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Some Properties of Triebel–Lizorkin and Besov Spaces Associated with Zygmund Dilations

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Abstract. In this paper, using Calderón's reproducing formula and almost orthogonality estimates, we prove the lifting property and the embedding theorem of the Triebel–Lizorkin and Besov spaces associated with Zygmund dilations.

1 Introduction and Statement of Main Results

Ricci and Stein [9] introduced a class of singular integral operators $\mathcal{T}_Z$ associated with Zygmund dilations and showed that $\mathcal{T}_Z$ is bounded on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$. When the weight ω satisfies an analogous condition of Muckenhoupt weight associated with Zygmund dilations, Fefferman and Pipher [2] further studied that $\mathcal{T}_Z$ is bounded on $L^p_\omega(\mathbb{R}^3)$ for all $1 < p < \infty$.

Applying the discrete Calderón's reproducing formula and Littlewood–Paley–Stein theory, Han and Lu [7] introduced Hardy spaces $H^p_Z(\mathbb{R}^3)$ and Carleson measure spaces $CMO^p_Z(\mathbb{R}^3)$ related to Zygmund dilations. For $0 < p \le 1$, they presented that the boundedness of $\mathcal{T}_Z$ from $H^p_Z(\mathbb{R}^3)$ to $H^p_Z(\mathbb{R}^3)$ and from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$. Liao and Liu [8] extended the boundedness of $\mathcal{T}_Z$ to the Triebel–Lizorkin and Besov spaces associated with Zygmund dilations.

The main aim of this note is to present the lifting property and the embedding theorem of the Triebel–Lizorkin and Besov spaces associated with Zygmund dilations that were constructed in [8].

Throughout the paper, we use C to denote positive constants, whose value may change from one occurrence to the next. Constants with subscripts, such as C_1, do not change in different settings. We write $f \sim g$ if there exists a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Let χ_R denote the characteristic function of R.

Suppose that $\mathcal{S}$ denotes the Schwartz function space. Let $\psi^{(1)} \in \mathcal{S}(\mathbb{R})$ and satisfy

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}x)|^2 = 1$$

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for all $x \in \mathbb{R}\backslash\{0\}$, and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$ and satisfy

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}y, 2^{-k}z)|^2 = 1$$

for all $(y,z) \in \mathbb{R}^2\backslash\{(0,0)\}$, and the cancellation conditions

$$\int_{\mathbb{R}} \psi^{(1)}(x)x^{\alpha}\,dx = \int_{\mathbb{R}^2} \psi^{(2)}(y,z)y^{\beta}z^{\gamma}\,dy\,dz = 0$$

for all nonnegative integers α, β, and γ.

By taking the Fourier transform, it is easy to see that the following continuous Calderón's reproducing formula holds on $L^2(\mathbb{R}^3)$,

$$f(x,y,z) = \sum_{j,k\in\mathbb{Z}} \psi_{j,k} * \psi_{j,k} * f(x,y,z),$$

where

$$\psi_{j,k}(x,y,z) = 2^{2(j+k)}\psi^{(1)}(2^j x)\psi^{(2)}(2^k y, 2^{j+k}z). \tag{1.1}$$

We now recall the product test function on $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$.

Definition 1.1 ([7]) A Schwartz function $f(x)$ defined on $\mathbb{R}^3$ is said to be a *test function* in $\mathcal{S}_Z(\mathbb{R}^3)$ if $f \in \mathcal{S}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}} f(x,y,z)x^{\alpha}\,dx = \int_{\mathbb{R}^2} f(x,y,z)y^{\beta}z^{\gamma}\,dy\,dz = 0$$

for all indices α, β, γ of nonnegative integers.

We endow $\mathcal{S}_Z(\mathbb{R}^3)$ with the same topology as $\mathcal{S}(\mathbb{R}^3)$. We denote by $(\mathcal{S}_Z(\mathbb{R}^3)'$ the dual of $\mathcal{S}_Z(\mathbb{R}^3)$.

The definitions of the Triebel–Lizorkin and Besov spaces associated with Zygmund dilations are as follows.

Definition 1.2 ([8]) Let $0 < p, q < \infty$, $s = (s_1, s_2) \in \mathbb{R}^2$ and let $\psi_{j,k}$ be the same as in (1.1) for $j, k \in \mathbb{Z}$. The Triebel–Lizorkin space $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$ associated with Zygmund dilations is the collection of all $f \in (\mathcal{S}_Z(\mathbb{R}^3))'$ such that

$$\|f\|_{\dot{F}^{s,q}_{p,Z}} = \Big\|\Big\{\sum_{j,k\in\mathbb{Z}}\Big(2^{js_1}2^{ks_2}|\psi_{j,k} * f|\Big)^q\Big\}^{1/q}\Big\|_p < \infty.$$

The Besov space $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$ associated with Zygmund dilations is the collection of all $f \in (\mathcal{S}_Z(\mathbb{R}^3))'$ such that

$$\|f\|_{\dot{B}^{s,q}_{p,Z}} = \Big\{\sum_{j,k\in\mathbb{Z}}\Big(2^{js_1}2^{ks_2}\|\psi_{j,k} * f\|_p\Big)^q\Big\}^{1/q} < \infty.$$

Using Calderón's reproducing formula associated with Zygmund dilations and almost orthogonality estimates, the authors [8] proved that the spaces in Definition 1.2 are independent of the choice of $\psi_{j,k}$. A rectangle $R = I \times J \times K$ on $\mathbb{R}^3$ is called a *Zygmund rectangle* if

$$|I| = 2^{-j}, \quad |J| = 2^{-k}, \quad \text{and} \quad |K| = 2^{-(j+k)} \tag{1.2}$$

for some $j,k\in\mathbb{Z}$. In what follows, let $\mathcal{R}_Z^{-j,-k}$ be the set of Zygmund rectangles that satisfies (1.2) for given $j,k\in\mathbb{Z}$, and let $\mathcal{R}_Z$ be the set of all Zygmund rectangles.

Lemma 1.3 ([7]) *Given any large positive integer N, suppose that $\psi_{j,k}$ is the same as in (1.1) for $j,k\in\mathbb{Z}$. Let $x_R\in R$ and $x\in\mathbb{R}^3$. Then Calderón's reproducing formulas*

$$f(x)=\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_Z^{-j-N,-k-N}}|R|\psi_{j,k}(x-x_R)(\psi_{j,k}*f)(x_R), \tag{1.3}$$

$$f(x)=\sum_{j,k\in\mathbb{Z}}\psi_{j,k}*\psi_{j,k}*f(x) \tag{1.4}$$

converge in both norm of $\mathcal{S}_Z(\mathbb{R}^3)$ and $(\mathcal{S}_Z(\mathbb{R}^3))'$.

Lemma 1.4 ([7]) *Suppose that $\psi_{j,k}$ and $\phi_{j',k'}$ are as in (1.1) for $j,k,j',k'\in\mathbb{Z}$. Then*

$$|\psi_{j,k}*\phi_{j',k'}(x,y,z)|\le C2^{-|j-j'|L}2^{-|k-k'|L}\frac{2^{-M(j\wedge j')}}{(2^{-(j\wedge j')}+|x|)^{1+M}}\frac{2^{-M(k\wedge k')}}{2^{-j^*}(2^{-(k\wedge k')}+|y|+2^{j^*}|z|)^{2+M}}$$

for any $L,M>0$, where $j^*=j$ if $k\le k'$ and $j^*=j'$ if $k>k'$.

From Lemmas 1.3 and 1.4, the authors in [8] also obtained the following conclusion.

