 0, 1, 2, \ldots$. In particular, for any projective family $X = (X_i)_I$ such that $L = -\sum X_i^2$ is $\mathcal{T}'_L$-$\mathcal{C}^k_X$-hypoelliptic for each $k = 0, 1, 2, \ldots$.

Proof This statement readily follows from Theorem 3.8 and Proposition 4.1. ■

Remark Let $-L$ be the infinitesimal generator of a symmetric central Gaussian semigroup. Let Z be a left invariant vector field and assume that $Z \in \mathcal{H}(L)$. Then by Lemma 2.5, $Z\mathcal{T}_L \subset \mathcal{T}_L$. A useful necessary and sufficient condition for this last property (equivalently (3.1)) is given in the next lemma taken from [15, Lemma 5.2].

Lemma 4.3 *Let $-L$ be the infinitesimal generator of a symmetric central Gaussian semigroup. Let Z be a left invariant vector field. Assume that there are constants m and C such that*

$$\forall\, f \in \mathcal{B}(G), \quad \|Zf\|_\infty \leq CM^m_L(f). \tag{4.2}$$

Then condition (3.1) *is satisfied, that is, $Z\mathcal{T}_L \subset \mathcal{T}_L$.*

Remark Inequality (4.2) does not imply $Z \in \mathcal{H}(L)$. See Theorem 6.2.

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4.2 The Case $P = L + Z$

In view of Theorems 3.8 and 4.2, the question arises of whether or not one can extend Theorem 4.2 to operators of the form $P = L+Z$ as in Theorem 3.8. The only difficulty comes from the fact that Z may not commute with other vector fields. This seems to be a rather serious technical difficulty in general, but it can be easily solved in certain cases by using the following notion.

Let Y be a projective family. Denote by $\check{Y}_i$ the right invariant vector field which coincides with Y_i at the neutral element. This leads in a natural way to the definition of a right invariant version of the space $\mathcal{C}_Y^k$ which we denote by $\mathcal{RC}_Y^k$. In order to be able to talk simultaneously about left and right invariant vector fields it is convenient to introduce a reasonable notion of (non-invariant) vector field. A continuous vector field on G is a linear map $Z\colon \mathcal{B}(G) \to \mathcal{C}(G)$ with the property that $Z(fg) = fZg + gZf$, *i.e.*, a derivation. It is not hard to see that given a projective basis $Y = (Y_i)_I$, any continuous vector field Z can be written

$$Z = \sum_i a_i Y_i, \text{ that is, for } f \in \mathcal{B}(G), \quad Zf(x) = \sum_i a_i(x) Y_i f(x),$$

where the coefficients a_i are continuous functions. Conversely, each such expression obviously yields a vector field on G.

Definition 4.4 A (left invariant) projective family $Y = (Y_i)_{i\in I}$ is called a *special projective family* if there are functions $a_{\overrightarrow{i,j}}, a_{\overleftarrow{i,j}} \in \mathcal{B}(G)$, $i, j \in I$, and finite sets $J(i)$, $i \in I$, such that $a_{\overrightarrow{i,j}} \equiv a_{\overleftarrow{i,j}} \equiv 0$ for all (i, j) with $j \notin J(i)$ and

$$\check{Y}_i = \sum_{j\in\mathcal{J}} a_{\overrightarrow{i,j}} Y_j, \quad Y_i = \sum_{j\in\mathcal{J}} a_{\overleftarrow{i,j}} \check{Y}_j.$$

Obviously, if Y is a special projective family, then the spaces $\mathcal{C}_Y^k$ and $\mathcal{RC}_Y^k$ coincide. Also, if G is abelian, any projective family is special. The following proposition follows easily from the structure theory of compact connected groups [27] and the structure of bi-invariant second order operators, see [8]. See also Section 6.2.

Proposition 4.5 *Let G be a compact connected group. Let $-L$ be the infinitesimal generator of a symmetric central Gaussian semigroup. Then there exists a special projective family $X = (X_i)$ such that $L = -\sum_{i\in I} X_i^2$.*

Theorem 4.6 *Let $-L$ be the bi-invariant infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t^L)_{t>0}$ which satisfies* (CK$*$). *Let Z be a left invariant vector field satisfying* (3.1) *and set $P = L + Z$. Let Y be a special projective family satisfying* (4.1). *Then P is $\mathcal{T}_L'$-$\mathcal{C}_Y^k$-hypoelliptic, $k = 0, 1, \dots$.*

5 $\mathcal{T}_L'$-$\mathcal{S}$-Hypoellipticity, $\mathcal{S} = \mathcal{S}_Y^k, \mathcal{T}_Y^k, \dots$

We now want to study $\mathcal{T}_L'$-$\mathcal{S}$-hypoellipticity when $\mathcal{S} = \mathcal{S}_Y^k$ or other spaces of this type. As mentioned in the introduction, it seems important in the present setting to have as much flexibility as possible in the choice of the spaces of smooth functions $\mathcal{S}$. Having identified a large space of distributions, namely $\mathcal{T}'_L$, for which $\mathcal{T}'_L$-$\mathcal{C}(G)$-hypoellipticity holds, turns out to be one of the keys to the desired flexibility concerning the choice of $\mathcal{S}$.

5.1 Smooth Function Spaces

The following definition gives a very flexible tool to define spaces of smooth functions. Recall that any finite order left invariant differential operator on G is a continuous operator on $\mathcal{B}(G)$ and on $\mathcal{B}'(G)$. Let I denote the identity operator acting on functions. In what follows we consider I as an invariant differential operator of order 0.

Definition 5.1 Let $\mathbb{N} = \{0, 1, 2, \dots\}$. For any $n \in \mathbb{N}$, let $\mathcal{O}(n)$ be a set of continuous linear operators on $\mathcal{B}(G)$, each of which admits a continuous extension to $\mathcal{B}'(G)$. Assume further that $I \in \mathcal{O}(0)$ and $\mathcal{O}(n) \subset \mathcal{O}(k)$ if $n \leq k$. Denote this data by $\mathcal{O}$. Set

$$\mathcal{B}_{\mathcal{O}}(G) = \Big\{ f \in \mathcal{B}(G) : \forall\, k \in \mathbb{N},\ \sup_{Q \in \mathcal{O}(k)} \|Qf\|_\infty < +\infty \Big\}$$

and consider the norms defined by

$$\forall\, f \in \mathcal{B}_{\mathcal{O}}(G), \quad p_k(f) = \sup_{Q \in \mathcal{O}(k)} \|Qf\|_\infty.$$

Let $\mathcal{S}^k(\mathcal{O}) \subset \mathcal{C}(G)$ be the closure of $\mathcal{B}_{\mathcal{O}}(G)$ for the topology defined by the norm p_k. Set $\mathcal{S}(\mathcal{O}) = \bigcap_0^\infty \mathcal{S}^k(\mathcal{O})$ equipped with the topology defined by the the family of norms p_k, $k = 0, 1, \dots$.

It follows from this definition that each $Q \in \mathcal{O}(k)$ can also be viewed as a continuous linear operator

$$Q\colon \mathcal{S}^k(\mathcal{O}) \to \mathcal{C}(G).$$

The following examples illustrates this definition.

Example 5.2 Fix a projective family $X = (X_i)$ and set $L = -\sum_{i \in I} X_i^2$.

(a) Enumerate $\bigcup_{n=0}^k I^n$ as $\{\ell_0, \ell_1, \dots\}$ with $\ell_0 = \varnothing$ and set (with the convention $X^\varnothing = I$)

$$\mathcal{O}(m) = \{X^\ell : \ell = \ell_0, \dots, \ell_m\}.$$

Then $p_m(f) = \sup_{\ell \in \{\ell_0, \dots, \ell_m\}} \|X^\ell f\|_\infty$ and $S(\mathcal{O}) = \mathcal{C}_X^k$ (one can take $k = \infty$).

