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ON THE DIVISIBILITY OF SUMS OF q-SUPER CATALAN NUMBERS

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Abstract

The integrality of the numbers $A_{n,m} = (2n)!\,(2m)!/n!\,m!\,(n+m)!$ was observed by Catalan as early as 1874 and Gessel named $A_{n,m}$ the super Catalan numbers. The positivity of the q-super Catalan numbers (q-analogue of the super Catalan numbers) was investigated by Warnaar and Zudilin ['A q-rious positivity', *Aequationes Math.* **81** (2011), 177–183]. We prove the divisibility of sums of q-super Catalan numbers, which establishes a q-analogue of Apagodu's congruence involving super Catalan numbers.

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1. Introduction

The Catalan numbers, given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0,$$

occur in various counting problems. For instance, C_n is the number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal, and is also the number of permutations of $\{1, \ldots, n\}$ that avoid the permutation pattern 123, that is, with no three-term increasing subsequence. We refer to [12] for many different combinatorial interpretations of the Catalan numbers.

Although the Catalan numbers naturally arise in combinatorics, they also possess rich arithmetic properties. One of the remarkable examples is the following congruence due to Sun and Tauraso [13]:

$$\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2}\left(\frac{p}{3}\right) - \frac{1}{2} \pmod{p^2}. \tag{1.1}$$

Here and in what follows, $p \geq 5$ is a prime and $(\frac{\cdot}{\cdot})$ denotes the Legendre symbol. We remark that Tauraso [14, Theorem 6.1] established an interesting q-analogue of

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the modulo p version of (1.1), which was further generalised to a q-analogue of the modulo p^2 version by the first author [9, Theorem 1].

In 1874, Catalan [3] observed that the numbers $A_{n,m} = (2n)!\,(2m)!/n!\,m!\,(n+m)!$ are always integral. Since $A_{n,1}/2$ coincides with the Catalan number C_n, these $A_{n,m}$ were named the super Catalan numbers by Gessel [6]. The integrality of $A_{n,m}$ can also be deduced from Von Szily's identity [15]:

$$A_{n,m} = \sum_{k=-\infty}^{\infty} (-1)^k \binom{2n}{n+k}\binom{2m}{m+k}. \tag{1.2}$$

There are interpretations of $A_{n,m}$ for some special values of m (see, for example, [1, 4, 11]). However, it is still an open problem to find a general combinatorial interpretation for the super Catalan numbers.

In 2018, Apagodu [2, Conjecture 2] proposed two conjectural congruences on double sums of super Catalan numbers, one of which is

$$\sum_{i=0}^{p-1}\sum_{j=0}^{p-1} A_{i,j} \equiv \left(\frac{p}{3}\right) \pmod p. \tag{1.3}$$

The first author [8] confirmed the conjectural congruence (1.3) using combinatorial identities which were proved by Zeilberger's algorithm [10].

It is natural to consider the q-counterpart for $A_{n,m}$. The q-super Catalan numbers are defined as

$$A_{n,m}(q) = \frac{[2n]!\,[2m]!}{[n]!\,[m]!\,[n+m]!},$$

where the q-factorial $[n]! = \prod_{k=1}^{n}(1-q^k)/(1-q)$. Warnaar and Zudilin [16] obtained the remarkable result that the $A_{n,m}(q)$ are polynomials with nonnegative coefficients (positive polynomials) and Guo *et al.* [7] obtained another interesting positivity result related to $A_{n,m}(q)$.

As mentioned in [16], one can obtain a q-analogue of Von Szily's identity (1.2),

$$A_{n,m}(q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}+k^2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}\begin{bmatrix} 2m \\ m+k \end{bmatrix}, \tag{1.4}$$

by taking $(a,b,c) \mapsto (1,\infty,q^{-m})$ in the very-well poised ${}_6\phi_5$ summation [5, (II.21)]. Here and throughout the paper, the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

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where the q-shifted factorials are given by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geq 1$ and $(a;q)_0 = 1$. The nth cyclotomic polynomial is given by

$$\Phi_n(q) = \prod_{\substack{1\leq k\leq n\\(n,k)=1}} (q-\zeta^k),$$

where ζ denotes a primitive nth root of unity.

Our interest concerns a q-analogue of Apagodu's congruence (1.3) as follows.