Lemma 1.5 *Let $0<p,q<\infty$ and $s=(s_1,s_2)\in\mathbb{R}^2$. Then we have*

$$\|f\|_{\dot F^{s,q}_{p,Z}}\sim\Big\|\Big\{\sum_{j,k\in\mathbb{Z}}\Big(\sum_{R\in\mathcal{R}_Z^{-j-N,-k-N}}|R|2^{js_1}2^{ks_2}|(\psi_{j,k}*f)(x_R)|\chi_R\Big)^q\Big\}^{1/q}\Big\|_p,$$

$$\|f\|_{\dot B^{s,q}_{p,Z}}\sim\Big\{\sum_{j,k\in\mathbb{Z}}\Big(2^{js_1}2^{ks_2}\Big\|\sum_{R\in\mathcal{R}_Z^{-j-N,-k-N}}|R||(\psi_{j,k}*f)(x_R)|\chi_R\Big\|_p\Big)^q\Big\}^{1/q},$$

where $x_R,\psi_{j,k},R,N$ are the same as in Lemma 1.3.

We now introduce the Riesz potential related to Zygmund dilations as follows. For more results about the Riesz potential, see [3–5, 11].

Definition 1.6 Let $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ be as in (1.1) with

$$\operatorname{supp}\widehat{\psi^{(1)}}(x)\subset\{x:1/2\le|x|<1\},$$
$$\operatorname{supp}\widehat{\psi^{(2)}}(y,z)\subset\{(y,z):1/2\le|(y,z)|<1\}.$$

Set $\alpha=(\alpha_0,\alpha_1)\in\mathbb{R}^2$. For $f\in(\mathcal{S}_Z(\mathbb{R}^3))'$. Then the *Riesz potential* I_α is defined by

$$I_\alpha(f)(x,y,z)=\sum_{l_1,l_2\in\mathbb{Z}}2^{-l_1\alpha_0}2^{-l_2\alpha_1}(\psi_{l_1,l_2}*f)(x,y,z)$$

for $(x,y,z)\in\mathbb{R}^3$.

The following result is the lifting property of the Triebel–Lizorkin and Besov spaces associated with Zygmund dilations.

Theorem 1.7 *Let N_0 be any given nonnegative integer and $0 < p, q < \infty$. Suppose that $s = (s_0, s_1) \in \mathbb{R}^2$ and $\alpha = (\alpha_0, \alpha_1)$ with $|\alpha_0| < N_0$, $|\alpha_1| < N_0$.*

(i) *For all $f \in \dot{F}^{s,q}_{p,Z}$, we have $\|f\|_{\dot{F}^{s,q}_{p,Z}} \sim \|I_\alpha(f)\|_{\dot{F}^{s+\alpha,q}_{p,Z}}$.*

(ii) *For all $f \in \dot{B}^{s,q}_{p,Z}$, we have $\|f\|_{\dot{B}^{s,q}_{p,Z}} \sim \|I_\alpha(f)\|_{\dot{B}^{s+\alpha,q}_{p,Z}}$.*

Remark 1.8 Yang [13] introduced the Riesz potential related to flag singular integrals and showed the lifting property of Triebel–Lizorkin and Besov spaces associated with flag singular integrals. In that paper, $|\alpha_0|$ and $|\alpha_1|$ must tend to zero. In this note, we only need to assume that $|\alpha_0|$ and $|\alpha_1|$ are less than any given positive integer. Thus, the range of α_0, α_1 in this paper is larger than the related parameters in [13].

The embedding theorem of the Triebel–Lizorkin and Besov spaces associated with Zygmund dilations can be stated as follows. For embedding theorem on the Triebel–Lizorkin and Besov spaces in the setting of single parameter, see [1, 6, 10, 12].

Theorem 1.9 *Suppose that $a = (a_0, a_1)$ and $s = (s_0, s_1)$ with $-\infty < s_0 < a_0 < \infty$, $-\infty < s_1 < a_1 < \infty$, $-\infty < a_0 - 2/p_0 = s_0 - 2/p_1$, $a_1 - 2/p_0 = s_1 - 2/p_1 < \infty$. Then*

(i) $\dot{F}^{a,q_0}_{p_0,Z} \to \dot{F}^{s,q_1}_{p_1,Z}$ *for $1 < p_0, p_1, q_0, q_1 < \infty$;*

(ii) $\dot{B}^{a,q}_{p_0,Z} \to \dot{B}^{s,q}_{p_1,Z}$ *for $1 < p_0, p_1, q < \infty$.*

2 Proof of Theorem 1.7

Let us first formulate some basic results before we verify Theorem 1.7. The maximal operator M_Z associated with Zygmund dilations is defined by

$$M_Z(f)(x) := \sup_{R \in \mathcal{R}_Z, R \ni x} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

The following lemma is the Fefferman–Stein vector-valued maximal inequality associated with Zygmund dilations.

Lemma 2.1 ([8]) *Suppose that $1 < p, q < \infty$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^3} \Big(\sum_{j \in \mathbb{Z}} |M_Z(f_j)(x)|^q \Big)^{p/q} dx \le C \int_{\mathbb{R}^3} \Big(\sum_{j \in \mathbb{Z}} |f_j(x)|^q \Big)^{p/q} dx.$$

We now describe a fundamental estimate as follows. We omit the proof here.

Proposition 2.2 *Let $j, j', k, k' \in \mathbb{Z}$ and $1 \le p < \infty$. For any $M > 0$, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^3} \Big[\frac{2^{-M(j\wedge j')}}{(2^{-(j\wedge j')} + |x_1|)^{1+M}} \frac{2^{-M(k\wedge k')}}{2^{-j*}(2^{-(k\wedge k')} + |x_2| + 2^{j*}|x_3|)^{2+M}} \Big]^p dx_1\, dx_2\, dx_3 \le \frac{1}{2^{-(j\wedge j')(p-1)}} \frac{1}{2^{-j*(p-1)} 2^{-2(k\wedge k')(p-1)}},$$

where $j^* = j$ if $k \le k'$ and $j^* = j'$ if $k' < k$.

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The next result is the almost orthogonality estimate.

Proposition 2.3 *Suppose that ψ_{j_i,k_i} is the same as in (1.1) with $j_i, k_i \in \mathbb{Z}$ for $i = 1,2,3,4$. Then we have*

$$
\begin{aligned}
(2.1)\qquad &|\psi_{j_1,k_1} * \psi_{j_2,k_2} * \psi_{j_3,k_3} * \psi_{j_4,k_4}(x,y,z)| \\
&\le C 2^{-[|j_1-j_2|+|j_2-j_3|+|j_3-j_4|+|j_1-j_3|+|j_1-j_4|+|j_2-j_4|]L} \\
&\quad\times 2^{-[|k_1-k_2|+|k_2-k_3|+|k_3-k_4|+|k_1-k_3|+|k_1-k_4|+|k_2-k_4|]L} \\
&\quad\times \frac{2^{-(j_1\wedge j_2\wedge j_3\wedge j_4)M}}{(2^{-(j_1\wedge j_2\wedge j_3\wedge j_4)}+|x|)^{1+M}} \frac{2^{-(k_1\wedge k_2\wedge k_3\wedge k_4)M}}{2^{-j^*}(2^{-(k_1\wedge k_2\wedge k_3\wedge k_4)}+|y|+2^{j^*}|z|)^{2+M}}
\end{aligned}
$$

for any positive integers M and L, where, for $i = 1,2,3,4$, if $\min\{k_1,k_2,k_3,k_4\} = k_i$, then $j^* = j_i$.