(b) Set $\mathcal{O}(k) = \big\{ Q_a^n = \sum_{I^n} a_\ell X^\ell : \sum_{I^n} a_\ell^2 \leq 1,\ n \leq k \big\}$. Then $p_k(f) = S_X^k(f)$, $\mathcal{S}^k(\mathcal{O}) = \mathcal{S}_X^k = \mathcal{S}_L^k$ and $\mathcal{S}(\mathcal{O}) = \mathcal{S}_L^\infty$.

(c) Set $\mathcal{O}(k) = \{L^n : n = 0, \dots, k\}$. Then $p_k(f) = D_L^{2k}(f)$ and $\mathcal{S}^k(\mathcal{O}) = \mathcal{D}_L^{2k}$ is the $\mathcal{C}(G)$-domain of L^k.

(d) Set

$$\mathcal{O}(k) = \Big\{ Q_a^{n,\lambda} = \sum_{I^n} a_\ell P^{\ell,\lambda} : \sum_{I^n} a_\ell^2 \leq 1, \lambda \in \Lambda(n,m), n + 2m \leq k \Big\}.$$

Then $p_k(f) = M_L^k(f)$ and $\mathcal{S}^k(\mathcal{O}) = \mathcal{T}_L^k$.

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(e) Fix $p \in [1, +\infty]$ and let q be the Hölder conjugate of p, *i.e.*, $1/p + 1/q = 1$. Set $\mathcal{O}(k) = \{\sum_{I^n} a_\ell X^\ell : \sum a_\ell^q \leq 1, n \leq k\}$. Then

$$p_k(f) = \sup_{n \leq k} \Big\| \Big(\sum_{I^k} |X^\ell f|^p \Big)^{1/p} \Big\|_\infty .$$

We set $\mathcal{S}^k(\mathcal{O}) = \mathcal{S}_X^{p,k}$. Thus we have $\mathcal{S}_X^{2,k} = \mathcal{S}_X^k$, whereas $\mathcal{S}_X^{\infty,k}$ corresponds to having a uniform control over the sup-norm of all derivatives $X^i f$ with $i \in I^m$, $m \leq k$.

(f) Fix a positive function $\omega \colon \mathbb{N} \to (0, \infty)$ and set

$$\mathcal{O} = \mathcal{O}(k) = \{\omega(|\ell|)^{-1} X^\ell,\ \ell \in \bigcup_0^\infty I^k\}$$

where $|\ell| = k$ if $\ell \in I^k$. Then

$$p(f) = p_k(f) = \sup_{\ell \in \bigcup_0^\infty I^k} \omega(|\ell|)^{-1} \|X^\ell f\|_\infty .$$

A function f is in $\mathcal{S}(\mathcal{O})$ if and only if it is in $\mathcal{C}_X^\infty$ and there is a constant C such that $\|X^\ell f\|_\infty \leq C\omega(|\ell|)$. Examples of potential interest are $\omega(k) = (Ak)^{ak}$ with some $A > 0$ and $a \geq 1$.

(g) Fix a positive function $\omega \colon \mathbb{N} \to (0, \infty)$ and set $\mathcal{O} = \mathcal{O}(k) = \{\omega(n)^{-1} L^n,\ n \in \mathbb{N}\}$. Then $p(f) = p_k(f) = \sup_{n \in \mathcal{N}} \omega(n)^{-1} \|L^n f\|_\infty$. For instance, assume that $-L$ is the infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t)_{t>0}$ satisfying (CK). Set $\omega(k) = (k)^{-k}$. Then it is not hard to see (*e.g.*, [10, Lemma 4.5]) that for all t small enough, μ_t belongs to the corresponding space $\mathcal{S}(\mathcal{O})$, that is, $\|L^k \mu_t\|_\infty \leq C(t) k^k$.

(h) In the above examples, we can replace the family of left invariant vector fields X by its right invariant counter part $\check{X}$. One can also use both left and right vector fields.

We will need a local version of Definition 5.1. Fix an open set $\Omega \subset G$ and $B \subset \Omega$. Fix $k = 1, 2, \dots$. If U is a Bruhat distribution such that $QU \in \mathcal{C}(\Omega)$ for all $Q \in \mathcal{O}(k)$ then set

$$p_{B,k}(U) = \sup_{Q \in \mathcal{O}(k)} \|QU\|_{B,\infty} .$$

Definition 5.3 Let $\mathcal{O}$ be as in Definition 5.1. Fix an open set $\Omega \subset G$ and $k = 1, 2, \dots$. We say that a Bruhat distribution U belongs to $\mathcal{S}^k(\Omega, \mathcal{O})$ if $QU \in \mathcal{C}(\Omega)$ for all $Q \in \mathcal{O}(k)$ and there exists a sequence of functions $f_i \in \mathcal{S}^k(\mathcal{O})$ such that, for any compact $K \subset \Omega$,

$$\lim_{i \to \infty} p_{K,n}(U - f_i) = 0.$$

The following lemma relates the previous definition to the use of cut-off functions. We say that Q is local if for any Bruhat distribution U, $U = 0$ in an open set Ω implies $QU = 0$ in Ω.

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Lemma 5.4 *Fix $\mathcal{O}$ as in Definition* 5.1 *and $k = 0, 1, \ldots$.*

(i) *Assume that each $Q \in \mathcal{O}(n)$, $n = 0, 1, 2, \ldots$, is a local operator. Let U be a Bruhat distribution. If $\phi U \in \mathcal{S}^k(\mathcal{O})$ for all $\phi \in \mathcal{B}_0(\Omega)$ then $U \in \mathcal{S}^k(\Omega, \mathcal{O})$.*

(ii) *Assume that, for any open set Ω and any $\phi \in \mathcal{B}_0(\Omega)$, we have*

$$\forall f \in \mathcal{S}^k(\mathcal{O}), \quad \phi f \in \mathcal{S}^k(\mathcal{O}) \quad \textit{and} \quad p_k(\phi f) \leq C(\Omega, \phi) p_{\Omega,k}(f). \tag{5.1}$$

Then, for any open set Ω and any Bruhat distribution U,

$$U \in \mathcal{S}^k(\Omega, \mathcal{O}) \Rightarrow \forall \phi \in \mathcal{B}_0(\Omega), \quad \phi U \in \mathcal{S}^k(\mathcal{O}).$$

Proof Assume $\phi U \in \mathcal{S}^k(\mathcal{O})$ for any $\phi \in \mathcal{B}_0(\Omega)$. Exhaust Ω by an increasing sequence of open subsets Ω_i with $\overline{\Omega_i} \subset \Omega$. For each i, pick $\phi_i \in \mathcal{B}_0(\Omega)$ with $\phi_i \equiv 1$ on Ω_i. Then $f_i = \phi_i U \in \mathcal{S}^k(\mathcal{O})$, and for each compact $K \subset \Omega$, $p_{K,k}(U - f_i) = 0$ for i large enough (we use here that each Q is local). This proves (i).

