COMPOSITIO MATHEMATICA

A mirror theorem for toric stacks

Tom Coates, Alessio Corti, Hiroshi Iritani and Hsian-Hua Tseng

Compositio Math. **151** (2015), 1878–1912.

doi:10.1112/S0010437X15007356

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY 150 YEARS

Compositio Math. 151 (2015) 1878–1912
doi:10.1112/S0010437X15007356

A mirror theorem for toric stacks

Tom Coates, Alessio Corti, Hiroshi Iritani and Hsian-Hua Tseng

Abstract

We prove a Givental-style mirror theorem for toric Deligne–Mumford stacks $\mathcal{X}$. This determines the genus-zero Gromov–Witten invariants of $\mathcal{X}$ in terms of an explicit hypergeometric function, called the I-function, that takes values in the Chen–Ruan orbifold cohomology of $\mathcal{X}$.

Contents

1. Introduction

In this paper we prove a mirror theorem that determines the genus-zero Gromov–Witten invariants of smooth toric Deligne–Mumford stacks. Toric Deligne–Mumford stacks are generalizations of toric varieties [BCS05], and our mirror theorem generalizes Givental's mirror theorem for toric manifolds [Giv98]. Following Givental [Giv04], the genus-zero Gromov–Witten theory of a toric Deligne–Mumford stack $\mathcal{X}$ can be encoded in a Lagrangian cone $\mathcal{L}_{\mathcal{X}}$ contained in an infinite-dimensional symplectic vector space $\mathcal{H}$. Universal properties of Gromov–Witten invariants of $\mathcal{X}$ translate into geometric properties of $\mathcal{L}_{\mathcal{X}}$. See § 2 for an overview. In §§ 5–7 of this paper we establish a mirror theorem for a smooth toric Deligne–Mumford stack $\mathcal{X}$. Roughly speaking, our result states that the *extended I-function*, which is a hypergeometric function defined in terms of the combinatorial data defining $\mathcal{X}$, lies on the Lagrangian cone $\mathcal{L}_{\mathcal{X}}$. The precise statement is Theorem 31 below.

Our mirror theorem (Theorem 31) has a number of applications. It has been used to give explicit formulas for genus-zero Gromov–Witten invariants of toric Deligne–Mumford stacks and, when combined with the quantum Lefschetz theorem [CG07, CCIT09], to prove a mirror

Received 7 February 2014, accepted in final form 2 October 2014, published online 1 June 2015.
2010 Mathematics Subject Classification 14N35 (primary), 14A20, 14M25, 14J33, 53D45 (secondary).
Keywords: Gromov–Witten theory, toric Deligne–Mumford stacks, orbifolds, quantum cohomology, mirror symmetry, Givental's symplectic formalism, hypergeometric functions.

theorem for convex toric complete intersection stacks [CCIT14]. Special cases of Theorem 31 have been used (as conjectures, proven here) to construct an integral structure on quantum orbifold cohomology of toric Deligne–Mumford stacks, to study Ruan's Crepant Resolution Conjecture, to compute open-closed Gromov–Witten invariants [Iri09, CCLT14, FLT12], to prove mirror theorems for open Gromov–Witten invariants [CCLT13], and to prove mirror theorems for certain toric complete intersection stacks [Iri11]. Theorem 31 will have further applications in the future: it allows a full proof of the Crepant Resolution Conjecture in the toric case, and a description of the quantum D-module of a toric Deligne–Mumford stack. We will discuss these applications elsewhere. Theorem 31 extends previous works on the Gromov–Witten theory of certain classes of toric stacks, including weighted projective spaces [AGV08, CLCT09, Man08, GS14, CG11], one-dimensional toric Deligne–Mumford stacks [Joh14, MT08], toric orbifolds of the form $[\mathbb{C}^n/G]$ [CC09, BC10, JPT11, BC11], and the ambient space for the mirror quintic [LS14].

Since our original announcement of Theorem 31, in February 2007 [Cor07], the Gromov–Witten theory of toric Deligne–Mumford stacks has matured considerably, and the proof that we give here relies heavily on two recent advances. The first is the beautiful characterization of the Lagrangian cone $\mathcal{L}_X$ for a toric variety (or toric bundle) X in terms of recursion relations [Bro14]; we establish the analogous result for toric Deligne–Mumford stacks in § 6. The second is Liu's virtual localization formula for toric Deligne–Mumford stacks [Liu13, Theorem 9.32]; this is the essential technical ingredient that allows us to characterize $\mathcal{L}_{\mathcal{X}}$ for a toric Deligne–Mumford stack $\mathcal{X}$.

A significant generalization of our Theorem 31 has recently been announced by Ciocan *et al.* [CK14, CCFK14]. Also one major application of our theorem, the calculation of the quantum cohomology ring of smooth toric Deligne–Mumford stacks with projective coarse moduli space, has been achieved directly by Gonzalez and Woodward, using the theory of gauged Gromov–Witten invariants [GW12a, GW12b, Woo12, Woo14a, Woo14b]. We feel that it is nonetheless worth presenting our argument here, in part because it is based on fundamentally different ingredients (on Givental's recursive characterization of $\mathcal{L}_X$, rather than on the the theory of quasimaps or gauged Gromov–Witten theory), in part because it gives explicit mirror formulas that have important applications, and in part to reduce our embarrassment at the long gap between our announcement of the mirror theorem and its proof.

The rest of this paper is organized as follows. Sections 2 and 3 contain reviews of Gromov–Witten theory and toric Deligne–Mumford stacks. In § 4 we introduce a notion of extended Picard group for a Deligne–Mumford stack. Our mirror theorem, Theorem 31, is stated in § 5. In § 6 we establish a criterion for points to lie on the Lagrangian cone $\mathcal{L}_{\mathcal{X}}$. In § 7 we prove Theorem 31 by showing that the extended I-function satisfies the criterion from § 6.

2. Gromov–Witten theory

Gromov–Witten theory for orbifold target spaces was first constructed in symplectic geometry by Chen and Ruan [CR02]. In algebraic geometry, the construction was established by Abramovich *et al.* [AGV02, AGV08]. In this section, we review the main ingredients of orbifold Gromov–Witten theory. We mostly follow the presentation of [Tse10]. More detailed discussions of the basics of orbifold Gromov–Witten theory from the viewpoint of Givental's formalism can be found in e.g. [Tse10, CIT09].

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

2.1 Chen–Ruan cohomology

Let $\mathcal{X}$ be a smooth Deligne–Mumford stack equipped with an action of an algebraic torus $\mathbb{T}$. Let X denote the coarse moduli space of $\mathcal{X}$. The inertia stack of $\mathcal{X}$ is defined as

$$I\mathcal{X} := \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

where $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal morphism. A point on $I\mathcal{X}$ is given by a pair (x, g) of a point $x \in \mathcal{X}$ and an element $g \in \mathrm{Aut}(x)$ of the isotropy group at x. As a module over $R_{\mathbb{T}} := H^{\bullet}_{\mathbb{T}}(\mathrm{pt}, \mathbb{C})$, the $\mathbb{T}$-equivariant *Chen–Ruan orbifold cohomology* of $\mathcal{X}$ is defined to be the $\mathbb{T}$-equivariant cohomology of the inertia stack:

$$H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) := H^{\bullet}_{\mathbb{T}}(I\mathcal{X}, \mathbb{C}).$$

When $\mathbb{T}$ is the trivial group, this is denoted by $H^{\bullet}_{\mathrm{CR}}(\mathcal{X})$. The work [CR04] equips $H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$ with a grading called the age grading and a product called the Chen–Ruan cup product. These are different from the usual ones on $H^{\bullet}_{\mathbb{T}}(I\mathcal{X}, \mathbb{C})$. There is an involution $\mathrm{inv} : I\mathcal{X} \to I\mathcal{X}$ given on points by $(x, g) \mapsto (x, g^{-1})$. When the $\mathbb{T}$-fixed set $\mathcal{X}^{\mathbb{T}}$ is proper, we can define the orbifold Poincaré pairing

$$(\alpha, \beta)_{\mathrm{CR}} := \int_{I\mathcal{X}}^{\mathbb{T}} \alpha \cup \mathrm{inv}^{\star} \beta$$

on $H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$ using the Atiyah–Bott localization formula; the pairing takes values in the fraction field $S_{\mathbb{T}}$ of $R_{\mathbb{T}} = H^{\bullet}_{\mathbb{T}}(\mathrm{pt})$.