Proof Let $\psi_j^{(1)}(x) = 2^j\psi^{(1)}(2^j x)$, $\psi_{j,k}^{(2)}(y,z) = 2^{2k+j}\psi^{(2)}(2^k y, 2^{k+j}z)$ and write

$$
\begin{aligned}
&\psi_{j_1,k_1} * \psi_{j_2,k_2} * \psi_{j_3,k_3} * \psi_{j_4,k_4}(x,y,z) \\
&\quad= \big(\psi_{j_1}^{(1)} * \psi_{j_2}^{(1)} * \psi_{j_3}^{(1)} * \psi_{j_4}^{(1)}(x)\big) \times \big(\psi_{j_1,k_1}^{(2)} * \psi_{j_2,k_2}^{(2)} * \psi_{j_3,k_3}^{(2)} * \psi_{j_4,k_4}^{(2)}(y,z)\big) \\
&\quad:= I_1 \times I_2.
\end{aligned}
$$

In order to verify (2.1), we only consider the case that $k_4 \ge k_3 \ge k_2 \ge k_1$ and $j_4 \ge j_3 \ge j_2 \ge j_1$; other cases can be proved by similar arguments. When $k_4 \ge k_3 \ge k_2 \ge k_1$, by the cancellation conditions, we have

$$
\begin{aligned}
|I_2| &= \big|\psi_{j_1,k_1}^{(2)} * \psi_{j_2,k_2}^{(2)} * \psi_{j_3,k_3}^{(2)} * \psi_{j_4,k_4}^{(2)}(y,z)\big| \\
&= \Big| \int_{\mathbb{R}^6} \big[\psi_{j_1,k_1}^{(2)}(y-u_1, z-v_1) - P_{L-1}[\psi_{j_1,k_1}^{(2)}(y,z)]\big] \\
&\quad\times \Big[\psi_{j_2,k_2}^{(2)}(u_1-u_2, v_1-v_2) - P_{L-1}\big[\psi_{j_2,k_2}^{(2)}(u_1,v_1)\big]\Big] \\
&\quad\times \big[\psi_{j_3,k_3}^{(2)}(u_2-u_3, v_2-v_3) - P_{L-1}[\psi_{j_3,k_3}^{(2)}(u_2,v_2)]\big] \\
&\quad\times \psi_{j_4,k_4}^{(2)}(u_3,v_3)\, du_3 dv_3 du_2 dv_2 du_1 dv_1 \Big| \\
&\le C \int_{\mathbb{R}^6} \Big(\frac{|u_1|}{2^{-k_1}} + \frac{|v_1|}{2^{-(j_1+k_1)}}\Big)^L \frac{2^{2k_1+j_1}}{(1+2^{k_1}|\xi_1|+2^{k_1+j_1}|\eta_1|)^{M_1}} \\
&\quad\times \Big(\frac{|u_2|}{2^{-k_2}} + \frac{|v_2|}{2^{-(j_2+k_2)}}\Big)^L \frac{2^{2k_2+j_2}}{(1+2^{k_2}|\xi_2|+2^{k_2+j_2}|\eta_2|)^{M_2}} \\
&\quad\times \Big(\frac{|u_3|}{2^{-k_3}} + \frac{|v_3|}{2^{-(j_3+k_3)}}\Big)^L \frac{2^{2k_3+j_3}}{(1+2^{k_3}|\xi_3|+2^{k_3+j_3}|\eta_3|)^{M_3}} \\
&\quad\times \frac{2^{2k_4+j_4}}{(1+2^{k_4}|u_3|+2^{k_4+j_4}|v_3|)^{M_4}}\, du_3 dv_3 du_2 dv_2 du_1 dv_1,
\end{aligned}
$$

for some (ξ_1,η_1), (ξ_2,η_2) and (ξ_3,η_3) on the segment joining $(y-u_1, z-v_1)$ to (y,z), (u_1-u_2, v_1-v_2) to (u_1,v_1) and (u_3-u_2, v_3-v_2) to (u_2,v_2), respectively, where

$$
P_{L-1}\big[\psi_{j_1,k_1}^{(2)}(y,z)\big], \quad P_{L-1}\big[\psi_{j_2,k_2}^{(2)}(u_1,v_1)\big], \quad \text{and} \quad P_{L-1}\big[\psi_{j_3,k_3}^{(2)}(u_2,v_2)\big]
$$

denote Taylor's polynomial of order $L-1$ of $\psi^{(2)}_{j_1,k_1}$, $\psi^{(2)}_{j_2,k_2}$, and $\psi^{(2)}_{j_3,k_3}$ at (y,z), (u_1,v_1) and (u_2,v_2), respectively.

Using the fact that $k_4 \ge k_3$ and the triangle inequality, we obtain

$$\frac{1}{1+2^{k_3}|\xi_3|+2^{k_3+j_3}|\eta_3|} \le C2^{|j_3-j_4|}\frac{1+2^{k_4}|u_3|+2^{k_4+j_4}|v_3|}{1+2^{k_3}|u_2|+2^{k_3+j_3}|v_2|}. \tag{2.2}$$

We also have

$$\Big(\frac{|u_3|}{2^{-k_3}}+\frac{|v_3|}{2^{-(j_3+k_3)}}\Big)^L \le C2^{(|j_3-j_4|+(k_3-k_4))L}\Big(\frac{|u_3|}{2^{-k_4}}+\frac{|v_3|}{2^{-(j_4+k_4)}}\Big)^L. \tag{2.3}$$

From (2.2), (2.3), and Proposition 2.2, we then have

$$\begin{aligned}
&\big|\psi^{(2)}_{j_1,k_1} * \psi^{(2)}_{j_2,k_2} * \psi^{(2)}_{j_3,k_3} * \psi^{(2)}_{j_4,k_4}(y,z)\big|\\
&\quad\le C2^{|j_3-j_4|(L+M_3)+(k_3-k_4)L}\int_{\mathbb{R}^6}\Big(\frac{|u_1|}{2^{-k_1}}+\frac{|v_1|}{2^{-(j_1+k_1)}}\Big)^L\frac{2^{2k_1+j_1}}{(1+2^{k_1}|\xi_1|+2^{k_1+j_1}|\eta_1|)^{M_1}}\\
&\qquad\times\Big(\frac{|u_2|}{2^{-k_2}}+\frac{|v_2|}{2^{-(j_2+k_2)}}\Big)^L\frac{2^{2k_2+j_2}}{(1+2^{k_2}|\xi_2|+2^{k_2+j_2}|\eta_2|)^{M_2}}\frac{2^{2k+j_3}}{(1+2^{k_3}|u_2|+2^{k_3+j_3}|v_2|)^{M_3}}\\
&\qquad\times\frac{2^{2k_4+j_4}}{(1+2^{k_4}|u_3|+2^{k_4+j_4}|v_3|)^{M_4-M_3-L}}\,du_3dv_3du_2dv_2du_1dv_1
\end{aligned}$$

$$\begin{aligned}
&\le C2^{|j_3-j_4|(L+M_3)+(k_3-k_4)L}\int_{\mathbb{R}^4}\Big(\frac{|u_1|}{2^{-k_1}}+\frac{|v_1|}{2^{-(j_1+k_1)}}\Big)^L\frac{2^{2k_1+j_1}}{(1+2^{k_1}|\xi_1|+2^{k_1+j_1}|\eta_1|)^{M_1}}\\
&\quad\times\Big(\frac{|u_2|}{2^{-k_2}}+\frac{|v_2|}{2^{-(j_2+k_2)}}\Big)^L\frac{2^{2k_2+j_2}}{(1+2^{k_2}|\xi_2|+2^{k_2+j_2}|\eta_2|)^{M_2}}\\
&\quad\times\frac{2^{2k_3+j_3}}{(1+2^{k_3}|u_2|+2^{k_3+j_3}|v_2|)^{M_3}}\,du_2dv_2du_1dv_1,
\end{aligned} \tag{2.4}$$

where $M_4-M_3-L>2$.