THEOREM 1.1. *For any positive integer n, modulo $\Phi_n(q)$,*

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q^{i+j}A_{i,j}(q) \equiv \begin{cases} \left(\frac{n}{3}\right)q^{n-1} & \text{if } n \not\equiv 0 \pmod 3\\ q^{n/3-1}(1-q^{n/3}) & \text{if } n \equiv 0 \pmod 3. \end{cases} \tag{1.5}$$

We remark that letting $n = p$ and $q \to 1$ in (1.5) leads us to (1.3). The proof of (1.5) relies on (1.4) and the following q-congruence due to Tauraso [14, Corollary 4.3]:

$$\sum_{i=0}^{n-1} q^i \begin{bmatrix} 2i\\ i+k \end{bmatrix} \equiv \left(\frac{n-k}{3}\right) q^{3r(r+1)/2+k(2r+1)} \pmod{\Phi_n(q)}, \tag{1.6}$$

where k is a nonnegative integer and $r = \lfloor 2(n-k)/3 \rfloor$.

As we will see, the proof of (1.5) is more natural than that of (1.3) in [8] and avoids using some exotic combinatorial identities. We shall present the proof of Theorem 1.1 in the next section.

2. Proof of Theorem 1.1

By (1.4),

$$\begin{aligned}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q^{i+j}A_{i,j}(q) &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q^{i+j} \sum_{k=-j}^{j} (-1)^k q^{\binom{k}{2}+k^2} \begin{bmatrix} 2j\\ j+k \end{bmatrix}\begin{bmatrix} 2i\\ i+k \end{bmatrix}\\ &= \sum_{k=1-n}^{n-1} (-1)^k q^{\binom{k}{2}+k^2} \sum_{i=0}^{n-1} q^i \begin{bmatrix} 2i\\ i+k \end{bmatrix} \sum_{j=0}^{n-1} q^j \begin{bmatrix} 2j\\ j+k \end{bmatrix}\\ &= \sum_{k=1-n}^{n-1} (-1)^k q^{\binom{k}{2}+k^2} \left(\sum_{i=0}^{n-1} q^i \begin{bmatrix} 2i\\ i+k \end{bmatrix} \right)^2. \end{aligned} \tag{2.1}$$

Let

$$a(k) = (-1)^k q^{\binom{k}{2}+k^2} \left(\sum_{i=0}^{n-1} q^i \begin{bmatrix} 2i\\ i+k \end{bmatrix} \right)^2$$

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and

$$b(k) = a(-k) = (-1)^k q^{\binom{k+1}{2}+k^2}\Bigg(\sum_{i=0}^{n-1} q^i \begin{bmatrix} 2i \\ i+k \end{bmatrix}\Bigg)^2.$$

We split the sum on the right-hand side of (2.1) into two pieces:

$$S_1 = \sum_{k=0}^{n-1} a(k) \quad \text{and} \quad S_2 = \sum_{k=1}^{n-1} b(k),$$

so that

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q^{i+j} A_{i,j}(q) = S_1 + S_2. \tag{2.2}$$

From (1.6), we deduce that

$$a(k) \equiv (-1)^k q^{\binom{k}{2}+k^2}\Big(\frac{n-k}{3}\Big)^2 q^{3r(r+1)+2k(2r+1)} \pmod{\Phi_n(q)} \tag{2.3}$$

and

$$b(k) \equiv (-1)^k q^{\binom{k+1}{2}+k^2}\Big(\frac{n-k}{3}\Big)^2 q^{3r(r+1)+2k(2r+1)} \pmod{\Phi_n(q)}, \tag{2.4}$$

where $r = \lfloor 2(n-k)/3 \rfloor$.

Next, we shall distinguish three cases to prove (1.5).

Case 1: $n \equiv 1 \pmod 3$.

If $k = 3m$, then $((n-k)/3) = 1$ and $r = 2(n-1)/3 - 2m$. It follows from (2.3) and (2.4) that

$$a(k) \equiv (-1)^m q^{2(2n+1)(n-1)/3+3m(m-1)/2} \pmod{\Phi_n(q)} \tag{2.5}$$

and

$$b(k) \equiv (-1)^m q^{2(2n+1)(n-1)/3+3m(m+1)/2} \pmod{\Phi_n(q)}. \tag{2.6}$$

If $k = 3m+1$, then $((n-k)/3) = 0$, and so

$$a(k) \equiv b(k) \equiv 0 \pmod{\Phi_n(q)}. \tag{2.7}$$

If $k = 3m+2$, then $((n-k)/3) = -1$ and $r = 2(n-1)/3 - 2m - 1$, and so

$$a(k) \equiv (-1)^m q^{2n(2n+1)/3+(m+1)(3m-2)/2} \pmod{\Phi_n(q)} \tag{2.8}$$

and

$$b(k) \equiv (-1)^m q^{2n(2n+1)/3+(m+2)(3m+1)/2} \pmod{\Phi_n(q)}. \tag{2.9}$$

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Combining (2.5)–(2.9) gives modulo $\Phi_n(q)$,