2.2 Gromov–Witten invariants and Gromov–Witten potentials

Gromov–Witten invariants are intersection numbers in moduli stacks of stable maps. Let $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ denote the moduli stack of n-pointed genus-g degree-d orbifold stable maps to $\mathcal{X}$ with sections to gerbes at the markings, where $d \in H_2(X, \mathbb{Z})$ (see [AGV08, § 4.5], [Tse10, § 2.4]). There are evaluation maps at the marked points

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) \to I\mathcal{X}, \quad 1 \leqslant i \leqslant n,$$

and, given $\vec{b} = (b(1), \ldots, b(n))$ where the $b(i)$ correspond to components $(I\mathcal{X})_{b(i)}$ of $I\mathcal{X}$, we set

$$\overline{\mathcal{M}}^{\vec{b}}_{g,n}(\mathcal{X}, d) = \bigcap_{i=1}^{n} \mathrm{ev}_i^{-1}(I\mathcal{X})_{b(i)} \quad \text{so that } \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) = \bigcup_{\vec{b}} \overline{\mathcal{M}}^{\vec{b}}_{g,n}(\mathcal{X}, d).$$

Let $\bar{\psi}_i \in H^2(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d), \mathbb{Q}), 1 \leqslant i \leqslant n$, denote the descendant classes [Tse10, § 2.5.1]. Suppose that $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ is proper. Then the moduli stack carries a weighted virtual fundamental class [AGV08], [Tse10, § 2.5.1]:

$$[\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)]^w \in H_{\bullet}(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d), \mathbb{Q}).$$

Given elements $a_1, \ldots, a_n \in H^{\bullet}_{\mathrm{CR}}(\mathcal{X})$ and non-negative integers $k_1, \ldots, k_n$, we define

$$\langle a_1 \bar{\psi}^{k_1}, \ldots, a_n \bar{\psi}^{k_n} \rangle_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)]^w} (\mathrm{ev}_1^{\star} a_1) \bar{\psi}_1^{k_1} \ldots (\mathrm{ev}_n^{\star} a_n) \bar{\psi}_n^{k_n}. \tag{1}$$

These are called the *descendant Gromov–Witten invariants* of $\mathcal{X}$.

When $\mathcal{X}$ is equipped with an action of an algebraic torus $\mathbb{T}$, there is an induced $\mathbb{T}$-action on the moduli space $\overline{\mathcal{M}}_{g,n,d}(\mathcal{X},d)$. The descendant classes $\bar{\psi}_i$ and the virtual fundamental class have canonical $\mathbb{T}$-equivariant lifts and we can define $\mathbb{T}$-equivariant Gromov–Witten invariants

$$\langle a_1\bar{\psi}^{k_1},\dots,a_n\bar{\psi}^{k_n}\rangle^{\mathbb{T}}_{g,n,d} = \int^{\mathbb{T}}_{[\overline{\mathcal{M}}_{g,n}(\mathcal{X},d)]^w} (\mathrm{ev}_1^\star a_1)\bar{\psi}_1^{k_1}\cdots(\mathrm{ev}_n^\star a_n)\bar{\psi}_n^{k_n}$$

for $a_1,\dots,a_n \in H^\bullet_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$. In this paper we consider the case where the moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{X},d)$ itself may not be proper, but the $\mathbb{T}$-fixed locus is proper. This happens for toric stacks. In this case, we define $\mathbb{T}$-equivariant descendant Gromov–Witten invariants by using the virtual localization formula (see [Liu13]); the invariants then take values in $S_{\mathbb{T}} = \mathrm{Frac}(R_{\mathbb{T}})$.

We package descendant Gromov–Witten invariants using generating functions. Let $\mathbf{t} = \mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \cdots \in H^\bullet_{\mathrm{CR}}(\mathcal{X})[z]$. Define

$$\langle \mathbf{t},\dots,\mathbf{t}\rangle_{g,n,d} = \langle \mathbf{t}(\bar{\psi}),\dots,\mathbf{t}(\bar{\psi})\rangle_{g,n,d} := \sum_{k_1,\dots,k_n \geqslant 0} \langle t_{k_1}\bar{\psi}^{k_1},\dots,t_{k_n}\bar{\psi}^{k_n}\rangle_{g,n,d}.$$

The *genus-g descendant potential* of $\mathcal{X}$ is

$$\mathcal{F}^g_{\mathcal{X}}(\mathbf{t}) := \sum_{n=0}^{\infty} \sum_{d\in \mathrm{NE}(\mathcal{X})} \frac{Q^d}{n!}\langle \mathbf{t},\dots,\mathbf{t}\rangle_{g,n,d}.$$

Here Q^d is an element of the Novikov ring $\Lambda_{\mathrm{nov}} := \mathbb{C}[\![\mathrm{NE}(X)\cap H_2(X,\mathbb{Z})]\!]$ (see [Tse10, Definition 2.5.4]), where $\mathrm{NE}(X) \subset H_2(X,\mathbb{R})$ denotes the cone generated by effective curve classes in X. Let us fix an additive basis $\{\phi_\alpha\}$ for $H^\bullet_{\mathrm{CR}}(\mathcal{X})$ consisting of homogeneous elements, and write

$$t_k = \sum_\alpha t_k^\alpha \phi_\alpha \in H^\bullet_{\mathrm{CR}}(\mathcal{X}), \quad k \geqslant 0.$$

The generating function $\mathcal{F}^g_{\mathcal{X}}(\mathbf{t})$ is a Λ_{nov}-valued formal power series in the variables t_k^α.

The definition readily extends to the $\mathbb{T}$-equivariant setting. The $\mathbb{T}$-equivariant descendant Gromov–Witten potential $\mathcal{F}^g_{\mathcal{X},\mathbb{T}}(\mathbf{t})$ is defined as a $\Lambda^{\mathbb{T}}_{\mathrm{nov}} := S_{\mathbb{T}}[\![\mathrm{NE}(X)\cap H_2(X,\mathbb{Z})]\!]$-valued function of $\mathbf{t}(z) \in H^\bullet_{\mathrm{CR},\mathbb{T}}(\mathcal{X})[z]$. Choosing a homogeneous basis $\{\phi_\alpha\}$ of $H^\bullet_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}$ over $S_{\mathbb{T}}$, we write $\mathbf{t}(z) = \sum_{k\geqslant 0} t_k z^k = \sum_{k\geqslant 0}\sum_\alpha t_k^\alpha \phi_\alpha z^k$.

2.3 Givental's symplectic formalism

Next we describe Givental's symplectic formalism for genus-zero Gromov–Witten theory [Giv01, Giv04]. We present the $\mathbb{T}$-equivariant version, following the presentation in [Tse10, § 3.1], [CIT09] and [CCIT09] for the non-equivariant case.