By an analogous argument to (2.2), (2.3), and (2.4), for $k_4 \ge k_3 \ge k_2 \ge k_1$, it is easy to obtain that

$$\begin{aligned}
&\big|\psi^{(2)}_{j_1,k_1} * \psi^{(2)}_{j_2,k_2} * \psi^{(2)}_{j_3,k_3} * \psi^{(2)}_{j_4,k_4}(y,z)\big|\\
&\quad\le C2^{|j_3-j_4|(L+M_3)}2^{|j_2-j_3|(L+M_2)}2^{|j_1-j_2|(L+M_1)}2^{(k_3-k_4)L}2^{(k_2-k_3)L}2^{(k_1-k_2)L}\\
&\qquad\times\frac{2^{2k_1}}{2^{-j_1}(1+2^{k_1}|y|+2^{k_1+j_1}|z|)^{M_1}}\\
&\quad\le C2^{-[|k_1-k_2|+|k_2-k_3|+|k_3-k_4|+|k_1-k_3|+|k_1-k_4|+|k_2-k_4|]L_0}\\
&\qquad\times 2^{(|j_3-j_4|+|j_2-j_3|+|j_2-j_1|)(L+M)}\frac{2^{-k_1M}}{2^{-j_1}(2^{-k_1}+|y|+2^{j_1}|z|)^{2+M}},
\end{aligned} \tag{2.5}$$

where $M_3-M_2-L>2$, $M_2-M_1-L>2$ and $L=3L_0$.

Similar to the estimate of (2.5), when $j_4 \ge j_3 \ge j_2 \ge j_1$,

$$|I_1| \le C2^{-[|j_1-j_2|+|j_2-j_3|+|j_3-j_4|+|j_1-j_3|+|j_1-j_4|+|j_2-j_4|]L_1}\frac{2^{-j_1M}}{(2^{-j_1}+|x|)^{1+M}} \tag{2.6}$$

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for any $M > 0$ and large positive integer L_1. From (2.5) and (2.6), the proof of Proposition 2.3 is concluded. ∎

In order to prove Theorem 1.7, we also need the following result.

Proposition 2.4 *Given any nonnegative integer N, let $j_i, k_i \in \mathbb{Z}$ for $i = 1,2,3,4$ and $R \in \mathcal{R}_Z^{-j_1-N,-k_1-N}$, $R' \in \mathcal{R}_Z^{-j_2-N,-k_2-N}$. Suppose that $\{a_{R'}\}$ is any sequence and $x_R = (x_I, x_J, x_K)$ is any point in R'. For $u = (u_1,u_2,u_3), v = (v_1,v_2,v_3) \in R$,*

$$\begin{aligned}
&\sum_{R'\in\mathcal{R}_Z^{-j_2-N,-k_2-N}} \frac{2^{-(j_1\wedge j_2\wedge j_3\wedge j_4)M}}{(2^{-(j_1\wedge j_2\wedge j_3\wedge j_4)}+|u_1-x_I|)^{1+M}}\\
&\quad\times\frac{2^{-(k_1\wedge k_2\wedge k_3\wedge k_4)M}}{2^{-j^*}(2^{-(k_1\wedge k_2\wedge k_3\wedge k_4)}+|u_2-x_J|+2^{j^*}|u_3-x_K|)^{2+M}}|R'||a_{R'}|\\
&\quad\le C2^{4N(\frac{1}{r}-1)}2^{[|j_1-j_2|+|j_2-j_3|+|j_3-j_4|+|j_1-j_3|+|j_1-j_4|+|j_2-j_4|]}2^{2[|j_1-j_2|+|j_3-j_2|+|j_4-j_2|](\frac{1}{r}-1)}\\
&\quad\times 2^{2[|k_1-k_2|+|k_3-k_2|+|k_4-k_2|](\frac{1}{r}-1)}\Big(M_Z\Big(\sum_{R'\in\mathcal{R}^{-j_2-N,-k_2-N}}|a_{R'}|^r\chi_{R'}\Big)(v)\Big)^{1/r},
\end{aligned}$$

where $\frac{2}{1+M} < r \le 1$, and $j^* = j_i$ if $\min\{k_1,k_2,k_3,k_4\} = k_i$ for $i = 1,2,3,4$.

The proof of Proposition 2.4 is essentially contained in the proof of [7]. We leave the details to the interested reader.

Now we are ready to give the proof of Theorem 1.7.

Proof of Theorem 1.7 We first consider

$$\|I_\alpha f\|_{\dot F^{s+\alpha,q}_{p,Z}} \le C\|f\|_{\dot F^{s,q}_{p,Z}}, \tag{2.7}$$

$$\|I_\alpha f\|_{\dot B^{s+\alpha,q}_{p,Z}} \le C\|f\|_{\dot B^{\alpha,q}_{p,Z}}. \tag{2.8}$$

Given any nonnegative integer N, let $j_i, k_i \in \mathbb{Z}$ for $i = 1,2,3$, $R \in \mathcal{R}_Z^{-j_3-N,-k_3-N}$ and $(u,v,w) \in R$. By the definition of Riesz potential, Calderón's reproducing formula (1.3) and almost orthogonality estimates, we have

$$\begin{aligned}
&|\psi_{j_1,k_1} * I_\alpha(f)(x,y,z)| \qquad (2.9)\\
&\le \sum_{j_2,k_2}\sum_{j_3,k_3}\sum_R |R|2^{-j_2\alpha_0}2^{-k_2\alpha_1}|\psi_{j_3,k_3} * f(u,v,w)|\\
&\quad\times|\psi_{j_1,k_1} * \psi_{j_2,k_2} * \psi_{j_3,k_3}(x-u,y-v,z-w)|\\
&\le C\sum_{j_2,k_2}\sum_{j_3,k_3}\sum_R |R|2^{-j_2\alpha_0}2^{-k_2\alpha_1}2^{-[|j_1-j_2|+|j_2-j_3|+|j_1-j_3|]L}2^{-[|k_1-k_2|+|k_2-k_3|+|k_1-k_3|]L}\\
&\quad\times\frac{2^{-(j_1\wedge j_2\wedge j_3)M}}{(2^{-(j_1\wedge j_2\wedge j_3)}+|x-u|)^{1+M}}\frac{2^{-(k_1\wedge k_2\wedge k_3)M}|\psi_{j_3,k_3} * f(u,v,w)|}{2^{j^*}(2^{-(k_1\wedge k_2\wedge k_3)}+|y-v|+2^{-j^*}|z-w|)^{2+M}}\\
&\le C\sum_{j_2,k_2}\sum_{j_3,k_3}2^{-j_2\alpha_0}2^{-k_2\alpha_1}2^{-[|j_1-j_2|+|j_2-j_3|+|j_1-j_3|](L+1-2/r)}\\
&\quad\times 2^{-[|k_1-k_2|+|k_2-k_3|+|k_1-k_3|](L+2-2/r)}\Big(M_Z\Big(\sum_R|\psi_{j_3,k_3} * f|^r\chi_R\Big)(x,y,z)\Big)^{1/r},
\end{aligned}$$