$$a(k) \equiv \begin{cases} (-1)^m q^{2(2n+1)(n-1)/3+3m(m-1)/2} & \text{if } k = 3m \\ 0 & \text{if } k = 3m+1 \\ (-1)^m q^{2n(2n+1)/3+(m+1)(3m-2)/2} & \text{if } k = 3m+2 \end{cases} \tag{2.10}$$

and

$$b(k) \equiv \begin{cases} (-1)^m q^{2(2n+1)(n-1)/3+3m(m+1)/2} & \text{if } k = 3m \\ 0 & \text{if } k = 3m+1 \\ (-1)^m q^{2n(2n+1)/3+(m+2)(3m+1)/2} & \text{if } k = 3m+2. \end{cases} \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\begin{aligned} S_1 = \sum_{k=0}^{n-1} a(k) &= \sum_{m=0}^{\lfloor (n-1)/3 \rfloor} a(3m) + \sum_{m=0}^{\lfloor (n-2)/3 \rfloor} a(3m+1) + \sum_{m=0}^{\lfloor (n-3)/3 \rfloor} a(3m+2) \\ &\equiv q^{2(2n+1)(n-1)/3} \sum_{m=0}^{(n-1)/3} (-1)^m q^{3m(m-1)/2} \\ &\quad + q^{2n(2n+1)/3} \sum_{m=0}^{(n-4)/3} (-1)^m q^{(m+1)(3m-2)/2} \pmod{\Phi_n(q)} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} S_2 = \sum_{k=1}^{n-1} b(k) &= \sum_{m=1}^{\lfloor (n-1)/3 \rfloor} b(3m) + \sum_{m=0}^{\lfloor (n-2)/3 \rfloor} b(3m+1) + \sum_{m=0}^{\lfloor (n-3)/3 \rfloor} b(3m+2) \\ &\equiv q^{2(2n+1)(n-1)/3} \sum_{m=1}^{(n-1)/3} (-1)^m q^{3m(m+1)/2} \\ &\quad + q^{2n(2n+1)/3} \sum_{m=0}^{(n-4)/3} (-1)^m q^{(m+2)(3m+1)/2} \pmod{\Phi_n(q)}. \end{aligned} \tag{2.13}$$

Combining (2.2), (2.12) and (2.13), we arrive at

$$\begin{aligned} &\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{i+j} A_{i,j}(q) \\ &\equiv q^{2(2n+1)(n-1)/3} \bigg(\sum_{m=0}^{(n-1)/3} (-1)^m q^{3m(m-1)/2} + \sum_{m=1}^{(n-1)/3} (-1)^m q^{3m(m+1)/2} \bigg) \\ &\quad + q^{2n(2n+1)/3} \bigg(\sum_{m=0}^{(n-4)/3} (-1)^m q^{(m+1)(3m-2)/2} + \sum_{m=0}^{(n-4)/3} (-1)^m q^{(m+2)(3m+1)/2} \bigg) \pmod{\Phi_n(q)}. \end{aligned}$$

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Noting that

$$\sum_{m=0}^{(n-1)/3}(-1)^m q^{3m(m-1)/2} = -\sum_{m=-1}^{(n-4)/3}(-1)^m q^{3m(m+1)/2}$$

and

$$\sum_{m=0}^{(n-4)/3}(-1)^m q^{(m+2)(3m+1)/2} = -\sum_{m=1}^{(n-1)/3}(-1)^m q^{(m+1)(3m-2)/2},$$

we obtain

$$\begin{aligned}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q^{i+j}A_{i,j}(q) &\equiv q^{2n(2n+1)/3-1} - (-1)^{(n-1)/3}q^{(n+1)(3n-2)/2} + (-1)^{(n-1)/3}q^{(n-1)(3n+2)/2}\\ &= q^{2n(2n+1)/3-1} + (-1)^{(n-1)/3}q^{(n-1)(3n+2)/2}(1-q^n)\\ &\equiv q^{-1} \equiv q^{n-1} \pmod{\Phi_n(q)},\end{aligned}$$

where we have used the fact $q^n \equiv 1 \pmod{\Phi_n(q)}$. This completes the proof of the case $n \equiv 1 \pmod 3$ of (1.5).