Since our target space $\mathcal{X}$ is not necessarily proper, we work over the field $S_{\mathbb{T}} \cong \mathbb{C}(\chi_1,\dots,\chi_d)$ of fractions of $H^\bullet_{\mathbb{T}}(\mathrm{pt})$, where $\{\chi_1,\dots,\chi_d\}$ is a basis of characters of the torus $\mathbb{T} \cong (\mathbb{C}^\times)^d$. Recall that the $\mathbb{T}$-equivariant Novikov ring is

$$\Lambda^{\mathbb{T}}_{\mathrm{nov}} = S_{\mathbb{T}}[\![\mathrm{NE}(X)\cap H_2(X,\mathbb{Z})]\!]. \tag{2}$$

Givental's symplectic vector space is the $\Lambda^{\mathbb{T}}_{\mathrm{nov}}$-module

$$\mathcal{H} := H_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}(\!(z^{-1})\!)[\![\mathrm{NE}(X)\cap H_2(X,\mathbb{Z})]\!]$$

equipped with the symplectic form:

$$\Omega(f,g) = -\mathrm{Res}_{z=\infty}(f(-z),g(z))_{\mathrm{CR}}\,dz, \quad \text{for } f,g\in\mathcal{H}.$$

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

The coefficient of Q^d in an element of $\mathcal{H}$ is a formal Laurent series in z^{-1}, i.e. a power series of the form $\sum_{n=n_0}^{\infty} a_n z^{-n}$ for some $n_0 \in \mathbb{Z}$. The symplectic form Ω is given by the coefficient of z^{-1} of the orbifold Poincaré pairing $(f(-z), g(z))_{\mathrm{CR}}$; the minus sign reflects the fact that we take the residue at $z = \infty$ rather than $z = 0$. Consider the polarization

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

where

$$\begin{aligned}\mathcal{H}_+ &:= H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}[z][\![\mathrm{NE}(X) \cap H_2(X,\mathbb{Z})]\!],\\ \mathcal{H}_- &:= z^{-1} H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}[\![z^{-1}]\!][\![\mathrm{NE}(X) \cap H_2(X,\mathbb{Z})]\!].\end{aligned}$$

The subspaces $\mathcal{H}_\pm$ are maximally isotropic with respect to Ω, and the symplectic form Ω induces a non-degenerate pairing between $\mathcal{H}_+$ and $\mathcal{H}_-$. Thus we can regard $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ as the total space of the cotangent bundle $T^*\mathcal{H}_+$ of $\mathcal{H}_+$.

Let $\{\phi^\mu\} \subset H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}$ be the $S_{\mathbb{T}}$-basis dual to $\{\phi_\nu\}$ with respect to the orbifold Poincaré pairing, so that $(\phi^\mu, \phi_\nu)_{\mathrm{CR}} = \delta^\mu_\nu$. A general point in $\mathcal{H}$ takes the form

$$\sum_{a=0}^{\infty} \sum_{\mu} p_{a,\mu} \phi^\mu (-z)^{-a-1} + \sum_{b=0}^{\infty} \sum_{\nu} q_b^\nu \phi_\nu z^b, \tag{3}$$

and this defines Darboux coordinates $\{p_{a,\mu}, q_b^\nu\}$ on $(\mathcal{H}, \Omega)$ which are compatible with the polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Put $p_a = \sum_\mu p_{a,\mu}\phi^\mu$, $q_b = \sum_\nu q_b^\nu \phi_\nu$, and denote

$$\begin{aligned}\mathbf{p} = \mathbf{p}(z) &:= \sum_{k=0}^{\infty} p_k (-z)^{-k-1} = p_0(-z)^{-1} + p_1(-z)^{-2} + \cdots,\\ \mathbf{q} = \mathbf{q}(z) &:= \sum_{k=0}^{\infty} q_k z^k = q_0 + q_1 z + q_2 z^2 + \cdots.\end{aligned}$$

We relate the coordinates $\mathbf{q}$ on $\mathcal{H}_+$ to the variables $\mathbf{t}$ of the descendant potential $\mathcal{F}^g_{\mathcal{X}}(\mathbf{t})$ by $\mathbf{q}(z) = \mathbf{t}(z) - 1z$; this identification is called the *dilaton shift* [Giv01].

The genus-zero descendant potential $\mathcal{F}^0_{\mathcal{X},\mathbb{T}}$ defines a formal germ of a Lagrangian submanifold

$$\mathcal{L}_{\mathcal{X}} := \{(\mathbf{p}, \mathbf{q}) \in \mathcal{H}_+ \oplus \mathcal{H}_- : \mathbf{p} = d_{\mathbf{q}} \mathcal{F}^0_{\mathcal{X},\mathbb{T}}\} \subset T^*\mathcal{H}_+ \cong \mathcal{H}$$

given by the graph of the differential of $\mathcal{F}^0_{\mathcal{X},\mathbb{T}}$. The submanifold $\mathcal{L}_{\mathcal{X}}$ may be viewed as a formal subscheme of the formal neighbourhood of $-1z$ in $\mathcal{H}$ cut out by the equations

$$p_{a,\mu} = \frac{\partial \mathcal{F}^0_{\mathcal{X},\mathbb{T}}}{\partial q_a^\mu}.$$

Let $x = (x_1, \ldots, x_m)$ be formal variables. Instead of giving a rigorous definition of $\mathcal{L}_{\mathcal{X}}$ as a formal scheme (cf. [CCIT09, Appendix B]) we define the notion of a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x]\!]$-valued point on $\mathcal{L}_X$. By a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x]\!]$*-valued point* of $\mathcal{L}_{\mathcal{X}}$, we mean an element of $\mathcal{H}[\![x]\!]$ of the form

$$-1z + \mathbf{t}(z) + \sum_{n=0}^{\infty} \sum_{d \in \mathrm{NE}(X)} \sum_{\alpha} \frac{Q^d}{n!} \left\langle \mathbf{t}(\bar\psi), \ldots, \mathbf{t}(\bar\psi), \frac{\phi_\alpha}{-z - \bar\psi} \right\rangle^{\mathbb{T}}_{0,n+1,d} \phi^\alpha \tag{4}$$

for some $\mathbf{t}(z) \in \mathcal{H}_+[\![x]\!]$ satisfying

$$\mathbf{t}|_{x=Q=0} = 0. \tag{5}$$

Here the expression $\phi_\alpha/(-z - \bar\psi)$ should be expanded as a power series $\sum_{n=0}^{\infty}(-z)^{-n-1}\phi_\alpha \bar\psi^n$ in z^{-1}. The condition (5) ensures that expression (4) converges in the (Q, x)-adic topology.

Remark 1. As we shall see in § 6, using localization in $\mathbb{T}$-equivariant cohomology, expression (4) lies in a rational version of Givental's symplectic space,

$$\mathcal{H}_{\mathrm{rat}} := H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}\times\mathbb{C}^\times}[\![\mathrm{NE}(X) \cap H_2(X,\mathbb{Z})]\!],$$

where $S_{\mathbb{T}\times\mathbb{C}^\times} = \mathrm{Frac}(H^{\bullet}_{\mathbb{T}\times\mathbb{C}^\times}(\mathrm{pt})) \cong \mathbb{C}(\chi_1, \ldots, \chi_d, z)$ and z is identified with the $\mathbb{C}^\times$-equivariant parameter. The space $\mathcal{H}_{\mathrm{rat}}$ is embedded into $\mathcal{H}$ by the Laurent expansion at $z = \infty$. This fact plays an important role in the characterization of points on $\mathcal{L}_{\mathcal{X}}$ in § 6.

The Lagrangian submanifold $\mathcal{L}_{\mathcal{X}}$ has very special geometric properties.