where $j^* = j_i$ when $\min\{k_1, k_2, k_3\} = k_i$ for $i = 1, 2, 3$. When $\frac{2}{1+M} < r < \min\{1, p, q\}$,

$$\begin{aligned}
&\Big\{ \sum_{j_1,k_1} \Big(2^{j_1(s_0+\alpha_0)} 2^{k_1(s_1+\alpha_1)} |\psi_{j_1,k_1} * I_\alpha(f)(x,y,z)| \Big)^q \Big\}^{1/q} \\
&\quad \le C \Big\{ \sum_{j_1,k_1} \Big(2^{j_1(s_0+\alpha_0)} 2^{k_1(s_1+\alpha_1)} \sum_{j_2,k_2} \sum_{j_3,k_3} 2^{-j_2\alpha_1} 2^{-k_2\alpha_2} 2^{-[|j_1-j_2|+|j_2-j_3|+|j_1-j_3|](L+1-2/r)} \\
&\qquad \times 2^{-[|k_1-k_2|+|k_2-k_3|+|k_1-k_3|](L+2-2/r)} \Big(M_Z\Big(\sum_R |\psi_{j_3,k_3} * f|^r \chi_R \Big)(x,y,z) \Big)^{1/r} \Big)^q \Big\}^{1/q} \\
&\quad \le C \Big\{ \sum_{j_1,k_1} \Big(\sum_{j_2,k_2} \sum_{j_3,k_3} 2^{(j_1-j_2)\alpha_0} 2^{(k_1-k_2)\alpha_1} 2^{-|j_1-j_2|(L+1-2/r)} 2^{-|k_1-k_2|(L+2-2/r)} \\
&\qquad \times 2^{(j_1-j_3)s_0} 2^{(k_1-k_3)s_1} 2^{-|j_1-j_3|(L+1-2/r)} 2^{-|k_1-k_3|(L+2-2/r)} \\
&\qquad \times 2^{j_3 s_0} 2^{k_3 s_1} \Big(M_Z\Big(\sum_R |\psi_{j_3,k_3} * f|^r \chi_R \Big)(x,y,z) \Big)^{1/r} \Big)^q \Big\}^{1/q} \\
&\quad \le C \Big\{ \sum_{j_3,k_3} \Big(2^{j_3 s_0} 2^{k_3 s_1} \Big(M_Z\Big(\sum_R |\psi_{j_3,k_3} * f|^r \chi_R \Big)(x,y,z) \Big)^{1/r} \Big)^q \Big\}^{1/q},
\end{aligned}$$

where $|\alpha_0| < L$, $|\alpha_1| < L$, $|s_0| < L + 1 - 2/r$, and $|s_1| < L + 2 - 2/r$.

By the Fefferman–Stein vector-valued maximal inequality, we get

$$\begin{aligned}
(2.10)\quad \|I_\alpha(f)\|_{\dot F^{s+\alpha,q}_{p,Z}} &\le C \Big\| \Big\{ \sum_{j_3,k_3} \Big(2^{j_3 s_0} 2^{k_3 s_1} \Big(M_Z\Big(\sum_R |\psi_{j_3,k_3} * f|^r \chi_R \Big) \Big)^{1/r} \Big)^q \Big\}^{1/q} \Big\|_p \\
&\le C \Big\| \Big\{ \sum_{j_3,k_3} \Big(2^{j_3 s_0} 2^{k_3 s_1} \sum_R |\psi_{j_3,k_3} * f| \chi_R \Big)^q \Big\}^{1/q} \Big\|_p \le C \|f\|_{\dot F^{s,q}_{p,Z}}.
\end{aligned}$$

For (2.8), using the fact that $\frac{2}{1+M} < r < \min\{1, p, q\}$ and the estimate (2.9), we have

$$\begin{aligned}
\|\psi_{j_1,k_1} * I_\alpha(f)\|_p \le C \sum_{j_2,k_2} \sum_{j_3,k_3} & 2^{-j_2\alpha_0} 2^{-k_2\alpha_1} 2^{-[|j_1-j_2|+|j_2-j_3|+|j_1-j_3|](L+1-2/r)} \\
& \times 2^{-[|k_1-k_2|+|k_2-k_3|+|k_1-k_3|](L+2-2/r)} \Big\| \sum_R |\psi_{j_3,k_3} * f| \chi_R \Big\|_p.
\end{aligned}$$

Similar to the estimate of (2.10), it is easy to obtain that

$$\|I_\alpha(f)\|_{\dot B^{s+\alpha,q}_{p,Z}} \le C \|f\|_{\dot B^{s,q}_{p,Z}}$$

with $|\alpha_0|, |\alpha_1| < L$, $|s_0| < L + 1 - 2/r$, and $|s_1| < L + 2 - 2/r$.

In order to prove the inverse inequalities of (2.7) and (2.8), we need to show that the operator $I_{-\alpha} I_\alpha$ is invertible in $\dot F^{s,q}_{p,Z}$ and $\dot B^{s,q}_{p,Z}$ when α_0, α_1 satisfy certain conditions. Suppose that I is the identity operator and

$$T = I - I_{-\alpha} I_\alpha = \sum_{j_2,k_2 \in \mathbb{Z}} \sum_{j_3,k_3 \in \mathbb{Z}} (1 - 2^{j_3\alpha_0} 2^{-k_3\alpha_1}) \psi_{j_2,k_2} * \psi_{j_2+j_3,k_2+k_3}.$$

Thus, it suffices to verify that the operator T is bounded on $\dot F^{s,q}_{p,Z}$ and $\dot B^{s,q}_{p,Z}$ with operator norm less than 1.

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Given large positive integer N, let $j_i, k_i \in \mathbb{Z}$ for $i = 1,2,3,4$, $R \in \mathcal{R}_Z^{-j_4-N,-k_4-N}$, using Lemma 2.3 and Proposition 2.4. Then

$$\begin{aligned}
&\Big|\psi_{j_1,k_1} * (I - I_{-\alpha} I_\alpha) * f(x,y,z)\Big| \\
&\quad \le C \sum_{j_2,k_2} \sum_{j_3,k_3} \sum_{j_4,k_4} |1 - 2^{j_3\alpha_0} 2^{-k_3\alpha_1}| 2^{-[|j_1-j_2|+|j_3|+|j_1-j_4|](L-2/r+1)} \\
&\qquad \times 2^{-[|k_1-k_2|+|k_3|+|k_1-k_4|(L-2/r+2)} \Big(M_Z\Big(\sum_R |\psi_{j_4,k_4} * f|^r \chi_R\Big)(x,y,z)\Big)^{1/r} \\
&\quad \le C \sum_{j_3,k_3} \sum_{j_4,k_4} |1 - 2^{j_3\alpha_0} 2^{-k_3\alpha_1}| 2^{-[|j_3|+|j_1-j_4|](L-2/r+1)} 2^{-[|k_3|+|k_1-k_4|](L-2/r+2)} \\
&\qquad \times \Big(M_Z\Big(\sum_R |\psi_{j_4,k_4} * f|^r \chi_R\Big)(x,y,z)\Big)^{1/r}.
\end{aligned}$$