Concerning (ii), observe that the assumption implies that $\mathcal{B}_0(G)$ acts on $\mathcal{S}^k(\mathcal{O})$ by pointwise multiplication. Suppose $U \in \mathcal{S}^k(\Omega, \mathcal{O})$ and let f_i be as in Definition 5.3. Let $\phi \in \mathcal{B}_0(\Omega)$. Then ϕ is supported in a compact $K \subset \Omega$. Let Ω_1 be a relatively compact neighborhood of K in Ω. By (5.1) we have

$$p_k(\phi(f_j - f_i)) \leq C(\Omega_1, \phi) p_{\Omega_1,k}(f_j - f_i).$$

By construction,

$$\lim_{i \to \infty} p_{\Omega_1,k}(U - f_i) = 0.$$

It follows that ϕf_i is a Cauchy sequence in $S^k(\mathcal{O})$. As $\mathcal{S}^k(\mathcal{O})$ is complete, there exists $v \in \mathcal{S}^k(\mathcal{O})$ such that $\lim_{i\to\infty} \phi f_i = v$ in $\mathcal{S}^k(\mathcal{O})$. By assumption, we also know that ϕf_i converges uniformly to ϕU. It follows that $\phi U = v \in \mathcal{S}^k(\mathcal{O})$ as desired. ∎

Example 5.5 Referring to Example 5.2(a), (b), (d) and (e), the hypothesis (5.1) of Lemma 5.4(2) is satisfied because these spaces are algebras. See [10, 12, 15]. In the case of Example 5.2(c), whether or not (5.1) is satisfied is not known in general. It is satisfied when $G = \mathbb{T}^\infty$, see [15]. In Example 5.2(f), (5.1) does not always hold. Indeed, as the constant function 1 is in $\mathcal{S}^k(\mathcal{O})$, it follows that (5.1) implies $\mathcal{B}(G) \subset \mathcal{S}^k(\mathcal{O})$. But in general, in Example 5.2(f), $\mathcal{B}(G) \not\subset \mathcal{S}^k(\mathcal{O})$.

5.2 Applications to Hypoellipticity

The following theorem provides a convenient way to obtain many hypoellipticity results.

Theorem 5.6 *Let $-L$ be the infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t^L)_{t>0}$. Let Z be a left invariant vector field and set $P = L + Z$. Let $\mathcal{O}$ be as in Definition* 5.1. *Assume that $L, Z, \mathcal{O}$ have the following properties:*

(i) *The semigroup $(\mu_t^L)_{t>0}$ satisfies* (CK$*$) *and Z satisfies* (3.1).

(ii) *For any* $Q \in \bigcup_0^\infty \mathcal{O}(k)$, *$Q$ is a left invariant differential operator of finite order and* $ZQ = QZ$.

(iii) *For any* $n = 0, 1, \dots$, *there exist* $m = m(n)$ *and* $C = C(n)$ *such that,*

$$\forall\, Q \in \mathcal{O}(n),\ \forall\, \phi \in \mathcal{T}_L, \quad Q\phi \in \mathcal{C}(G) \quad \textit{and} \quad \|Q\phi\|_\infty \le CM_L^m(\phi).$$

(iv) *For any* $Q \in \bigcup_0^\infty \mathcal{O}(n)$, $Q^*\mathcal{T}_L \subset \mathcal{T}_L$ *and, for any* $n, k = 0, 1, \dots$ *there exist* $m(n,k)$ *and* $C(n,k)$ *such that*

$$\forall\, Q \in \mathcal{O}(n),\ \forall\, \phi \in \mathcal{T}_L, \quad M_L^k(Q^*\phi) \le C(n,k)M_L^{m(n,k)}(\phi).$$

Fix $k = 1, 2, \dots$ *and an open set* Ω. *Let* $U, F \in \mathcal{T}_L'$ *be such that* $PU = F$ *in* $\mathcal{B}'(G)$. *Then* $F \in \mathcal{S}^k(\Omega, \mathcal{O})$ *implies* $U \in \mathcal{S}^k(\Omega, \mathcal{O})$.

Proof Let B and r be such that

$$\forall\, \phi \in \mathcal{T}_L, \quad \max\{|U(\phi)|, |F(\phi)|\} \le BM_L^r(\phi).$$

By hypothesis, for any n and any $Q \in \mathcal{O}(n)$

$$PQU = QF \text{ in } \mathcal{B}'(G).$$

Moreover, for any $Q \in \mathcal{O}(k)$, $QF \in \mathcal{C}(\Omega)$ and for any set Ω_0 with closure contained in Ω,

$$\|QF\|_{\Omega_0,\infty} \le p_{\Omega_0,k}(F).$$

Also, for $Q \in \mathcal{O}(k)$, $QU \in \mathcal{T}_L'$ because $QU(\phi) = U(Q^*\phi)$ and

$$M_L^q(Q^*\phi) \le C(k,q)M_L^{m(k,q)}(\phi)$$

with $C(k,q), m(k,q)$ as in hypothesis (iv) of the theorem. The same line of reasoning shows that $QF \in \mathcal{T}_L'$. Moreover, we have

$$\forall\, \phi \in \mathcal{T}_L, \quad \max\{|QU(\phi)|, |QF(\phi)|\} \le BC(k,r)M_L^{m(k,r)}(\phi).$$

Thus Theorem 3.8 shows that $QU \in \mathcal{C}(\Omega)$. Moreover, by Theorem 3.8, for any open sets Ω_0, Ω_1 with $\overline{\Omega_1} \subset \Omega_0$ and $\overline{\Omega_0} \subset \Omega$,

$$\forall\, t \in (0,1), \quad \|QU - (QU)^t\|_{\Omega_1,\infty} \le tC_1(\Omega_0, \Omega_1)(p_{\overline{\Omega_0},k}(F) + B)$$

where $U^t = U * \check{\mu}_t$, $(\mu_t)_{t>0}$ being the Gaussian convolution semigroup generated by $-P$ as in Section 3.1. By hypothesis (ii), $(QU)^t = QU^t$. Hence

$$\lim_{t \to 0} p_{\overline{\Omega_1},n}(U - U^t) = 0.$$

Since $U^t \in \mathcal{T}_L$ and, by hypothesis (iii), $\mathcal{T}_L \subset \mathcal{S}^k(\mathcal{O})$, this proves that $U \in \mathcal{S}^k(\Omega, \mathcal{O})$ as desired. ∎

Remarks

(1) The conclusion of Theorem 5.6 resembles closely $\mathcal{T}'_L$-$\mathcal{S}^k(\mathcal{O})$-hypoellipticity but is different in general. This is because the localization procedure used in Theorem 5.6 is not always equivalent to the localization by multiplication by functions in $\mathcal{B}(G)$ used in Definition 1.1. However, if we assume that (5.1) holds, then the conclusion of Theorem 5.6 is equivalent to saying that P is $\mathcal{T}'_L$-$\mathcal{S}^k(\mathcal{O})$-hypoelliptic. See the next two theorems.

(2) The hypothesis that the operators Q are differential operators of finite order is made for convenience only. It enables us to consider the adjoint Q^* without additional explanations.

(3) Note that the hypothesis that $ZQ = QZ$ is automatically satisfied if $Z = 0$.

The following results are corollaries of Lemma 5.4 and Theorem 5.6.

Theorem 5.7 *Assume that $(\mu_t)_{t>0}$ is a symmetric central Gaussian semigroup satisfying* (CK$*$). *Let $-L$ denotes its infinitesimal generator. Let $Y = (Y_i)_{i\in I}$ be a fixed projective family in $\mathfrak{G}$.*

(i) *Referring to Example* 5.2(e), *fix $p \in [1,\infty]$ and assume that for any $k \in \mathbb{N}$ there exist $C(k)$ and $n(k)$ such that*

$$S_Y^{p,k}(f) \leq C(k) M_L^{n(k)}(f), \tag{5.2}$$

Then L is $\mathcal{T}'_L$-$\mathcal{S}_Y^{p,k}$-hypoelliptic for each integer k (for $p = 2$, $\mathcal{S}_Y^{p,k} = \mathcal{S}_Y^k$).

(ii) *Assume that for any $k \in \mathbb{N}$, there exist $C(k)$ and $n(k)$ such that*

$$M_Y^k(f) \leq C(k) M_L^{n(k)}(f). \tag{5.3}$$

Then L is $\mathcal{T}'_L$-$\mathcal{T}_Y^k$-hypoelliptic for each integer k.

Note that (5.2) is automatically satisfied when L is related to Y by $L = -\sum Y_i^2$ and $p \in [2,\infty]$. Similarly, (5.3) is automatically satisfied when $L = -\sum Y_i^2$.