Case 2: $n \equiv 2 \pmod 3$.

By using the same method as in the previous case, we can evaluate $a(k)$ and $b(k)$ modulo $\Phi_n(q)$:

$$a(k) \equiv \begin{cases}(-1)^m q^{2(2n-1)(n+1)/3+3m(m-1)/2} & \text{if } k = 3m\\ (-1)^{m+1} q^{2n(2n-1)/3+(3m+2)(m-1)/2} & \text{if } k = 3m+1\\ 0 & \text{if } k = 3m+2\end{cases}$$

and

$$b(k) \equiv \begin{cases}(-1)^m q^{2(2n-1)(n+1)/3+3m(m+1)/2} & \text{if } k = 3m\\ (-1)^{m+1} q^{2n(2n-1)/3+(3m+5)m/2} & \text{if } k = 3m+1\\ 0 & \text{if } k = 3m+2.\end{cases}$$

It follows that

$$\begin{aligned}S_1 = \sum_{k=0}^{n-1} a(k) &= \sum_{m=0}^{\lfloor (n-1)/3\rfloor} a(3m) + \sum_{m=0}^{\lfloor (n-2)/3\rfloor} a(3m+1) + \sum_{m=0}^{\lfloor (n-3)/3\rfloor} a(3m+2)\\ &\equiv q^{2(2n-1)(n+1)/3}\sum_{m=0}^{(n-2)/3}(-1)^m q^{3m(m-1)/2}\\ &\quad - q^{2n(2n-1)/3}\sum_{m=0}^{(n-2)/3}(-1)^m q^{(3m+2)(m-1)/2} \pmod{\Phi_n(q)}\end{aligned}$$

and

$$S_2 = \sum_{k=1}^{n-1} b(k) = \sum_{m=1}^{\lfloor (n-1)/3 \rfloor} b(3m) + \sum_{m=0}^{\lfloor (n-2)/3 \rfloor} b(3m+1) + \sum_{m=0}^{\lfloor (n-3)/3 \rfloor} b(3m+2)$$
$$\equiv q^{2(2n-1)(n+1)/3} \sum_{m=1}^{(n-2)/3} (-1)^m q^{3m(m+1)/2}$$
$$- q^{2n(2n-1)/3} \sum_{m=0}^{(n-2)/3} (-1)^m q^{(3m+5)m/2} \pmod{\Phi_n(q)}.$$

Thus,

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{i+j} A_{i,j}(q) = S_1 + S_2$$
$$\equiv q^{2(2n-1)(n+1)/3} \Big(\sum_{m=0}^{(n-2)/3} (-1)^m q^{3m(m-1)/2} + \sum_{m=1}^{(n-2)/3} (-1)^m q^{3m(m+1)/2} \Big)$$
$$- q^{2n(2n-1)/3} \Big(\sum_{m=0}^{(n-2)/3} (-1)^m q^{(3m+2)(m-1)/2} + \sum_{m=0}^{(n-2)/3} (-1)^m q^{(3m+5)m/2} \Big) \pmod{\Phi_n(q)}. \tag{2.14}$$

Furthermore, note that

$$\sum_{m=0}^{(n-2)/3} (-1)^m q^{3m(m-1)/2} = - \sum_{m=-1}^{(n-5)/3} (-1)^m q^{3m(m+1)2} \tag{2.15}$$

and

$$\sum_{m=0}^{(n-2)/3} (-1)^m q^{(3m+2)(m-1)/2} = - \sum_{m=-1}^{(n-5)/3} (-1)^m q^{(3m+5)m/2}. \tag{2.16}$$

Combining (2.14)–(2.16) gives

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{i+j} A_{i,j}(q) \equiv (-1)^{(n-2)/3} q^{(n+1)(3n-2)/2} - q^{2n(2n-1)/3-1} - (-1)^{(n-2)/3} q^{(3n+2)(n-1)/2}$$
$$= -q^{2n(2n-1)/3-1} + (-1)^{(n-2)/3} q^{(3n+2)(n-1)/2} (q^n - 1)$$
$$\equiv -q^{-1} \equiv -q^{n-1} \pmod{\Phi_n(q)},$$

which is the case $n \equiv 2 \pmod 3$ of (1.5).