Following the above estimates, we then have

$$\begin{aligned}
&\Big(\sum_{j_1,k_1} \Big(2^{j_1 s_0} 2^{k_1 s_1} |\psi_{j_1,k_1} * (I - I_{-\alpha} I_\alpha) * f(x,y,z)|\Big)^q\Big)^{1/q} \\
&\le \Big(\sum_{j_1,j_2} \Big(2^{(j_1-j_4)s_0} 2^{(k_1-k_4)s_1} \sum_{j_3,k_3} \sum_{j_4,k_4} |1 - 2^{j_3\alpha_0} 2^{-k_3\alpha_1}| 2^{-[|j_3|+|j_1-j_4|](L-2/r+1)} \\
&\quad \times 2^{-[|k_3|+|k_1-k_4|](L-2/r+2)} 2^{j_4 s_0} 2^{k_4 s_1} \Big(M_Z\Big(\sum_R |\psi_{j_4,k_4} * f|^r \chi_R\Big)(x,y,z)\Big)^{1/r}\Big)^q\Big)^{1/q} \\
&\le C \sum_{j_3,k_3} |1 - 2^{j_3\alpha_0} 2^{-k_3\alpha_1}| 2^{-|j_3|(L-2/r+1)} 2^{-|k_3|(L-2/r+2)} \\
&\quad \times \Big(\sum_{j_4,k_4} \Big(2^{j_4 s_0} 2^{k_4 s_1} M_Z\Big(\sum_R |\psi_{j_4,k_4} * f|^r \chi_R\Big)(x,y,z)\Big)^{q/r}\Big)^q,
\end{aligned}$$

where $|s_0| < L - 2/r + 1, |s_1| < L - 2/r + 2$. Let $\frac{2}{1+M} < r < \min\{1, p, q\}$. By the Fefferman–Stein vector-valued maximal inequality, we have

$$\begin{aligned}
\|T(f)\|_{\dot F^{s,q}_{p,Z}} &\le C \sum_{j_3,k_3} |1 - 2^{j_3\alpha_0} 2^{-k_3\alpha_1}| 2^{-|j_3|(L-2/r+1)} 2^{-|k_3|(L-2/r+2)} \\
&\qquad \times \Big\| \Big(\sum_{j_4,k_4} \Big(2^{j_4 s_0} 2^{k_4 s_1} \sum_R |\psi_{j_4,k_4} * f| \chi_R\Big)^q\Big)^{1/q}\Big\|_p,
\end{aligned}$$

where C is a positive constant independent of α_0, α_1. For any given $N_0 \in \mathbb{N}$, let $|\alpha_0| < N_0$ and $|\alpha_1| < N_0$, and we choose L large enough such that

$$C \sum_{j_3,k_3} |1 - 2^{j_3\alpha_0} 2^{-k_3\alpha_1}| 2^{-|j_3|(L-2/r+1)} 2^{-|k_3|(L-2/r+2)} < 1.$$

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Namely, we obtain $\|T(f)\|_{\dot F^{s,q}_{p,Z}} \le C_1\|f\|_{\dot F^{s,q}_{p,Z}}$ with the operator norm $C_1 < 1$. That is, T is bounded on $\dot F^{s,q}_{p,Z}$ with an operator norm less than 1, so $I_{-\alpha}I_\alpha$ is an invertible operator on $\dot F^{s,q}_{p,Z}$. By (2.7), for all $f \in \dot F^{s,q}_{p,Z}$, then we have

$$\|f\|_{\dot F^{s,q}_{p,Z}} = \|(I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}I_\alpha(f)\|_{\dot F^{s,q}_{p,Z}} \le C\|I_{-\alpha}I_\alpha(f)\|_{\dot F^{s,q}_{p,Z}} \le C\|I_\alpha(f)\|_{\dot F^{s+\alpha,q}_{p,Z}}.$$

In the same way, we can get $\|f\|_{\dot B^{s,q}_{p,Z}} \le \|I_\alpha(f)\|_{\dot B^{s+\alpha,q}_{p,Z}}$. Therefore, we complete the proof of Theorem 1.7. ∎

3 Proof of Theorem 1.9

Without loss of generality, we take $\|f\|_{\dot F^{s,q_0}_{p_0,Z}} = 1$. Note that L in this section can be large enough. By Calderón's reproducing formula (1.4), Hölder's inequality, almost orthogonality estimates and Proposition 2.2, we have

$$\begin{aligned}(3.1)\quad & |\psi_{j,k}(f)| \\ &\le C\sum_{j',k'\in\mathbb{Z}} 2^{-|j'-j|L}2^{-|k-k'|L}2^{(j\wedge j')(1-\frac{1}{p_0'})}2^{j^*(1-\frac{1}{p_0'})}2^{2(k\wedge k')(1-\frac{1}{p_0'})}\|\phi_{j',k'}(f)\|_{p_0} \\ &\le C\sum_{j',k'\in\mathbb{Z}} 2^{-|j'-j|L}2^{-|k-k'|L}2^{(j\wedge j')/p_0}2^{j^*/p_0}2^{2(k\wedge k')/p_0}2^{-j'a_0}2^{-k'a_1} \\ &\quad\times\Big\|\Big\{\sum_{j',k'\in\mathbb{Z}}\Big(2^{j'a_0}2^{k'a_1}|\phi_{j',k'}(f)|\Big)^q\Big\}^{1/q}\Big\|_{p_0} \\ &= C\sum_{j',k'} 2^{-|j'-j|L}2^{-|k-k'|L}2^{(j\wedge j')/p_0}2^{j^*/p_0}2^{2(k\wedge k')/p_0}2^{-j'a_0}2^{-k'a_1}.\end{aligned}$$

We first claim that

$$\begin{aligned}(3.2)\quad & \Big\{\sum_{j=-\infty}^{N}\sum_{k=N}^{\infty}\Big(2^{js_0}2^{ks_1}|\psi_{j,k}(f)|\Big)^{q_1}\Big\}^{1/q_1} \\ &\le C\Big\{\sum_{j=-\infty}^{N}\sum_{k=N}^{\infty}\Big(2^{js_0}2^{ks_1}\sum_{j',k'\in\mathbb{Z}}|\psi_{j,k}*\phi_{j',k'}*\phi_{j',k'}(f)|\Big)^{q_1}\Big\}^{1/q_1} \\ &\le C2^{4N/p_1}.\end{aligned}$$

We consider (3.2) in the following four cases:

Case 1. $j > j'$ and $k > k'$;

Case 2. $j > j'$ and $k \le k'$;

Case 3. $j \le j'$ and $k > k'$;

Case 4. $j \le j'$ and $k \le k'$.

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We first deal with Case 1. By the estimate (3.1) and the fact that $a_0 - \frac{2}{p_0} = s_0 - \frac{2}{p_1}$, $a_1 - \frac{2}{p_0} = s_1 - \frac{2}{p_1}$, we then have

(3.3)
$$\begin{aligned}
&\Big\{ \sum_{j=-\infty}^{N} \sum_{k=N}^{\infty} \Big(2^{js_0} 2^{ks_1} |\psi_{j,k}(f)| \Big)^{q_1} \Big\}^{1/q_1} \\
&\leq C \Big\{ \sum_{j=-\infty}^{N} \sum_{k=N}^{\infty} \Big(2^{js_0} 2^{ks_1} \sum_{j>j', k>k'} 2^{-|j'-j|L} 2^{-|k-k'|L} 2^{2j'/p_0} 2^{2k'/p_0} 2^{-j' a_0} 2^{-k' a_1} \Big)^{q_1} \Big\}^{1/q_1} \\
&= C \Big\{ \sum_{j=-\infty}^{N} \sum_{k=N}^{\infty} \Big(\sum_{j>j', k>k'} 2^{-|j'-j|L} 2^{(j'-j)(\frac{2}{p_0} - a_0)} 2^{2j/p_1} 2^{-|k-k'|L} 2^{(k-k')s_1} 2^{2k'/p_1} \Big)^{q_1} \Big\}^{1/q_1} \\
&\leq C \sum_{j=-\infty}^{N} 2^{2j/p_1} \sum_{k'=-\infty}^{N} \sum_{k>k'} 2^{-|k-k'|L} 2^{(k-k')s_1} 2^{2k'/p_1} \\
&\leq C 2^{2N/p_1} \sum_{k'=-\infty}^{N} 2^{2k'/p_1} \\
&\leq C 2^{4N/p_1},
\end{aligned}$$

where $-L < \frac{2}{p_0} - a_0 < L$ and $-L/2 < s_1 < L/2$.