Proof Consider the case of the spaces $\mathcal{S}_Y^k$ (the case of $\mathcal{S}_Y^{p,k}$, $p \neq 2$, and $\mathcal{T}_Y^k$ are very similar). To apply Theorem 5.6 we need to check that (5.2) implies Theorem 5.6(i) and (ii). According to Example 5.2(b), it suffices to show that for any n, k and any $a = (a_\ell)_{I^k}$ with $\sum_\ell |a_\ell|^2 \leq 1$, we have

$$M_L^n(Qf) \leq C(n,k) M_L^{m(n,k)}(f)$$

where $Q = \sum_\ell a_\ell Y^\ell$. This can be proved by the argument of the proof of [15, Lemma 5.2], using the family of right invariant vectors $\check{X} = (\check{X}_i)$ corresponding to X. ∎

Next we consider the case of $P = L + Z$.

Theorem 5.8 *Let $-L$ be the bi-invariant infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t^L)_{t>0}$. Let Z be a left invariant vector field and set $P = L+Z$. Assume that $(\mu_t^L)_{t>0}$ satisfies* (CK$*$) *and that Z satisfies* (3.1).

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(i) *The operator P is is $\mathcal{T}'_L$-$\mathcal{S}^k_L$-hypoelliptic as well as $\mathcal{T}'_L$-$\mathcal{T}^k_L$-hypoelliptic for each* $k \in \{0, 1, \dots, \infty\}$.

(ii) *Fix an integer k and a real $p \in [1, \infty]$. Let $Y = (Y_i)_{i \in I}$ be a fixed special projective family (see Definition* 4.4*). If Y satisfies* (5.2), *(resp.,* (5.3)*), then the operator P is $\mathcal{T}'_L$-$\mathcal{S}^{p,k}_Y$-hypoelliptic (resp., $\mathcal{T}'_L$-$\mathcal{T}^k_Y$-hypoelliptic).*

Proof We need to modify the proof of Theorem 5.7 because the Q's used there do not necessarily commute with Z hence with P. To prove (i), observe that the spaces $\mathcal{S}^k_L$ can be defined as is in Example 5.2(e), but using right invariant vector fields (because L is bi-invariant). Thus we write $L = -\sum X_i^2$ where the $X = (X_i)_{i \in I}$ is a projective family of left invariant vector fields and consider the (right invariant) operators $Q^k_a = \sum_{I^k} a_\ell \check{X}^\ell$, $\sum_{I^k} a_\ell^2 \leq 1$. The advantage is that $Q = Q^k_a$ commutes with P since any left invariant operator commutes with any right invariant operator. All the other assumptions of Theorem 5.6 are trivially satisfied in this case. This proves (i) in the case of the spaces $\mathcal{S}^k_L$. The proof for $\mathcal{T}^k_L$ is similar. Note that this proof does not work for the spaces $\mathcal{S}^{p,k}_X$ of Example 5.2(e) with $p \neq 2$ because it is not clear that the right and left versions of these spaces coincide.

To prove (ii), we use the same trick as above (defining the relevant function spaces using right invariant vectors instead of left invariant vectors). This works well because we assume that the projective family is special and this implies that $\check{Y} = (\check{Y}_i)_I$ yields the same function spaces as Y. This ends the proof of Theorem 5.8. ■

5.3 Applications to Harmonic Functions

Let $-L$ be the infinitesimal generator of a Gaussian semigroup $(\mu_t)_{t>0}$. A distribution U is harmonic in a domain Ω if $LU = 0$ in Ω. In the present setting, an important question is to decide whether or not such a distribution can be represented in Ω by a continuous function. This is of course closely related to hypoellipticity. Let $\mathcal{A}$ be a space of distributions (in particular, $\mathcal{A}$ is stable by multiplication by Bruhat test functions). Following [12, Definition 1.4], we say that the operator L is $\mathcal{A}$-regular if for any Bruhat distribution U and domain Ω,

$$\forall\, \phi \in \mathcal{B}_0(\Omega), \phi U \in \mathcal{A} \text{ and } \phi L U = 0 \quad \Rightarrow \quad \forall\, \phi \in \mathcal{B}_0(\Omega), \phi U \in \mathcal{C}(G).$$

Under the mild assumptions that $\mathcal{S}$ is a topological space of continuous functions whose topology is not weaker than the uniform topology and that $\mathcal{C}(G) \subset \mathcal{A}$, [12, Proposition 3.4] asserts that L is $\mathcal{A}$-$\mathcal{S}$-hypoelliptic if and only if L is $\mathcal{A}$-regular. By Theorem 4.2 and the result of [12], we obtain the following statement.

Theorem 5.9 *Let $-L$ be the infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t)_{t>0}$. Then L is $\mathcal{T}'_L$-regular if and only if $(\mu_t)_{t>0}$ satisfies* (CK$*$).

By the closed graph theorem, hypoellipticity translates easily into more quantitative statements such as the following corollary of Theorem 5.7.

Theorem 5.10 *Assume that $(\mu_t)_{t>0}$ is a symmetric central Gaussian semigroup satisfying* (CK$*$). *Denote by $-L$ its infinitesimal generator and let X be a projective family* *such that $L = -\sum_I X_i^2$. Fix a domain $\Omega \subset G$ and a compact subset $K \subset \Omega$. Let $u \in L^1(\Omega)$ be a solution of $Lu = 0$ in Ω in the sense of Bruhat distributions. Then, for any $k \in \mathbb{N}$ there exists a constant $C = C(L, \Omega, K, k)$ such that*

$$\sup_K \Big(\sum_{\ell \in I^k} |X^\ell u|^2 \Big)^{1/2} \le C \int_\Omega |u| \, d\nu.$$

A more subtle question concerning harmonic functions in a domain Ω is to ask in which sense does Lu exist and vanish in Ω. More precisely, assume that $L = -\sum_1^\infty X_i^2$. The best one might hope for is that for every compact $K \subset \Omega$, $\sum_1^\infty \sup_K |X_i^2 u| < \infty$. A slightly weaker statement is that $Lu = -\sum_1^\infty X_i^2 u$ converges absolutely locally uniformly in Ω. We can only prove the following even weaker result.

Theorem 5.11 *Assume that $(\mu_t)_{t>0}$ is a symmetric central Gaussian semigroup satisfying* (CK$*$). *Denote by $-L$ its infinitesimal generator and let X be a projective family such that $L = -\sum_I X_i^2$. Fix a domain $\Omega \subset G$, and a continuous harmonic function u in Ω. Then, for any compact subset K of Ω, $\sum_I X_i^2 u$ is summable to* 0 *in $\mathcal{C}(K)$.*

Proof First we need some simple facts concerning the Riesz transform in $L^2(G)$. Let $L_0^2 = L_0^2(G)$ be the orthogonal complement of the constants in $L^2(G)$. On L_0^2, $L^{-1/2}$ is well defined by spectral theory. The i-th Riesz transform is $R_i = X_i L^{-1/2}$ (here X_i and $L^{-1/2}$ commute because L is bi-invariant). The vector Riesz transform of a function $f \in L_0^2$ is $Rf = (R_i f)_I$. By integration by parts, we have

$$\forall\, f \in L_0^2, \quad \sum_1^\infty \|R_i f\|_2^2 = \|f\|_2^2. \tag{5.4}$$