THEOREM 2 [Giv04, CCIT09, Tse10]. *$\mathcal{L}_{\mathcal{X}}$ is the formal germ of a Lagrangian cone with vertex at the origin such that each tangent space T to the cone is tangent to the cone exactly along zT.*

In other words, if N is a formal neighbourhood in $\mathcal{H}$ of $-1z \in \mathcal{L}_{\mathcal{X}}$, then we have the following statements:

$$\begin{aligned}&\text{(a) } T \cap \mathcal{L}_{\mathcal{X}} = zT \cap N;\\ &\text{(b) for each } \mathbf{f} \in zT \cap N, \text{ the tangent space to } \mathcal{L}_{\mathcal{X}} \text{ at } \mathbf{f} \text{ is } T;\\ &\text{(c) if } T = T_{\mathbf{f}}\mathcal{L}_{\mathcal{X}} \text{ then } \mathbf{f} \in zT \cap N.\end{aligned} \tag{6}$$

Givental has proven that these statements are equivalent to the string equation, dilaton equation, and topological recursion relations [Giv04, Theorem 1]. The statements (6) imply that:

- the tangent spaces T of $\mathcal{L}_{\mathcal{X}}$ are closed under multiplication by z;
- $\mathcal{L}_{\mathcal{X}}$ is the union of the (finite-dimensional) family of germs of linear subspaces

$$\{zT \cap N : T \text{ is a tangent space of } \mathcal{L}_{\mathcal{X}}\}.$$

Remark 3. A finite-dimensional slice of the Lagrangian submanifold $\mathcal{L}_{\mathcal{X}}$ is given by the so-called J-function [Giv04], [Tse10, Definition 3.1.2]

$$J_{\mathcal{X}}(t,z) = 1z + t + \sum_{n=0}^{\infty} \sum_{d \in \mathrm{NE}(\mathcal{X})} \sum_{\alpha} \frac{Q^d}{n!} \left\langle t, \ldots, t, \frac{\phi_\alpha}{z - \bar{\psi}} \right\rangle_{0,n+1,d} \phi^\alpha$$

which is a formal power series in coordinates t^α of $t = \sum_\alpha t^\alpha \phi_\alpha \in H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}$ taking values in $\mathcal{H}$. The J-function $J_{\mathcal{X}}(t,-z)$ gives a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![t]\!]$-valued point of the Lagrangian submanifold $\mathcal{L}_{\mathcal{X}}$.

2.4 Twisted Gromov–Witten invariants

We will need also to consider Gromov–Witten invariants twisted by the $\mathbb{T}$-equivariant inverse Euler class [CG07, Tse10]. In this section we assume that the torus $\mathbb{T}$ acts on the target space $\mathcal{X}$ trivially. This is sufficient for our purposes, as in § 6 we consider twisted Gromov–Witten theory for a $\mathbb{T}$-fixed point of a toric stack. Givental's symplectic formalism for the twisted theory has a subtle but important difference from that in the previous section: we need to work with formal Laurent series in z rather than z^{-1}.

Let $E \to \mathcal{X}$ be a vector bundle equipped with a $\mathbb{T}$-linearization; as mentioned above, $\mathbb{T}$ here acts trivially on the base $\mathcal{X}$. Consider the virtual vector bundle $E_{g,n,d} = R\pi_\star \mathrm{ev}^\star E \in K^0_{\mathbb{T}}(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d))$ where $\pi : \mathcal{C}_{g,n,d} \to \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ and $\mathrm{ev} : \mathcal{C}_{g,n,d} \to \mathcal{X}$ give the universal family of stable maps:

$$\begin{tikzcd} \mathcal{C}_{g,n,d} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & \mathcal{X} \\ \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) & \end{tikzcd}$$

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

Let $e_{\mathbb{T}}^{-1}(\cdot)$ denote the inverse of the $\mathbb{T}$-equivariant Euler class. Twisted Gromov–Witten invariants

$$\langle a_1\bar{\psi}^{k_1},\dots,a_n\bar{\psi}^{k_n}\rangle_{g,n,d}^{e_{\mathbb{T}}^{-1},E}$$

are defined by replacing the weighted virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(\mathcal{X},d)]^w$ in (1) by $[\overline{\mathcal{M}}_{g,n}(\mathcal{X},d)]^w \cap e_{\mathbb{T}}^{-1}(E_{g,n,d})$. The *twisted genus-$g$ descendant potential* is

$$\mathcal{F}^g_{e_{\mathbb{T}}^{-1},E}(\mathbf{t}) := \sum_{n=0}^{\infty}\sum_{d\in \mathrm{NE}(\mathcal{X})}\frac{Q^d}{n!}\langle \mathbf{t},\dots,\mathbf{t}\rangle_{g,n,d}^{e_{\mathbb{T}}^{-1},E}.$$

In the twisted theory, we work with the twisted orbifold Poincaré pairing

$$(\alpha,\beta)_{\mathrm{CR}}^{e_{\mathbb{T}}^{-1},E} := \int_{I\mathcal{X}} \alpha\cup \mathrm{inv}^{\star}\beta\cup e_{\mathbb{T}}^{-1}(IE)$$

where IE is the inertia stack of the total space of E; IE is a vector bundle over $I\mathcal{X}$ such that the fibre over $(x,g)\in I\mathcal{X}$ is the g-fixed subspace of E_x. Givental's symplectic vector space for twisted theory is the $\Lambda^{\mathbb{T}}_{\mathrm{nov}}$-module

$$\mathcal{H}^{\mathrm{tw}} = H^{\bullet}_{\mathrm{CR}}(\mathcal{X})\otimes S_{\mathbb{T}}(\!(z)\!)[\![\mathrm{NE}(X)\cap H_2(X,\mathbb{Z})]\!]$$

equipped with the symplectic form:

$$\Omega(f,g) = \mathrm{Res}_{z=0}(f(-z),g(z))_{\mathrm{CR}}^{e_{\mathbb{T}}^{-1},E}\,dz.$$

The polarization $\mathcal{H}^{\mathrm{tw}} = \mathcal{H}^{\mathrm{tw}}_+\oplus\mathcal{H}^{\mathrm{tw}}_-$ of $\mathcal{H}^{\mathrm{tw}}$ is given by

$$\begin{aligned}\mathcal{H}^{\mathrm{tw}}_+ &= H^{\bullet}_{\mathrm{CR}}(\mathcal{X})\otimes S_{\mathbb{T}}[\![z]\!][\![\mathrm{NE}(X)\cap H_2(X,\mathbb{Z})]\!],\\ \mathcal{H}^{\mathrm{tw}}_- &= H^{\bullet}_{\mathrm{CR}}(\mathcal{X})\otimes S_{\mathbb{T}}[z^{-1}][\![\mathrm{NE}(X)\cap H_2(X,\mathbb{Z})]\!].\end{aligned}$$

Let $\{\phi_\mu\}$, $\{\phi^\mu\}$ be dual bases of $H^{\bullet}_{\mathrm{CR}}(\mathcal{X})\otimes S_{\mathbb{T}}$ with respect to the twisted orbifold Poincaré pairing. They define Darboux coordinates $\{p_{a,\mu},q_a^\mu\}$ on $\mathcal{H}^{\mathrm{tw}}$ as in (3). The Lagrangian submanifold $\mathcal{L}_{\mathrm{tw}}$ of the twisted theory is then defined similarly: a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x]\!]$-valued point of $\mathcal{L}_{\mathrm{tw}}$ is an element of $\mathcal{H}^{\mathrm{tw}}[\![x]\!]$ of the form

$$-1z+\mathbf{t}(z)+\sum_{n=0}^{\infty}\sum_{d\in\mathrm{NE}(X)}\sum_{\alpha}\frac{Q^d}{n!}\left\langle \mathbf{t}(\bar{\psi}),\dots,\mathbf{t}(\bar{\psi}),\frac{\phi_\alpha}{-z-\bar{\psi}}\right\rangle_{0,n+1,d}^{e_{\mathbb{T}}^{-1},E}\phi^{\alpha} \tag{7}$$

for some $\mathbf{t}(z)\in\mathcal{H}^{\mathrm{tw}}_+[\![x]\!]$ satisfying $\mathbf{t}|_{x=Q=0}=0$. Note that expression (7) makes sense as an element of $\mathcal{H}^{\mathrm{tw}}[\![x]\!]$. We use here the fact that, as $\mathbb{T}$ acts trivially on $\mathcal{X}$, the descendant classes $\bar{\psi}_i$ are nilpotent on each moduli space $\overline{\mathcal{M}}_{0,n}(\mathcal{X},d)$; therefore $\mathbf{t}(\bar{\psi})=\sum_{k=0}^{\infty}t_k\bar{\psi}^k$ and $\phi_\alpha/(-z-\bar{\psi}) = \sum_{n=0}^{\infty}\phi_\alpha\bar{\psi}^n(-z)^{-n-1}$ truncate to finite series on each moduli space $\overline{\mathcal{M}}_{0,n}(\mathcal{X},d)$.