In order to estimate Case 2, by an analogous argument to (3.3), we obtain

$$\begin{aligned}
&\Big\{ \sum_{j=-\infty}^{N} \sum_{k=N}^{\infty} \Big(2^{js_0} 2^{ks_1} |\psi_{j,k}(f)| \Big)^{q_1} \Big\}^{1/q_1} \\
&\quad \leq C \Big\{ \sum_{j=-\infty}^{N} \sum_{k=N}^{\infty} \Big(2^{js_0} 2^{ks_1} \sum_{j>j'} \sum_{k\leq k'} 2^{-|j'-j|L} 2^{-|k-k'|L} 2^{2j/p_0} 2^{2k/p_0} 2^{-j' a_0} 2^{-k' a_1} \Big)^{q_1} \Big\}^{1/q_1} \\
&\quad \leq \Big\{ \sum_{j=-\infty}^{N} \sum_{k=N}^{\infty} \Big(\sum_{j>j'} \sum_{k\leq k'} 2^{-|j'-j|L} 2^{2j/p_1} 2^{(j-j')a_0} 2^{-|k-k'|L} 2^{ks_1} 2^{2k/p_0} 2^{-k' a_1} \Big)^{q_1} \Big\}^{1/q_1} \\
&\quad \leq C 2^{2N/p_1} \Big\{ \sum_{k=N}^{\infty} \Big(\sum_{k\leq k'} 2^{-|k-k'|L} 2^{ks_1} 2^{2k/p_0} 2^{-k' a_1} \Big)^{q_1} \Big\}^{1/q_1}.
\end{aligned}$$

Therefore, we need to verify

$$\Big\{ \sum_{k=N}^{\infty} \Big(\sum_{k\leq k'} 2^{-|k-k'|2L} 2^{ks_1} 2^{2k/p_0} 2^{-k' a_1} \Big)^{q_1} \Big\}^{1/q_1} \leq C 2^{2N/p_1}.$$

When $k' - 2k > -N$, we then have

$$\Big\{ \sum_{k=N}^{\infty} \Big(\sum_{k\le k'} 2^{-|k-k'|L} 2^{ks_1} 2^{2k/p_0} 2^{-k'a_1} \Big)^{q_1} \Big\}^{1/q_1} \tag{3.4}$$
$$= \Big\{ \sum_{k=N}^{\infty} \Big(\sum_{k\le k'} 2^{-|k-k'|L/2} 2^{(k-k')a_1} 2^{(2k-k')L/2} 2^{-k(L/2-2/p_1)} \Big)^{q_1} \Big\}^{1/q_1}$$
$$\le C \Big\{ \sum_{k=N}^{\infty} \Big(2^{NL/2} 2^{-k(L/2-2/p_1)} \Big)^{q_1} \Big\}^{1/q_1} \le C 2^{2N/p_1}$$

with $-L/2 < a_1 < L/2$ and $2/p_1 < L/2$.

When $k' - 2k < -N$, if $-L/2 < a_1 < L/2$ and $4/p_1 < L/2$, then

$$\Big\{ \sum_{k=N}^{\infty} \Big(\sum_{k\le k'} 2^{-|k-k'|L} 2^{ks_1} 2^{2k/p_0} 2^{-k'a_1} \Big)^{q_1} \Big\}^{1/q_1} \tag{3.5}$$
$$\le \Big\{ \sum_{k=N}^{\infty} \Big(\sum_{k\le k'} 2^{-|k-k'|L/2} 2^{(k-k')a_1} 2^{(k-k')L/2} 2^{2k'/p_1} \Big)^{q_1} \Big\}^{1/q_1}$$
$$= \Big\{ \sum_{k=N}^{\infty} \Big(\sum_{k\le k'} 2^{-|k-k'|L/2} 2^{(k-k')a_1} 2^{(k'-2k)(2/p_1-L/2)} 2^{-k(L/2-4/p_1)} \Big)^{q_1} \Big\}^{1/q_1}$$
$$\le C 2^{2N/p_1}.$$

By analogue arguments to (3.3), (3.4), and (3.5), we can similarly obtain the desired results for Cases 3 and 4. Here we omit the details.

Similar to the proof of (3.2), we get

$$\Big\{ \sum_{j=N}^{\infty} \sum_{k=-\infty}^{N} \Big(2^{js_1} 2^{ka_1} |\psi_{j,k}(f)| \Big)^{q_1} \Big\}^{1/q_1} \le C 2^{4N/p_1} \tag{3.6}$$

with $-L < a_0, s_0, \frac{2}{p_0} - s_0, \frac{2}{p_0} - s_1 < L$, $-L/2 < a_1, s_1 < L/2$, and $4/p < L/2$.

By the estimate (3.1) and the fact that $a_0 - \frac{2}{p_0} = s_0 - \frac{2}{p_1}$, $a_1 - \frac{2}{p_0} = s_1 - \frac{2}{p_1}$, we obtain

(3.7)
$$\Big\{ \sum_{j=-\infty}^{N} \sum_{k=-\infty}^{N} \Big(2^{js_0} 2^{ks_1} |\psi_{j,k}(f)| \Big)^{q_1} \Big\}^{1/q_1}$$
$$\le \Big\{ \sum_{j=-\infty}^{N} \sum_{k=-\infty}^{N}$$
$$\Big(2^{js_0} 2^{ks_1} \sum_{j',k'} 2^{-|j'-j|L} 2^{-|k-k'|L} 2^{(j\wedge j')/p_0} 2^{j^*/p_0} 2^{2(k\wedge k')/p_0} 2^{-j'a_0} 2^{-k'a_1} \Big)^{q_1} \Big\}^{1/q_1}$$
$$\le C 2^{4N/p_1},$$

where $-L < \frac{2}{p_0} - a_0, \frac{2}{p_0} - a_1 < L$, and $-L < a_0, a_1 < L$.

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Since $s_0 < a_0$ and $s_1 < a_1$,

$$
\begin{aligned}
(3.8)\qquad &\Big\{\sum_{j=N}^{\infty}\sum_{k=N}^{\infty}\Big(2^{js_0}2^{ks_1}|\psi_{j,k}(f)|\Big)^{q_1}\Big\}^{1/q_1}\\
&=\Big\{\sum_{j=N}^{\infty}\sum_{k=N}^{\infty}\Big(2^{j(s_0-a_0)}2^{k(s_1-a_1)}2^{ja_0}2^{ka_1}|\psi_{j,k}(f)|\Big)^{q_1}\Big\}^{1/q_1}\\
&\le\Big\{\sum_{j=N}^{\infty}\sum_{k=N}^{\infty}\Big(2^{j(s_0-a_0)}2^{k(s_1-a_1)}\Big)^{q_1}\Big\}^{1/q_1}\sup_{j,k}2^{ja_0}2^{ka_1}|\psi_{j,k}(f)|\\
&\le 2^{N(s_0-a_0)}2^{N(s_1-a_1)}\Big\{\sum_{j,k\in\mathbb{Z}}\Big(2^{ja_0}2^{ka_1}|\psi_{j,k}(f)|\Big)^{q_0}\Big\}^{1/q_0}\\
&=2^{4N(\frac{1}{p_1}-\frac{1}{p_0})}\Big\{\sum_{j,k\in\mathbb{Z}}\Big(2^{ja_0}2^{ka_1}|\psi_{j,k}(f)|\Big)^{q_0}\Big\}^{1/q_0}.
\end{aligned}
$$