For any finite set $J \subset I$, let $R_J f = (R_i f)_{i \in J}$. It is clear that for any fixed $f \in L_0^2$ and for any $\epsilon > 0$ there exists a finite set J_0 such that, for any finite set $J \subset I \setminus J_0$, we have

$$\sum_{j \in J} \|R_i f\|_2^2 < \epsilon. \tag{5.5}$$

Let u be harmonic in Ω. Let K be compact subset of Ω and Ω_1 be an open set such that $K \subset \Omega_1$ and $\overline{\Omega_1} \subset \Omega$. Let $\phi \in \mathcal{B}_0(\Omega)$ with $\phi \equiv 1$ on Ω_1. Set $v = \phi u - \int_G \phi u \, d\nu$. Observe that v is a continuous function in L_0^2 and that, by the hypoellipticity results of Theorem 5.7, $Lv = (L\phi)u + \phi u + \Gamma(\phi, u)$ is also a continuous function in L_0^2. We claim that the series $\sum X_i^2 v$ is summable in $L^2(G)$. To see this, let w be an arbitrary function in L_0^2 and compute $|\int w(\sum_J X_i^2 v)\, d\nu|$. We have

$$\begin{aligned}
\Big| \int w(\sum_J X_i^2 v) \, d\nu \Big| &= \Big| \sum_J \int v(X_i^2 L^{-1} L v) \, d\nu \Big| \\
&= \Big| \int \sum_J (X_i L^{-1/2} w)(X_i L^{-1/2} L v) \, d\nu \Big| \\
&\le \Big(\int \sum_J |R_i w|^2 \, d\nu \Big)^{1/2} \Big(\int \sum_J |R_i L v|^2 \, d\nu \Big)^{1/2}.
\end{aligned}$$

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Now, by (5.4), $\int \sum_J |R_i w|^2 \, d\nu \leq \|w\|_2^2$. Hence

$$\Big\| \sum_J X_i^2 v \Big\|_2^2 \leq \sum_J \|R_i L v\|_2^2.$$

As Lv is in L_0^2, (5.5) shows that $\sum_I X_i^2 v$ is summable in $L^2(G)$. To finish the proof of the theorem, we observe that for any finite set J, $\sum_J X_i^2 v$ is a harmonic function in Ω_1 (recall that L commutes with each X_i). Applying Theorem 5.10, we obtain

$$\begin{aligned}\sup_{x\in K}\Big|\sum_J X_i^2 u\Big| &= \sup_{x\in K}\Big|\sum_J X_i^2 v\Big| \leq C\int_{\Omega_1}\Big|\sum_J X_i^2 v\Big|\,d\nu\\ &\leq C\Big(\int\Big|\sum_J X_i^2 v\Big|^2 d\nu\Big)^{1/2} = C\Big\|\sum_J X_i^2 v\Big\|_2.\end{aligned}$$

As $\sum X_i^2 v$ is summable in $L^2(G)$ it follows that $\sum X_i^2 u$ is summable in $\mathcal{C}(K)$. ∎

6 Examples

6.1 Laplacians on $\mathbf{T} = \mathbb{T}^\infty$

Let $G = \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ where $\mathbf{R} = \mathbb{R}^\infty$ and $\mathbf{Z} = \mathbb{Z}^\infty$. Thus, $\mathbf{T}$ is the countable product of circle groups, each isomorphic to $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For the following discussion it is important to observe that the product structure is not intrinsically attached to $\mathbf{T}$. More precisely, writing $\mathbf{T}$ as an infinite product is equivalent to choosing a projective basis $E = (E_i)$ of its Lie algebra $\mathbf{R} = \mathbb{R}^\infty$, with the property that

$$\mathbf{Z} = \Big\{ z = \sum z_i E_i : z_i \in \mathbb{Z} \Big\}. \tag{6.1}$$

Once this product structure is fixed, $E_i = \partial_i$ can be identified with partial differentiation in the i-th coordinate. Any symmetric non-degenerate Gaussian semigroup $(\mu_t)_{t>0}$ is determined by a symmetric positive definite matrix $A = (a_{i,j})$, so that

$$L = -\sum_{i,j} a_{i,j}\partial_i\partial_j.$$

See [3, 4, 7, 14]. Set $\mathbf{Z}_0 = \Big\{ z = \sum z_i E_i : z_i \in \mathbb{Z},\ z_i = 0 \text{ for all but finitely many } i \Big\}$. Define

$$W(s) = \#\{\theta \in \mathbf{Z}_0 : \sum a_{i,j}\theta_i\theta_j < s\}. \tag{6.2}$$

Then by [7, Theorem 5.13], the semigroup $(\mu_t)_{t>0}$ has the property (CK$*$) if and only if

$$\lim_{s\to\infty} s^{-1/2}\log W(s) = 0. \tag{6.3}$$

By Theorems 5.7 and 1.5, it follows that L is $\mathcal{T}'_L$-$\mathcal{S}^k_L$-hypoelliptic, (resp., $\mathcal{T}'_L$-$\mathcal{T}^k_L$-hypoelliptic) if and only if (6.3) holds. It should be noted that this statement is rather abstract in general since the condition (6.3) is hard to check, and the spaces $\mathcal{S}_L$, $\mathcal{T}_L$ and $\mathcal{T}'_L$ do not have very explicit descriptions in terms of the matrix A. From this viewpoint it is worth mentioning that (6.3) is equivalent to $\mathcal{M}(G)$-$\mathcal{C}(G)$-hypoellipticity. We are going now to describe more explicit examples.

Example 6.1 (The symmetric diagonal case) Assume that E is a projective basis of $\mathbf{R}$ such that (6.1) holds. We say that $(\mu_t)_{t>0}$ is E-diagonal if A is a diagonal matrix with $a_{i,i} = a_i$, that is,

$$L = -\sum a_i \partial_i^2.$$

In what follows we always assume that $a_i > 0$ for each i. Much is known about the properties of $(\mu_t)_{t>0}$ in this case. See [3, 7, 16]. In particular, $\mathcal{H}(L)$ is the Hilbert space contained in $\mathbf{R}$ with orthonormal Hilbert basis

$$(a_i^{1/2}\partial_i)_{i\in I}.$$

Instead of the function W defined at (6.2), it is easier to work with

$$N(s) = \#\{i : a_i \le s\}. \tag{6.4}$$

In terms of the function N, the E-diagonal semigroup $(\mu_t)_{t>0}$ associated to the sequence (a_i) is (CK$*$) if and only if

$$N(s) = o(s) \text{ at infinity.}$$

Let $\mathcal{S}$ be one of the spaces $\mathcal{C}^k_E$, $\mathcal{S}^k_L$ or $\mathcal{T}^k_L$, $k = 0, 1, 2, \dots, \infty$. Then $\mathcal{T}'_L$-$\mathcal{S}$-hypoellipticity holds if and only if $N(s) = o(s)$ at infinity. This improves upon the result in [3] by enlarging the distribution space from $\mathcal{M}(\mathbf{T})$ (Borel measures) to $\mathcal{T}'_L$ and allowing different spaces of smooth functions.

For any integers k, r, [15, Proposition 5.5] gives

$$\forall\, \phi \in \mathcal{B}(\mathbf{T}), \quad \|\partial_{\ell_1}\cdots\partial_{\ell_k}\phi\|_\infty \le a_{\ell_1}^{-r/2}\cdots a_{\ell_k}^{-r/2} S_L^{kr}(\phi). \tag{6.5}$$

The condition $N(s) = o(s)$ at infinity implies that $\sum a_i^{-1-\epsilon} < \infty$ for any $\epsilon > 0$. This and (6.5) show that

$$(\mathrm{CK}*) \Rightarrow \mathcal{S}^\infty_L = \mathcal{T}_L = \mathcal{D}^\infty_L.$$

We now record some further consequences of (6.5).

Theorem 6.2 *Let $L = -\sum a_i\partial_i^2$ be such that $N(s) = o(s)$ at infinity. Let $X = (X_i)$ be a projective family with $X_i = \sum x_{i,j}\partial_j$.*

(i) *Assume that for each i there exists $n_i > 0$ such that*

$$\sum_j |x_{i,j}| a_j^{-n_i} < \infty.$$

Then $X_i\mathcal{T}_L \subset \mathcal{T}_L$ and L is $\mathcal{T}'_L$-$\mathcal{C}^k_X$-hypoelliptic.

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(ii) *Let $\mathcal{S}$ be one of the spaces $\mathcal{S}_X^k$ or $\mathcal{T}_X^k$, $k = 0, 1, 2, \dots, \infty$. Assume that there exists $n > 0$ such that*

$$\sum_j \Big(\sum_i |x_{i,j}|^2 \Big)^{1/2} a_j^{-n} < \infty.$$

Then $\mathcal{T}_L'$-$\mathcal{S}$-hypoelliptic.