Remark 4. The analogue of Theorem 2 holds for $\mathcal{L}_{\mathrm{tw}}$.

3. Toric Deligne–Mumford stacks

In this section we discuss some background material on toric stacks. More details can be found in [BCS05, Iwa09a, Iwa09b, FMN10].

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

3.1 Basics

Following Borisov *et al.* [BCS05], a *toric Deligne–Mumford stack* is defined in terms of a stacky fan

$$\mathbf{\Sigma} = (N, \Sigma, \rho)$$

where N is a finitely generated abelian group, $\Sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a rational simplicial fan, and $\rho : \mathbb{Z}^n \to N$ is a homomorphism. We denote by ρ_i the image under ρ of the ith standard basis vector e_i of $\mathbb{Z}^n$. Let $\mathbb{L} \subset \mathbb{Z}^n$ be the kernel of ρ. The exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^n \xrightarrow{\ \rho\ } N$$

is called the *fan sequence*. By assumption, ρ has finite cokernel and the images $\bar{\rho}_i$, $1 \leqslant i \leqslant n$, of the ρ_i under the canonical map $N \to N_{\mathbb{Q}}$ generate one-dimensional cones of the simplicial fan Σ.

By abuse of notation, we sometimes identify a cone $\sigma \in \Sigma$ with the subset $\{i : \bar{\rho}_i \in \sigma\}$ of $\{1, \dots, n\}$ and write $i \in \sigma$ instead of $\bar{\rho}_i \in \sigma$. The set of *anti-cones* is defined to be

$$\mathcal{A} := \left\{ I \subset \{1, \dots, n\} : \sum_{i \notin I} \mathbb{R}_{\geqslant 0} \bar{\rho}_i \text{ is a cone in } \Sigma \right\}.$$

Let

$$\mathcal{U}_{\mathcal{A}} := \mathbb{C}^n \Big\backslash \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I$$

where $\mathbb{C}^I \subset \mathbb{C}^n$ is the subvariety determined by the ideal in $\mathbb{C}[Z_1, \dots, Z_n]$ generated by $\{Z_i : i \notin I\}$. Let $\rho^\vee : (\mathbb{Z}^*)^n \to \mathbb{L}^\vee$ be the Gale dual of ρ [BCS05]. Here $\mathbb{L}^\vee := H^1(\mathrm{Cone}(\rho)^*)$ is an extension of the ordinary dual $\mathbb{L}^* = \mathrm{Hom}(\mathbb{L}, \mathbb{Z})$ by a torsion subgroup. We have $\mathrm{Ker}(\rho^\vee) = N^*$. The exact sequence

$$0 \longrightarrow N^* \longrightarrow (\mathbb{Z}^*)^n \xrightarrow{\ \rho^\vee\ } \mathbb{L}^\vee \tag{8}$$

is called the *divisor sequence*.

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to $\rho^\vee$ gives a map

$$\alpha : G \to (\mathbb{C}^\times)^n \tag{9}$$

where $G := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{L}^\vee, \mathbb{C}^\times)$. The toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ associated to $\mathbf{\Sigma}$ is defined to be the quotient stack

$$\mathcal{X}(\mathbf{\Sigma}) := [\mathcal{U}_{\mathcal{A}} / G]$$

where G acts on $\mathcal{U}_{\mathcal{A}}$ via α.

Throughout this paper we assume that the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ has *semi-projective* coarse moduli space, i.e. that the coarse moduli space $X(\Sigma)$ is a toric variety that has at least one torus-fixed point, such that the natural map $X(\Sigma) \to \mathrm{Spec}\, H^0(X(\Sigma), \mathcal{O}_{X(\Sigma)})$ is projective. In terms of the fan Σ, this is equivalent [CLS11] to demanding that the support $|\Sigma|$ of the fan Σ is full-dimensional and convex, and that there exists a strictly convex piecewise linear function $\phi : |\Sigma| \to \mathbb{R}$.

Let N_{tor} denote the torsion subgroup of N, and set $\overline{N} := N/N_{\mathrm{tor}}$. For $c \in N$ we denote by $\overline{c} \in \overline{N}$ the image of c under the natural projection $N \to \overline{N}$. Given a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, one can consider the set Box defined as follows. For a cone $\sigma \in \Sigma$, define

$$\mathrm{Box}(\sigma) := \left\{ b \in N : \bar{b} = \sum_{i \in \sigma} a_i \bar{\rho}_i \text{ for some } a_i \text{ with } 0 \leqslant a_i < 1 \right\}$$

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

and set $\mathrm{Box}(\mathbf{\Sigma}) := \bigcup_{\sigma\in\Sigma}\mathrm{Box}(\sigma)$. Components of the inertia stack $I\mathcal{X}(\mathbf{\Sigma})$ are indexed by Box; we write $I\mathcal{X}(\mathbf{\Sigma})_b$ for the component corresponding to $b \in \mathrm{Box}$. The involution inv on $I\mathcal{X}(\mathbf{\Sigma})$ induces an involution $b \mapsto \hat{b}$ on $\mathrm{Box}(\mathbf{\Sigma})$.

Each cone $\sigma \in \Sigma$ defines a closed toric substack $\mathcal{X}(\mathbf{\Sigma})_\sigma \cong \mathcal{X}(\mathbf{\Sigma}/\sigma)$, where $\mathbf{\Sigma}/\sigma$ denotes the quotient stacky fan [BCS05, § 4] defined on the quotient space $N(\sigma) = N/\sum_{i\in\sigma}\mathbb{Z}\rho_i$. The component $I\mathcal{X}(\mathbf{\Sigma})_b$ of the inertia stack corresponding to $b \in \mathrm{Box}(\mathbf{\Sigma})$ is isomorphic to the toric substack $\mathcal{X}(\mathbf{\Sigma})_{\sigma(b)}$, where $\sigma(b)$ is the minimal cone containing $\bar{b}$.

3.2 Extended stacky fans

Following Jiang [Jia08], toric Deligne–Mumford stacks can also be described using extended stacky fans. Let $\mathbf{\Sigma} = (N, \Sigma, \rho)$ be a stacky fan, and let S be a finite set equipped with a map[1] $S \to N_\Sigma := \{c \in N : \bar{c} \in |\Sigma|\}$. We label the finite set S by $\{1, \ldots, m\}$, where $m = |S|$, and write $s_j \in N$ for the image of the jth element of S. The S*-extended stacky fan* is given by the same group N, the same fan Σ, and the fan map $\rho^S : \mathbb{Z}^{n+m} \to N$ defined by

$$\rho^S(e_i) = \begin{cases} \rho_i & 1 \leqslant i \leqslant n, \\ s_{i-n} & n+1 \leqslant i \leqslant n+m. \end{cases}$$

Given an S-extended stacky fan (N, Σ, ρ^S), an associated stack may be defined as follows. Define

$$\mathcal{U}_{\mathcal{A},S} := \mathcal{U}_{\mathcal{A}} \times (\mathbb{C}^\times)^m.$$

Let $\mathbb{L}^S$ be the kernel of $\rho^S : \mathbb{Z}^{n+m} \to N$. Applying Gale duality to the S-extended fan sequence $0 \to \mathbb{L}^S \to \mathbb{Z}^{n+m} \to N$ gives the S-extended divisor sequence

$$0 \longrightarrow N^* \longrightarrow (\mathbb{Z}^*)^{n+m} \xrightarrow{\rho^{S\vee}} \mathbb{L}^{S\vee}.$$

Applying $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^\times)$ to the S-extended divisor sequence gives a map $\alpha^S : G^S \to (\mathbb{C}^\times)^{n+m}$ where $G^S := \mathrm{Hom}_\mathbb{Z}(\mathbb{L}^{S\vee}, \mathbb{C}^\times)$. We consider the quotient stack

$$[\mathcal{U}_{\mathcal{A},S}/G^S] \tag{10}$$

where G^S acts on $\mathcal{U}_{\mathcal{A},S}$ via α^S. Jiang showed [Jia08] that this stack associated to the S-extended stacky fan (N, Σ, ρ^S) is isomorphic to the stack $\mathcal{X}(\mathbf{\Sigma})$.