Therefore, from (3.2), (3.6), (3.7), and (3.8), we have

$$
\begin{aligned}
\|f\|^{p_1}_{\dot F^{s,q_1}_{p_1}} &= p_1\int_0^{\infty}t^{p_1-1}\Big|\Big\{\Big(\sum_{j,k}\Big(2^{js_0}2^{ks_1}|\psi_{j,k}(f)|\Big)^{q_1}\Big)^{1/q_1}>t\Big\}\Big|\,dt\\
&\le\sum_{N=-\infty}^{\infty}p_1\int_{C2^{4N/p_1}}^{C2^{4(N+1)/p_1}}t^{p_1-1}\Big|\Big\{\Big(\sum_{j,k\in\mathbb{Z}}\Big(2^{js_0}2^{ks_1}|\psi_{j,k}(f)|\Big)^{q_1}\Big)^{1/q_1}>t\Big\}\Big|\,dt\\
&\le C\sum_{N=-\infty}^{\infty}p_1\int_{C2^{4N/p_1}}^{C2^{4(N+1)/p_1}}t^{p_1-1}\Big|\Big\{\Big(\sum_{j=N}^{\infty}\sum_{k=N}^{\infty}\Big(2^{js_0}2^{ks_1}|\psi_{j,k}(f)|\Big)^{q_1}\Big)^{1/q_1}>\frac{t}{4}\Big\}\Big|\,dt\\
&\le C\sum_{N=-\infty}^{\infty}p_1\int_{C2^{4N/p_1}}^{C2^{4(N+1)/p_1}}\\
&\qquad t^{p_1-1}\Big|\Big\{\Big(\sum_{j,k\in\mathbb{Z}}\Big(2^{ja_0}2^{ka_1}|\phi_{j,k}(f)|\Big)^{q_0}\Big)^{1/q_0}>\frac{1}{2}C2^{4N(\frac{1}{p_0}-\frac{1}{p_1})}t\Big\}\Big|\,dt\\
&\le C\sum_{N=-\infty}^{\infty}p_1\int_{C2^{4N/p_1}}^{C2^{4(N+1)/p_1}}t^{p_1-1}\Big|\Big\{\Big(\sum_{j,k\in\mathbb{Z}}\Big(2^{ja_0}2^{ka_1}|\phi_{j,k}(f)|\Big)^{q_0}\Big)^{1/q_0}>Ct^{p_1/p_0}\Big\}\Big|\,dt\\
&\le Cp_1\int_0^{\infty}t^{p_1-1}\Big|\Big\{\Big(\sum_{j,k\in\mathbb{Z}}\Big(2^{ja_0}2^{ka_1}|\phi_{j,k}(f)|\Big)^{q_0}\Big)^{1/q_0}>Ct^{p_1/p_0}\Big\}\Big|\,dt\\
&\le Cp_1\int_0^{\infty}t^{p_0-1}\Big|\Big\{\Big(\sum_{j,k\in\mathbb{Z}}\Big(2^{ja_0}2^{ka_1}|\phi_{j,k}(f)|\Big)^{q_0}\Big)^{1/q_0}>Ct\Big\}\Big|\,dt\\
&\le C\|f\|^{p_0}_{\dot F^{a,q_0}_{p_0}}\\
&\le C,
\end{aligned}
$$

which concludes Theorem 1.9(i).

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Suppose that $f \in \dot{B}^{s,q}_{p,Z}$. By Calderón's reproducing formula (1.4), Minkowski's inequality and almost orthogonality estimates, we have

$$\begin{aligned}
\|f\|_{\dot{B}^{s,q}_{p,Z}} &\le \Big\{ \sum_{j,k\in\mathbb{Z}} \Big(2^{js_0}2^{ks_1} \sum_{j',k'\in\mathbb{Z}} \|\psi_{j,k} * \phi_{j',k'} * \phi_{j',k'}(f)\|_{p_1} \Big)^q \Big\}^{1/q} \\
&\le \Big\{ \sum_{j,k\in\mathbb{Z}} \Big(2^{js_0}2^{ks_1} \sum_{j',k'\in\mathbb{Z}} \|\psi_{j,k} * \phi_{j',k'}\|_{r\to r} \|\phi_{j',k'}(f)\|_{p_0} \Big)^q \Big\}^{1/q} \\
&\le \Big\{ \sum_{j,k\in\mathbb{Z}} \Big(2^{js_0}2^{ks_1} \sum_{j',k'\in\mathbb{Z}} 2^{-|j'-j|L} 2^{-|k-k'|L} 2^{(j'\wedge j)(1-\frac{1}{r})} 2^{j^*(1-\frac{1}{r})} \\
&\qquad\qquad \times 2^{2(k'\wedge k)(1-\frac{1}{r})} \|\phi_{j',k'}(f)\|_{p_0} \Big)^q \Big\}^{1/q} \\
&\le \Big\{ \sum_{j,k\in\mathbb{Z}} \Big(2^{js_0}2^{ks_1} \sum_{j',k'\in\mathbb{Z}} 2^{-|j'-j|L} 2^{-|k-k'|L} 2^{(j'\wedge j)(\frac{1}{p_0}-\frac{1}{p_1})} 2^{j^*(\frac{1}{p_0}-\frac{1}{p_1})} \\
&\qquad\qquad \times 2^{-2(k'\wedge k)(\frac{1}{p_0}-\frac{1}{p_1})} \|\phi_{j',k'}(f)\|_{p_0} \Big)^q \Big\}^{1/q},
\end{aligned}$$

where $1 + 1/p_1 = 1/r + 1/p_0$. And we consider it in four cases as we did for the estimate (3.2). When $j' > j$ and $k' > k$, by Hölder's inequality and the fact that $a_0 - 2/p_0 = s_0 - 2/p_1$ and $a_1 - 2/p_0 = s_1 - 2/p_1$,

$$\begin{aligned}
&\Big\{ \sum_{j,k\in\mathbb{Z}} \Big(\sum_{j'>j,k'>k} 2^{js_0}2^{ks_1} 2^{-|j'-j|L} 2^{-|k-k'|L} 2^{2j(\frac{1}{p_0}-\frac{1}{p_1})} 2^{2k(\frac{1}{p_0}-\frac{1}{p_1})} \|\phi_{j',k'}(f)\|_{p_0} \Big)^q \Big\}^{1/q} \\
&\le C \Big\{ \sum_{j,k\in\mathbb{Z}} \Big(\sum_{j'>j,k'>k} 2^{-|j'-j|L} 2^{-|k-k'|L} 2^{(j-j')a_0} 2^{(k-k')a_1} 2^{j'a_0} 2^{k'a_1} \|\phi_{j',k'}(f)\|_{p_0} \Big)^q \Big\}^{1/q} \\
&\le C \Big\{ \sum_{j',k'\in\mathbb{Z}} \Big(2^{j'a_0} 2^{k'a_1} \|\phi_{j',k'}(f)\|_{p_0} \Big)^q \Big\}^{1/q},
\end{aligned}$$

where $|a_0| < L$ and $|a_1| < L$. We can similarly deal with another three cases with $|a_i|, |s_i| < L$ for $i = 1, 2$. We leave the details to the interested reader.

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