Proof Part (i) follows from (6.5) and Theorem 4.2. Part (ii) requires more work. To apply Theorem 5.7, it suffices to show that (a) $P = \sum X_i^2$ satisfies $P\mathcal{T}_L \subset \mathcal{T}_L$ and (b) for each integer k, there exists C and m such that

$$\Big\| \sum_{\ell \in I^k} b_\ell X^\ell \phi \Big\|_\infty \leq C M_L^m(\phi)$$

for all (b_ℓ) with $\sum_{I^k} b_\ell^2 \leq 1$. By (6.5), we have

$$\begin{aligned} |P\phi| = \Big| \sum_i^\infty X_i^2 \phi \Big| &\leq \sum_{j_1, j_2} \Big(\sum_i |x_{i,j_1} x_{i,j_2}| \Big) |\partial_{j_1} \partial_{j_2} \phi| \\ &\leq \sum_{j_1, j_2} \Big(\sum_i |x_{i,j_1} x_{i,j_2}| \Big) a_{j_1}^{-n} a_{j_2}^{-n} S_L^{4n}(\phi) \\ &\leq \Big[\sum_j \Big(\sum_i |x_{i,j}|^2 \Big)^{1/2} a_j^{-n} \Big]^2 S_L^{4n}(\phi). \end{aligned}$$

This proves that P has L-finite order. A similar argument shows that for any (b_ℓ) with $\sum_{I^k} b_\ell^2 \leq 1$, we have

$$\Big| \sum_{\ell \in I^k} b_\ell X^\ell \phi \Big| \leq \Big[\sum_j \Big(\sum_i |x_{i,j}|^2 \Big)^{1/2} a_j^{-n} \Big]^k S_L^{2kn}(\phi).$$

This proves (b). Theorem 6.2(2) then follows from Theorem 5.7. ∎

The following corollary of Theorem 5.7 and (6.5) should be compared with the much weaker but more general statement obtained in Theorem 5.11.

Theorem 6.3 *Let $L = -\sum a_i \partial_i^2$ be such that $N(s) = o(s)$ at infinity. Fix a domain $\Omega \subset \mathbf{T}$. Let u be a continuous function in Ω such that $Lu = 0$ in Ω (in the sense of distributions). Then for any integers k, n and any compact $K \subset \Omega$, there exists a constant $C(k, n, K)$ such that*

$$\forall\, \ell \in I^k, \quad \sup_K |\partial_{\ell_1} \cdots \partial_{\ell_k} u| \leq C(k, n, K) [a_{\ell_1} \cdots a_{\ell_k}]^{-n} \sup_\Omega \{|u|\}.$$

In particular, for any $\epsilon > 0$, the series $\sum_i |a_i \partial_i^2 u|^\epsilon$ converges locally uniformly in Ω.

Example 6.4 (The diagonal case with a first order term) Let us now consider some examples of diagonal but not symmetric Gaussian semigroups on $\mathbf{T}$. As above, let $E = (E_i)$, $E_i = \partial_i$. Let

$$L = -\sum a_i \partial_i^2, \quad Z = \sum b_i \partial_i, \quad P = L + Z.$$

Let N be defined as in (6.4). Theorems 4.6, 5.8, and 6.5 give the following result.

Theorem 6.5 *Assume that $N(s) = o(s)$ at infinity and that there exists an integer n such that $\sum |b_i| a_i^{-n} < \infty$. Let $\mathcal{S}$ be one of the spaces $\mathcal{C}_E^k$, $\mathcal{S}_L^k$, $\mathcal{T}_L^k$. Then the operator P is $\mathcal{T}_E'$-$\mathcal{S}$-hypoelliptic.*

Example 6.6 (Hidden diagonal cases) Recognizing whether or not there is a projective basis of $\mathbf{T}$ in which a given Gaussian semigroup $(\mu_t)_{t>0}$ is diagonal is not an easy problem. See the discussion in [14]. To illustrate this, consider an increasing sequence $(\alpha_i)_1^\infty$, $\alpha_i > 0$, and form the matrix $A = (a_{i,j})$ with $a_{i,j} = \min\{\alpha_i, \alpha_j\}$, that is,

$$A = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 & \cdots \\ \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \cdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_3 & \cdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We will verify below that A is positive definite. Let $(\mu_t)_{t>0}$ be the symmetric Gaussian semigroup associated to $L = -\sum_{i,j} a_{i,j} \partial_i \partial_j$. If we set

$$X_i = \sum_{j \geq i} E_i, \quad X = (X_i),$$

then X is a projective basis and, setting $\alpha_0 = 0$,

$$L = -\sum_i (\alpha_i - \alpha_{i-1}) X_i^2.$$

This shows that A is positive definite. Moreover, the lattice $\mathbf{Z}$ defining $\mathbf{T}$ equals the X-integer lattice

$$\mathbf{Z}_X = \left\{ z = \sum z_i X_i \in \mathbf{R} : z_i \in \mathbf{Z} \right\}.$$

Thus, viewed in the product structure of $\mathbf{T}$ induced by X, $(\mu_t)_{t>0}$ is diagonal, that is, $(\mu_t)_{t>0}$ is X-diagonal with coefficients $a_i = (\alpha_i - \alpha_{i-1})$. By Example 6.1 above, it follows that $(\mu_t)_{t>0}$ satisfies (CK$*$) if and only if

$$N(s) = \#\{i : \alpha_i - \alpha_{i-1} \leq s\} = o(s).$$

For instance, if $\alpha_i = i^\beta$ for some $\beta > 0$, then (CK$*$) holds if and only if $\beta > 2$. In terms of hypoellipticity, we see that L is $\mathcal{T}_L'$-$\mathcal{C}_X^k$-hypoelliptic if and only if $N(s) = o(s)$. Observe that the projective basis X is very different from our original basis E. However, for each i, $E_i = X_i - X_{i+1}$. Hence, $E_i \in \mathcal{H}_L$. By Theorem 4.2 and Lemma 4.3, we see that L is also $\mathcal{T}_L'$-$\mathcal{C}_E^k$-hypoelliptic.

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Example 6.7 Fix $\alpha \geq 0$, $\sigma \in \mathbb{R}$, and consider the tridiagonal matrix $A = (a_{i,j})$ with $a_{i,j} = a_{j,i}$ given for $i \leq j$ by

$$a_{i,j} = \begin{cases} i^{2\alpha} + \sigma^2(i+1)^{2\alpha} & \text{if } j = i, \\ \sigma i^{2\alpha} & \text{if } j = i+1, \\ 0 & \text{if } j > i+1. \end{cases}$$

We easily compute a projective basis $X = (X_i)$ in which $L = -\sum_{i,j} a_{i,j}\partial_i\partial_j$ has the form $L = -\sum X_i^2$. It suffices to take

$$X_i = i^\alpha(E_i + \sigma E_{i+1}) = i^\alpha E_i^\sigma \text{ where } E_i^\sigma = E_i + \sigma E_{i+1}.$$

Case 1 Let us first consider the case where $\sigma \in \mathbb{Z}$. In this case,

$$\mathbf{Z}_\sigma = \left\{ \sum z_i E_i^\sigma : (z_i) \in \mathbb{Z}^\infty \right\} = \mathbf{Z} \subset \mathbf{R}.$$

It follows that the semigroup $(\mu_t)_{t>0}$ is E^σ-diagonal with diagonal coefficients $i^{2\alpha}$. Thus this semigroup satisfies (CK$*$) if and only if $\alpha > 1/2$. It follows that L is $\mathcal{T}'_L$-$\mathcal{C}^\infty_{E^\sigma}$-hypoelliptic if and only if $\alpha > 1/2$. Observe that

$$E_i = \sum_{j \geq i}^{\infty} \frac{(-\sigma)^{j-1}}{j^\alpha} X_j. \tag{6.6}$$

Thus, if $\alpha > 1/2$ and $\sigma = 0, \pm 1$, then $E_i \in \mathcal{H}(L)$. In this case we see that L is $\mathcal{T}'_L$-$\mathcal{C}^\infty_E$-hypoelliptic. If instead σ is an integer with $|\sigma| > 1$, then $E_i \notin \mathcal{H}(L)$ and, in fact, is not of finite L-order. Thus we cannot conclude that L is $\mathcal{T}'_L$-$\mathcal{C}^\infty_E$-hypoelliptic.