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Case 3: $n \equiv 0 \pmod 3$.

By using the same method as in the first case, we find that modulo $\Phi_n(q)$,

$$a(k) \equiv \begin{cases} 0 & \text{if } k = 3m \\ (-1)^{m+1} q^{2n(2n+1)/3+(3m+2)(m-1)/2} & \text{if } k = 3m+1 \\ (-1)^m q^{2n(2n-1)/3+(3m-2)(m+1)/2} & \text{if } k = 3m+2 \end{cases}$$

and

$$b(k) \equiv \begin{cases} 0 & \text{if } k = 3m \\ (-1)^{m+1} q^{2n(2n+1)/3+m(3m+5)/2} & \text{if } k = 3m+1 \\ (-1)^m q^{2n(2n-1)/3+(m+2)(3m+1)/2} & \text{if } k = 3m+2. \end{cases}$$

It follows that

$$\begin{aligned} S_1 = \sum_{k=0}^{n-1} a(k) &= \sum_{m=0}^{\lfloor (n-1)/3 \rfloor} a(3m) + \sum_{m=0}^{\lfloor (n-2)/3 \rfloor} a(3m+1) + \sum_{m=0}^{\lfloor (n-3)/3 \rfloor} a(3m+2) \\ &\equiv q^{2n(2n-1)/3} \sum_{m=0}^{(n-3)/3} (-1)^m q^{(3m-2)(m+1)/2} \\ &\quad - q^{2n(2n+1)/3} \sum_{m=0}^{(n-3)/3} (-1)^m q^{(3m+2)(m-1)/2} \pmod{\Phi_n(q)} \end{aligned}$$

and

$$\begin{aligned} S_2 = \sum_{k=1}^{n-1} b(k) &= \sum_{m=1}^{\lfloor (n-1)/3 \rfloor} b(3m) + \sum_{m=0}^{\lfloor (n-2)/3 \rfloor} b(3m+1) + \sum_{m=0}^{\lfloor (n-3)/3 \rfloor} b(3m+2) \\ &\equiv q^{2n(2n-1)/3} \sum_{m=0}^{(n-3)/3} (-1)^m q^{(m+2)(3m+1)/2} \\ &\quad - q^{2n(2n+1)/3} \sum_{m=0}^{(n-3)/3} (-1)^m q^{m(3m+5)/2} \pmod{\Phi_n(q)}. \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{i+j} A_{i,j}(q) = S_1 + S_2 \\ &\equiv q^{2n(2n-1)/3} \bigg(\sum_{m=0}^{(n-3)/3} (-1)^m q^{(3m-2)(m+1)/2} + \sum_{m=0}^{(n-3)/3} (-1)^m q^{(m+2)(3m+1)/2} \bigg) \\ &\quad - q^{2n(2n+1)/3} \bigg(\sum_{m=0}^{(n-3)/3} (-1)^m q^{(3m+2)(m-1)/2} + \sum_{m=0}^{(n-3)/3} (-1)^m q^{m(3m+5)/2} \bigg) \pmod{\Phi_n(q)}. \end{aligned} \tag{2.17}$$

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Furthermore, note that

$$\sum_{m=0}^{(n-3)/3}(-1)^m q^{(3m+2)(m-1)/2} = -\sum_{m=-1}^{(n-6)/3}(-1)^m q^{m(3m+5)/2} \tag{2.18}$$

and

$$\sum_{m=0}^{(n-3)/3}(-1)^m q^{(3m-2)(m+1)/2} = -\sum_{m=-1}^{(n-6)/3}(-1)^m q^{(3m+1)(m+2)/2}. \tag{2.19}$$

Finally, combining (2.17)–(2.19), we arrive at

$$\begin{aligned}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} & q^{i+j}A_{i,j}(q)\\ &\equiv q^{2n(2n-1)/3-1} - q^{2n(2n+1)/3-1} + (-1)^{n/3}q^{(n+1)(3n-2)/2} + (-1)^{(n-3)/3}q^{(n-1)(3n+2)/2}\\ &= q^{n(4n-3)/3+n/3-1}(1-q^{n+n/3}) - (-1)^{n/3}q^{(n-1)(3n+2)/2}(1-q^n)\\ &\equiv q^{n/3-1}(1-q^{n/3}) \pmod{\Phi_n(q)},\end{aligned}$$

which confirms the case $n \equiv 0 \pmod 3$ of (1.5).

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