3.3 Torus action and line bundles

The inclusion $(\mathbb{C}^\times)^n \subset \mathcal{U}_\mathcal{A}$ induces an open embedding of the Picard stack $\mathcal{T} = [(\mathbb{C}^\times)^n/G]$ into $\mathcal{X}(\mathbf{\Sigma})$. We have $\mathcal{T} \cong \mathbb{T} \times BN_{\mathrm{tor}}$ with $\mathbb{T} := (\mathbb{C}^\times)^n/\mathrm{Im}\,\alpha \cong N \otimes \mathbb{C}^\times$ and $N_{\mathrm{tor}} \cong \mathrm{Ker}\,\alpha$, where α is given in (9). The Picard stack $\mathcal{T}$ acts naturally on $\mathcal{X}(\mathbf{\Sigma})$ and the $\mathcal{T}$-action restricts to the $\mathbb{T}$-action on $\mathcal{X}(\mathbf{\Sigma})$.

A line bundle on $\mathcal{X}(\mathbf{\Sigma})$ corresponds to a G-equivariant line bundle on $\mathcal{U}_\mathcal{A}$, and a $\mathcal{T}$-equivariant line bundle on $\mathcal{X}(\mathbf{\Sigma})$ corresponds to a $(\mathbb{C}^\times)^n$-equivariant line bundle on $\mathcal{U}_\mathcal{A}$. Thus we have natural identifications

$$\mathrm{Pic}(\mathcal{X}(\mathbf{\Sigma})) \cong \mathrm{Hom}(G, \mathbb{C}^\times) \cong \mathbb{L}^\vee,$$
$$\mathrm{Pic}_\mathcal{T}(\mathcal{X}(\mathbf{\Sigma})) \cong \mathrm{Hom}((\mathbb{C}^\times)^n, \mathbb{C}^\times) \cong (\mathbb{Z}^n)^*.$$

The natural map $\mathrm{Pic}_\mathcal{T}(\mathcal{X}(\mathbf{\Sigma})) \to \mathrm{Pic}(\mathcal{X}(\mathbf{\Sigma}))$ is identified with the divisor map $\rho^\vee : (\mathbb{Z}^n)^* \to \mathbb{L}^\vee$ in (8). We write $u_1, \ldots, u_n$ for the basis of $\mathcal{T}$-equivariant line bundles on $\mathcal{X}(\mathbf{\Sigma})$ corresponding to the standard basis of $(\mathbb{Z}^n)^*$ and write $D_1, \ldots, D_n$ for the corresponding non-equivariant line bundles,i.e. $D_i = \rho^\vee(u_i)$. Abusing notation, we also write u_i or D_i for the corresponding

[1] The reader can keep in mind the most basic case where S is a subset of $\mathrm{Box}(\mathbf{\Sigma})$.

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

($\mathbb{T}$-equivariant or non-equivariant) first Chern classes. These are the ($\mathbb{T}$-equivariant or non-equivariant) Poincaré duals of the toric divisors $[\{Z_i = 0\}/G] \subset [\mathcal{U}_{\mathcal{A}}/G]$.

3.4 Chen–Ruan cohomology

The Chen–Ruan orbifold cohomology (see § 2.1) of the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ associated to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$ has been computed by Borisov *et al.* [BCS05] and, in the semi-projective case, by Jiang and Tseng [JT08]:

$$H^{\bullet}_{\mathrm{CR}}(\mathcal{X}(\mathbf{\Sigma}), \mathbb{C}) \simeq \frac{\mathbb{C}[N_\Sigma]}{\{\sum_{i=1}^n \chi(\rho_i) y^{\rho_i} : \chi \in N^*\}}$$

where

(i) $\mathbb{C}[N_\Sigma] := \bigoplus_{c \in N_\Sigma} \mathbb{C} y^c$ with the product

$$y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \overline{c_1}, \overline{c_2} \in \sigma, \\ 0 & \text{otherwise;} \end{cases}$$

(ii) $N_\Sigma := \{c \in N : \bar{c} \in \sigma \text{ for some } \sigma \in \Sigma\}$.

Similarly, the $\mathbb{T}$-equivariant Chen–Ruan orbifold cohomology of the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is [Liu13]

$$H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}(\mathbf{\Sigma}), \mathbb{C}) \simeq \frac{R_{\mathbb{T}}[N_\Sigma]}{\{\chi - \sum_{i=1}^n \chi(\rho_i) y^{\rho_i} : \chi \in N^* \otimes \mathbb{C} \cong H^2_{\mathbb{T}}(\mathrm{pt})\}},$$

where

(i) $R_{\mathbb{T}} := H^{\bullet}_{\mathbb{T}}(\mathrm{pt}) = \mathrm{Sym}^{\bullet}_{\mathbb{C}}(N^* \otimes \mathbb{C})$;
(ii) $R_{\mathbb{T}}[N_\Sigma] := \bigoplus_{c \in N_\Sigma} R_{\mathbb{T}} y^c$ with the product

$$y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \overline{c_1}, \overline{c_2} \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

The ($\mathbb{T}$-equivariant or non-equivariant) classes u_i, D_i in § 3.3 correspond to y^{ρ_i} in the above descriptions. For $b \in \mathrm{Box}(\mathbf{\Sigma})$, y^b is the identity class supported on the twisted sector $I\mathcal{X}(\mathbf{\Sigma})_b$.

3.5 Maps to one-dimensional torus orbits

We next describe toric maps from certain very simple toric orbifolds $\mathbb{P}_{r_1,r_2}$ to the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$. This establishes notation that we will need to state and prove our mirror theorem.

Definition 5. Let r_1 and r_2 be positive integers. There is a unique Deligne–Mumford stack with coarse moduli space equal to $\mathbb{P}^1$, isotropy group μ_{r_1} at $0 \in \mathbb{P}^1$, isotropy group μ_{r_2} at $\infty \in \mathbb{P}^1$, and no other non-trivial isotropy groups. We call this stack $\mathbb{P}_{r_1,r_2}$.

Let $r = \mathrm{lcm}(r_1, r_2)$, and let r'_1 and r'_2 satisfy $r_1 r'_2 = r'_1 r_2 = r$ (i.e. $r'_i = r_i / \gcd(r_1, r_2)$). The stack $\mathbb{P}_{r_1,r_2}$ is a toric Deligne–Mumford stack with fan sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} r'_2 \\ r'_1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -r_1, r_2 \end{pmatrix}} \mathbb{Z}$$

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

Proposition 6 (cf. [Joh14, Lemma 2]). *Let $\mathcal{X}(\mathbf{\Sigma})$ be the toric Deligne–Mumford stack associated to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, and suppose that the fan Σ is complete and one-dimensional. Let $\sigma_1 = \langle \bar{\rho}_1 \rangle$, $\sigma_2 = \langle \bar{\rho}_2 \rangle$ be the one-dimensional cones of Σ, and assume without loss of generality that $\bar{\rho}_1 < 0$ and $\bar{\rho}_2 > 0$ in $N_{\mathbb{Q}} \simeq \mathbb{Q}$. Let $w_2\rho_1 + w_1\rho_2 = 0$ with $w_1, w_2 \in \mathbb{Z}_{>0}$ be the minimal integral relation between $\rho_1, \rho_2 \in N$. The following are equivalent:*