Case 2 In general, if σ is not an integer, we do not see how to express $(\mu_t)_{t>0}$ as a product semigroup (and one certainly should expect that, for some σ, this really cannot be done). However, $(\mu_t)_{t>0}$ can be compared efficiently with a diagonal semigroup as follows. Let D be the diagonal matrix with (i,i) entry $d_i = a_{i,i} = i^{2\alpha} + \sigma^2(i+1)^{2\alpha}$. Then, for all $\xi \in \mathbb{R}^{(\infty)}$,

$$\sum_i d_i\xi_i^2 - 2\sum_{i<j} |a_{i,j}||\xi_i||\xi_j| \leq \sum_{i,j} a_{i,j}\xi_i\xi_j \leq \sum_i d_i\xi_i^2 + 2\sum_{i<j} |a_{i,j}||\xi_i||\xi_j|.$$

For $j > i+1$, $a_{i,j} = 0$, and

$$|a_{i,i+1}| \leq \frac{|\sigma|}{1+\sigma^2}\big(|a_{i,i}||a_{i+1,i+1}|\big)^{1/2}.$$

Thus

$$\sum_{i<j} |a_{i,j}||\xi_i||\xi_j| \leq \frac{|\sigma|}{1+\sigma^2} \sum_i (|a_{i,i}||a_{i+1,i+1}|)^{1/2}|\xi_i||\xi_{i+1}|.$$

By Cauchy–Schwarz inequality, this yields

$$\Big(1-\frac{2|\sigma|}{1+\sigma^2}\Big)\sum_i d_i\xi_i^2 \le \sum_{i,j} a_{i,j}\xi_i\xi_j \le \Big(1+\frac{2|\sigma|}{1+\sigma^2}\Big)\sum_i d_i\xi_i^2.$$

It follows that, assuming as we may that $|\sigma| \neq 1$, the semigroup (μ_t) satisfies (CK$*$) if and only if $\#\{i : d_i \le s\} = o(s)$, that is, if and only if $\alpha > 1/2$. Thus, $\mathcal{T}'_L$-$\mathcal{C}^\infty_{E^\sigma}$-hypoellipticity holds for any σ and any $\alpha > 1/2$ and does not hold if $\alpha \le 1/2$. By (6.6), $\mathcal{T}'_L$-$\mathcal{C}^\infty_E$-hypoellipticity also holds if $|\sigma| < 1$ and $\alpha > 1/2$.

6.2 Semisimple Groups

Recall that a compact connected group G is called semisimple (see [27, Definition 9.5]) if its commutator $G' = [G, G]$ is equal to G itself. In the case of compact connected Lie groups, this definition coincides with other classical definitions. It is proved in [8] that many questions about symmetric central Gaussian semigroups on general compact connected groups can be split into a purely abelian part and a purely semisimple part. In particular, whether or not a symmetric central Gaussian semigroup satisfies the property (CK$*$) can be reduced to purely abelian and semisimple parts. See [13] for the treatment of some explicit examples. In this section, we describe in concrete terms how the hypoellipticity results of this paper apply in the case of semisimple groups.

Let G be a compact connected metrizable semisimple group. Then there exists a sequence (Σ_i) of compact simple Lie groups and a closed central subgroup H of $\Sigma = \prod \Sigma_i$ such that

$$G \cong \Sigma/H.$$

Since the center of Σ is a product of finite groups, H is totally disconnected and the projective Lie algebra $\mathfrak{G}$ of G equals the projective Lie algebra of Σ, that is, equals the product of the Lie algebras $\mathfrak{S}_i$ of the simple Lie groups Σ_i. Because G and Σ have the same projective Lie algebra, there is a natural one-to-one correspondence between finite order left invariant differential operators on G and on Σ. By Theorem 2.2, this induces a one-to-one correspondence between Gaussian convolution semigroups on G and on Σ. For each i, denote by Δ_i the Laplacian on Σ_i induced by the Killing form on $\mathfrak{S}_i$. This Laplacian Δ_i is sometimes called the Casimir operator. Up to scalar multiplication, it is the unique bi-invariant second order differential operator without constant term on Σ_i. For each i, let n_i be the dimension of Σ_i. Set $J_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i\}$. Denote by $(E_j)_{j\in J_i}$ a basis of $\mathfrak{S}_i$ such that

$$\Delta_i = -\sum_{j\in J_i} E_j^2.$$

Denote by E the projective basis of $\mathfrak{G}$ formed by the vectors E_j, $j = 1, \ldots$.

We can now describe the set of all symmetric central Gaussian semigroups on G. Namely, there is a one-to-one correspondence between symmetric central Gaussian

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semigroups $(\mu_t)_{t>0}$ on G and sequences $a = (a_i)$ of non-negative numbers such that the infinitesimal generator of $(\mu_t)_{t>0}$ is given by

$$L = \sum a_i \Delta_i = -\sum_i \sum_{j \in J_i} a_i E_j^2.$$

Set $\widetilde{a}_j = a_i$ if $j \in J_i$ and $\widetilde{A}_\ell = \widetilde{a}_{\ell_1} \cdots \widetilde{a}_{\ell_k}$ if $\ell = (\ell_1, \dots, \ell_k) \in \{1, 2, \dots\}^k$. In this notation,

$$L = -\sum_i a_i \Delta_i = -\sum_j \widetilde{a}_j E_j^2.$$

Note that the vectors $a_i^{1/2} E_j$, $j \in J_i, i = 1, 2, \dots$ form an orthonormal basis of $\mathcal{H}(L)$ and a special projective basis in the sense of Definition 4.4. Set

$$N(s) = \sum_{i: a_i \le s} n_i. \tag{6.7}$$

Then it is proved in [5] that $(\mu_t)_{t>0}$ satisfies (CK$*$) if and only if

$$N(s) = o(s) \text{ at infinity.} \tag{6.8}$$

In this setting, Theorems 4.2 and 5.7 give the following statement.

Theorem 6.8 *Referring to the notation introduced above, let $\mathcal{S}$ be one of the spaces $\mathcal{C}_E^k$, $\mathcal{S}_L^k$, $\mathcal{T}_L^k$. Then L is $\mathcal{T}_L'$-$\mathcal{S}$-hypoelliptic if and only if* (6.8) *holds true.*

The results of [15, §5] imply that assuming that N satisfies (6.8) (in particular, $n_i = O(a_i)$), we have

$$\forall\, \ell \in \{1, 2, \dots\}^k, \forall\, \phi \in \mathcal{S}_L^\infty, \quad \|E^\ell \phi\|_\infty \le C_{k,m}(\phi) A_\ell^{-m}.$$

This immediately implies that $\mathcal{S}_L^\infty = \mathcal{T}_L$ and yields the following results.

Theorem 6.9 *Let L be as above and assume that* (6.8) *is satisfied. Let $X = (X_i)$ be a projective family with $X_i = \sum x_{i,j} E_j$.*

(i) *Assume that for each i there exists $r_i > 0$ such that*

$$\sum_m (\max_{j \in J_m} |x_{i,j}|) a_m^{-r_i} < \infty.$$

Then $X_i \mathcal{T}_L \subset \mathcal{T}_L$ and L is $\mathcal{T}_L'$-$\mathcal{C}_X^k$-hypoelliptic.

(ii) *Let $\mathcal{S}$ be one of the spaces $\mathcal{S}_X^k$ or $\mathcal{T}_X^k$, $k = 0, 1, 2, \dots, \infty$. Assume that there exist $r > 0$ such that*

$$\sum_m \big(\max_{j \in J_m} |x_{i,j}| \big) a_m^{-r} < \infty.$$

Then L is $\mathcal{T}_L'$-$\mathcal{S}$-hypoelliptic.

(iii) *Let $Z = \sum z_i E_i$ and assume that there exist $s > 0$ such that*

$$\sum_m \big(\max_{j \in J_m} |z_j|\big)\, a_m^{-s} < \infty.$$

Then $L + Z$ is $\mathcal{T}'_L$-$\mathcal{C}^k_E$-hypoelliptic. Moreover, if X and $\mathcal{S}$ are as in (ii) *above, $L + Z$ is $\mathcal{T}'_L$-$\mathcal{S}$-hypoelliptic.*

The proof is similar to that of Theorem 6.2 (the first part of the statement in (iii) uses the fact that E is a special projective basis). We omit the details.

Theorem 6.10 *Let L be as above and assume that* (6.8) *is satisfied. Fix a domain $\Omega \subset G$. Let u be a continuous function in Ω such that $Lu = 0$ in Ω (in the sense of distributions). Then for any integers k, n and any compact $K \subset \Omega$, there exists a constant $C(k, n, K)$ such that*

$$\forall\, \ell \in \{1, 2, \dots\}^k, \quad \sup_K |E^\ell u| \le C(k, n, K)\widetilde{A}_\ell^{-n} \sup_\Omega \{|u|\}.$$

In particular, for any $\epsilon > 0$, the series $\sum_i |\widetilde{a}_i E_i^2 u|^\epsilon$ converges locally uniformly in Ω.

References

[1] D. Bakry *Transformation de Riesz pour les semigroupes symétriques. I. Étude de la dimension* 1. Séminaire de Probabilités XIX, Lecture Notes in Mathematics 1123, Springer, Berlin, 1985, pp. 130–174.

[2] A. Bendikov *Spatially homogeneous continuous Markov processes on abelian groups and harmonic structures.* (Russian) Uspehi Mat. Nauk **29**(1974), no. 5, 215–216.

[3] ———, *Potential Theory on Infinite-Dimensional Abelian Groups.* Walter De Gruyter, Berlin, 1995.

[4] ———, *Symmetric stable semigroups on the infinite dimensional torus.* Expo. Math. **13**(1995), 39–79.

[5] A. Bendikov A. and L. Saloff-Coste, *Elliptic diffusions on infinite products.* J. Reine Angew. Math. **493**(1997), 171–220.

[6] ———, *Potential theory on infinite products and locally compact groups.* Potential Anal. **11**(1999), no. 4, 325–358.

[7] ———, *On- and off-diagonal heat kernel behaviors on certain infinite dimensional local Dirichlet spaces.* Amer. J. Math. **122**(2000), no. 6, 1205–1263.

[8] ———, *Central Gaussian semigroups of measures with continuous density.* J. Funct. Anal. **186**(2001), no. 1, 206–268.

[9] ———, *On the absolute continuity of Gaussian measures on locally compact groups.* J. Theoret. Probab. **14**(2001), no. 3, 887–898.

[10] ———, *Gaussian bounds for derivatives of central Gaussian semigroups on compact groups.* Trans. Amer. Math. Soc. **354**(2001), no. 4, 1279–1298.

[11] ———, *Invariant local Dirichlet forms on locally compact groups.* Ann. Fac. Sci. Toulouse Math. **11**(2002), no. 3, 303–349.

[12] ———, *On the hypoellipticity of sub-Laplacians on infinite dimensional compact groups.* Forum Math. **15**(2003), no. 1, 135–163.

[13] ———, *Brownian motions on compact groups of infinite dimension.* In: Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces. Contemp. Math. 338, American Mathematical Society, Providence, RI, 2003, pp. 41–63.

[14] ———, *Central Gaussian convolution semigroups on compact groups: a survey.* Infin. Dimens. Anal. Quantum Probab. Rel. Top. **6**(2003), 629–659.

[15] ———, *Spaces of smooth functions and distributions on infinite dimensional compact groups.* J. Funct. Anal. **218**(2005), 168–218.

[16] C. Berg, *Potential theory on the infinite dimensional torus.* Invent. Math. **32**(1976), no. 1, 49–100.

[17] J. M. Bony, *Opérateurs elliptiques dégénérés associés aux axiomatiques de la théorie du potentiel.* In: Potential Theory. Edizioni Cremonese, Rome, 1970, pp. 69–119.

[18] È. Born, *Projective Lie algebra bases of a locally compact group and uniform differentiability.* Math. Z. **200**(1989), no.2, 279–292.

[19] ———, *An Explicit Lévy-Hinčin formula for convolution semigroups on locally compact groups.* J. Theoret. Probab. **2**(1989), no. 3, 325–342.

[20] N. Bourbaki, *Espaces vectoriels topologiques.* In: lments de mathmatique. Ch 1–5, Masson, Paris, 1981.

[21] F. Bruhat, *Distributions sur un groupe localement compact et application à l'étude des représentations des groupes p-adiques.* Bull. Soc. Math. France **89**(1961), 43–75.

[22] E. B. Davies, *Heat kernels and spectral theory.* Cambridge Tracts in Mathematics 92, Cambridge, Cambridge University Press, 1989.

[23] ———, *Non-Gaussian aspects of heat kernel behaviour.* J. London Math. Soc. **55**(1997), no. 1, 105–125.

[24] M. Fukushima, Y. Ōshima, and M. Takeda, *Dirichlet forms and Symmetric Markov processes,* de Gruyter Studies in Mathematics 19, W. De Gruyter, Berlin, 1994.

[25] V. N. Glushkov, *The structure of locally compact groups and Hilbert's Fifth Problem.* AMS Translations 15, 1960, 55–94.

[26] H. Heyer, *Probability Measures on Locally Compact Groups.* Ergebnisse der Mathematik und ihrer Grenzgebieter 94, Springer-Verlag, Berlin, 1977.

[27] K. Hofmann and S. Morris, *The structure of compact groups. A primer for the student—a handbook for the expert.* de Gruyter Studies in Mathematics 25. W. de Gruyter, Berlin, 1998.

[28] L. Hörmander, *Hypoelliptic second order differential equations.* Acta Math. **119**(1967), 147–171.

[29] G. A. Hunt, *Semi-groups of measures on Lie groups.* Trans. Amer. Math. Soc. **81**(1956), 264–293.

[30] S. Kusuoka and D. Stroock, *Applications of the Malliavin calculus, Part II.* J. Fac. Sci., Univ. Tokyo Sect. IA Math. **32**(1985), no. 1, 1–76.

[31] M. Ledoux, *L'algèbre de Lie des gradients itérés d'un générateur Markovien—développements de moyennes et entropies.* Ann. Sci. École Norm. Sup. (4) **28**(1995), no. 4, 435–460.

[32] E. Siebert, *Absolute continuity, singularity and supports of Gaussian semigroups on a Lie group.* Monats. Math. **93**(1982), no. 3, 239–253.

[33] K.-T. Sturm, *On the geometry defined by Dirichlet forms.* In: Seminar on Stochastic Processes, Random Fields and Applications, Progr. Probab. 36, Birkhäuser, Basel, 1995, pp. 231–242.

[34] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations.* Graduate Texts in Mathematics 102, Springer-Verlag, New York, 1984.

[35] N. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and geometry on groups.* Cambridge Tracts in Mathematics 100, Cambridge, Cambridge University Press, 1993.

Instytut Matematyczny
Uniwersytetu Wrocławskiego
Poland
e-mail: bendikov@math.uni.wroc.pl

Department of Mathematics
Cornell University
Ithaca, NY 14853-4201
U.S.A.
e-mail: lsc@math.cornell.edu