(a) *a representable toric morphism $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$ for some r_1, r_2 such that $f(0) = \mathcal{X}(\mathbf{\Sigma})_{\sigma_1}$ and $f(\infty) = \mathcal{X}(\mathbf{\Sigma})_{\sigma_2}$;*
(b) *two box elements $b_1 \in \mathrm{Box}(\sigma_1)$, $b_2 \in \mathrm{Box}(\sigma_2)$ and non-negative integers q_1, q_2 such that $q_1\rho_1 + q_2\rho_2 + b_1 + b_2 = 0$ in N;*
(c) *a box element $b_1 \in \mathrm{Box}(\sigma_1)$ and a strictly positive rational number l such that $w_2 l - f_1$ is a non-negative integer, where $\overline{b}_1 = f_1\bar{\rho}_1$.*

These data are related as follows: r_i is the order of b_i in $N/\mathbb{Z}\rho_i$; $l = (q_2 + f_2)/w_1 = (q_1 + f_1)/w_2$; $q_1 = \lfloor lw_2 \rfloor$, $q_2 = \lfloor lw_1 \rfloor$, $f_1 = \langle lw_2 \rangle$, $f_2 = \langle lw_1 \rangle$.

Proof. Let

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix} w_2 \\ w_1 \end{pmatrix}"] & \mathbb{Z}^2 \arrow[r, "\rho"] & N \end{tikzcd} \tag{11}$$

be the fan sequence for $\mathcal{X}(\mathbf{\Sigma})$. A representable toric morphism $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$ is given by a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix} r'_2 \\ r'_1 \end{pmatrix}"] \arrow[d, "m"'] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} -r_1 & r_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}"'] & \mathbb{Z} \arrow[d, "\eta"] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix} w_2 \\ w_1 \end{pmatrix}"'] & \mathbb{Z}^2 \arrow[r, "\rho"'] & N \end{tikzcd} \tag{12}$$

for some integers m_1, m_2, m and some map η. Given a morphism as in (a), and hence a commutative diagram (12), let b_1 be the unique element of $\mathrm{Box}(\sigma_1)$ such that $b_1 \equiv \eta(-1) \mod \langle \rho_1 \rangle$, and let b_2 be the unique element of $\mathrm{Box}(\sigma_2)$ such that $b_2 \equiv \eta(1) \mod \langle \rho_2 \rangle$. Then there exist unique non-negative integers q_1, q_2 such that

$$\eta(-1) = q_1\rho_1 + b_1, \quad \eta(1) = q_2\rho_2 + b_2,$$

and we have $q_1\rho_1 + q_2\rho_2 + v_1 + v_2 = 0$ in N. Thus a morphism as in (a) determines data as in (b).

Conversely, suppose that we are given v_1, v_2, q_1, q_2 as in (b). Define $\eta : \mathbb{Z} \to N$ by setting

$$\eta(1) = -b_1 - q_1\rho_1 = b_2 + q_2\rho_2.$$

Now set $r_i = \operatorname{ord} b_i$ in the group N/ρ_i; by definition there are integers k_1, k_2 such that $r_i b_i = k_i \rho_i$, and (for instance by looking at images in $\overline{N}$) we see that $0 \leqslant k_i < r_i$. Now set

$$m_1 = r_1 q_1 + k_1, \quad m_2 = r_2 q_2 + k_2.$$

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

The diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\begin{pmatrix} r_1' \\ r_2' \end{pmatrix}"] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} -r_1 & r_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}"'] & \mathbb{Z} \arrow[d, "\eta"] \\ & \mathbb{Z}^2 \arrow[r, "\rho"'] & N \end{tikzcd}$$

is commutative: $m_1\rho_1 = r_1q_1\rho_1 + k_1\rho_1 = r_1(-b_1+\eta(-1)) + r_1b_1 = \eta(-r_1)$, and similarly $m_2\rho_2 = \eta(r_2)$. Thus,

$$\begin{pmatrix} m_1r_2' \\ m_2r_1' \end{pmatrix} \in \operatorname{Ker}\rho.$$

The fan sequence (11) defining $\mathcal{X}(\mathbf{\Sigma})$ is exact at $\mathbb{Z}^2$, and we deduce that there exists an integer $m > 0$ such that

$$\begin{pmatrix} m_1r_2' \\ m_2r_1' \end{pmatrix} = \begin{pmatrix} w_2m \\ w_1m \end{pmatrix}$$

and hence that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix} r_2' \\ r_1' \end{pmatrix}"] \arrow[d, "m"'] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} -r_1 & r_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}"'] & \mathbb{Z} \arrow[d, "\eta"] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix} w_2 \\ w_1 \end{pmatrix}"'] & \mathbb{Z}^2 \arrow[r, "\rho"'] & N \end{tikzcd}$$

defines a stable representable morphism $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$.

It is almost immediate that the constructions $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$ are inverses of each other: the key point is that, if $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$ is representable, then r_i is the order of b_i in N/ρ_i.

The equivalence $(b) \Leftrightarrow (c)$ is immediate: we set $q_1 = w_2l - f_1$, write $w_1l = q_2 + f_2$ with $f_2 = \langle w_1l\rangle$ the fractional part and $q_2 = \lfloor w_1l\rfloor$ the integer part, and set $b_2 = -q_1\rho_1 - q_2\rho_2 - b_1$. $\square$

Remark 7. The box elements b_1, b_2 in the above proposition are given by the restrictions of f to $0, \infty \in \mathbb{P}_{r_1,r_2}$, respectively. The rational number $l > 0$ in (c) measures the 'degree' of the map f in the sense that $l = \int_{\mathbb{P}_{r_1,r_2}} c_1(f^*\mathcal{O}(1)) = m/\operatorname{lcm}(r_1, r_2)$, where $\mathcal{O}(1)$ is the positive generator of $\operatorname{Pic}(\mathcal{X}(\mathbf{\Sigma}))$ modulo torsion. The degree of the map between the coarse moduli spaces $\bar{f} : \mathbb{P}^1 \cong |\mathbb{P}_{r_1,r_2}| \to \mathbb{P}^1 \cong X(\Sigma)$ is given by $\overline{\eta(1)} = (q_2\bar{\rho}_2 + \bar{b}_2) = (q_2 + f_2)\bar{\rho}_2 = lw_1\bar{\rho}_2 = lw_2|\bar{\rho}_1| \in \mathbb{Z}$.

Notation 8. Let $\mathbf{\Sigma} = (N, \Sigma, \rho)$ be a stacky fan. We write $\sigma|\sigma'$ if σ, $\sigma' \in \Sigma$ are top-dimensional cones that meet along a codimension-one face. Whenever $\sigma|\sigma'$, we write j for the unique index such that $\bar{\rho}_j$ is in σ but not in σ', and j' for the unique index such that $\bar{\rho}_{j'}$ is in σ' but not in σ.

Notation 9. Let $\mathbf{\Sigma} = (N, \Sigma, \rho)$ be a stacky fan. Given $b \in \operatorname{Box}(\mathbf{\Sigma})$, we define $b_i \in [0, 1)$, $1 \leqslant i \leqslant n$, by the conditions:

– $\bar{b} = \sum_{i=1}^{n} b_i \bar{\rho}_i$;
– $b_i = 0$ if $i \notin \sigma(b)$, where $\sigma(b) \in \Sigma$ is the minimal cone containing $\bar{b}$.

Proposition 10. *Let $\mathcal{X}(\mathbf{\Sigma})$ be the toric Deligne–Mumford stack associated to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$. Suppose that σ, $\sigma' \in \Sigma$ satisfy $\sigma|\sigma'$ and let $b \in \mathrm{Box}(\sigma)$. The following are equivalent:*

(i) *a representable toric morphism $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$ such that $f(0) = \mathcal{X}(\mathbf{\Sigma})_\sigma$, $f(\infty) = \mathcal{X}(\mathbf{\Sigma})_{\sigma'}$ and the restriction $f|_0 : B\mu_{r_1} \to \mathcal{X}(\mathbf{\Sigma})_\sigma$ gives the box element $\hat{b} \in \mathrm{Box}(\sigma)$;*
(ii) *a positive rational number c such that $\langle c \rangle = \hat{b}_j$.*

Here j is as in Notation 8, and $\hat{b} = \mathrm{inv}(b)$ (see § 2.1).

Proof. Let

$$w_j \rho_j + \left(\sum_{i \in \sigma \cap \sigma'} w_i \rho_i \right) + w_{j'} \rho_{j'} = 0$$

be the minimal integral relation between $\{\rho_i : \bar{\rho}_i \in \sigma \cup \sigma'\}$ such that $w_j > 0$. Replacing Σ by Σ/τ, we reduce to the case where $\mathcal{X}(\Sigma)$ is one-dimensional. The result now follows from Proposition 6, with w_1 there equal to $w_{j'}$ here, w_2 there equal to w_j here, and c equal to $l w_j$. $\square$

Remark 11. Note that the choice of σ, σ', b and c in Proposition 10 determines the map $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$ uniquely, and hence determines both r_2 and the box element $b' \in \mathrm{Box}(\sigma')$ given by the restriction $f|_\infty : B\mu_{r_2} \to \mathcal{X}(\mathbf{\Sigma})_{\sigma'}$. More precisely, b' is the unique element of $\mathrm{Box}(\sigma')$ such that

$$\hat{b} + \lfloor c \rfloor \rho_j + q' \rho_{j'} + b' \equiv 0 \mod \bigoplus_{i \in \sigma \cap \sigma'} \mathbb{Z} \rho_i \tag{13}$$

for some $q' \in \mathbb{Z}_{\geqslant 0}$. Note the asymmetry between b and b': the restriction $f|_0$ gives $\hat{b} = \mathrm{inv}(b)$ and the restriction $f|_\infty$ gives b'. This convention is useful in our recursion analysis.

Definition 12. Let σ, $\sigma' \in \Sigma$ be top-dimensional cones satisfying $\sigma|\sigma'$. Let j, j' be as in Notation 8. Define $l(c, \sigma, j)$ to be the element of $\mathbb{L} \otimes \mathbb{Q} \cong H_2(X, \mathbb{Q})$ given by the unique relation of the form

$$c \bar{\rho}_j + \left(\sum_{i \in \sigma \cap \sigma'} c_i \bar{\rho}_i \right) + c' \bar{\rho}_{j'} = 0.$$

Remark 13. When we have a box element $b \in \mathrm{Box}(\sigma)$ satisfying $\langle c \rangle = \hat{b}_j$, $l(c, \sigma, j)$ is the degree of the representable toric morphism $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$ specified by a rational number $c > 0$ in Proposition 10(2). We have $D_j \cdot l(c, \sigma, j) = c$, $D_{j'} \cdot l(c, \sigma, j) = c'$, $D_i \cdot l(c, \sigma, j) = c_i$ for $i \in \sigma \cap \sigma'$ and $D_i \cdot l(c, \sigma, j) = 0$ for $i \notin \sigma \cup \sigma'$.

Definition 14. Let σ, $\sigma' \in \Sigma$ be top-dimensional cones satisfying $\sigma|\sigma'$. Let j, j' be as in Notation 8. Let $b \in \mathrm{Box}(\sigma)$ and $b' \in \mathrm{Box}(\sigma')$. Define $\Lambda E_{\sigma,b}^{\sigma',b'} \subset \mathbb{L} \otimes \mathbb{Q} \cong H_2(\mathcal{X}, \mathbb{Q})$ to be the set of degrees of representable toric morphisms $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$ such that $f(0) = \mathcal{X}(\Sigma)_\sigma$, $f(\infty) = \mathcal{X}(\Sigma)_{\sigma'}$ and $f|_0$ and $f|_\infty$ give respectively the box elements $\hat{b}$ and b'. In other words:

$$\Lambda E_{\sigma,b}^{\sigma',b'} = \left\{ l(c, \sigma, j) \in \mathbb{L} \otimes \mathbb{Q} : \begin{array}{l} c > 0 \text{ such that } \langle c \rangle = \hat{b}_j \text{ and} \\ \text{that (13) holds for some } q' \in \mathbb{Z}_{\geqslant 0} \end{array} \right\}.$$

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

4. The extended Picard group

In this section we introduce notions of extended Picard group for a Deligne–Mumford stack $\mathcal{X}$ and extended degree for an orbifold stable map $f : \mathcal{C} \to \mathcal{X}$. There is less here than meets the eye: the extended degree of f amounts in the end to a convenient way of packaging the extra discrete data attached to f, given by the elements of $\mathrm{Box}(\mathcal{X})$ associated to the marked points. In what follows we will use this material only when $\mathcal{X}$ is a toric Deligne–Mumford stack, but the definitions make sense for general Deligne–Mumford stacks and we give them in this context.

DEFINITION 15. The *box* of a Deligne–Mumford stack $\mathcal{X}$, denoted $\mathrm{Box}\,\mathcal{X}$, is the set of generic representable morphisms $b : B\mu_r \to \mathcal{X}$. In other words, it is the set of connected components of the inertia stack $I\mathcal{X}$. We write the *order* r of the box element b as r_b.

Remark 16. If $\mathcal{X}$ is a toric Deligne–Mumford stack then this reduces to the notion of $\mathrm{Box}(\mathbf{\Sigma})$ given in § 3.1.

DEFINITION 17. Let $\mathcal{X}$ be a Deligne–Mumford stack and let S be a finite set equipped with a map $S \to \mathrm{Box}\,\mathcal{X}$. Abusing notation, we denote an element of S and its image in $\mathrm{Box}\,\mathcal{X}$ by the same symbol b. The *S-extended Picard group* of $\mathcal{X}$, denoted by $\mathrm{Pic}^S \mathcal{X}$, is defined by the exact sequence

$$0 \longrightarrow \mathrm{Pic}^S \mathcal{X} \longrightarrow \mathrm{Pic}\,\mathcal{X} \oplus \bigoplus_{b\in S} r_b^{-1}\mathbb{Z} \longrightarrow \bigoplus_{b\in S} r_b^{-1}\mathbb{Z}/\mathbb{Z} \longrightarrow 0\,. \tag{14}$$

In other words, an element of $\mathrm{Pic}^S \mathcal{X}$ is a pair (L, φ) where $L \in \mathrm{Pic}\,\mathcal{X}$ is a line bundle on $\mathcal{X}$, and $\varphi : S \to \mathbb{Q}$ has the property that $\varphi(b) + \mathrm{age}_b(L) \in \mathbb{Z}$, where $\mathrm{age}_b(L)$ is the age of L at b, i.e. $\mathrm{age}_b(L) = k_b/r_b$ with $0 \leqslant k_b < r_b$ the character of the μ_{r_b}-representation $b^\star L$.

DEFINITION 18. Let $f : (\mathcal{C}, x_1, \ldots, x_k) \to \mathcal{X}$ be an orbifold stable map. An *S-decoration* of f is an assignment of $s_j \in S$ to each marking x_j such that the element of $\mathrm{Box}\,\mathcal{X}$ given by $f|_{x_j}$ coincides with the image of s_j in $\mathrm{Box}\,\mathcal{X}$. The *$S$-extended degree* of an S-decorated orbifold stable map f is an element of $(\mathrm{Pic}^S \mathcal{X})^*$ defined by

$$\deg^S(f)(L, \varphi) = \deg f^*L + \sum_{j=1}^{k} \varphi(s_j).$$

The Riemann–Roch theorem for orbifold curves [AGV08] shows that the right-hand side is an integer. The *S-extended Mori cone* is the cone $\mathrm{NE}^S(\mathcal{X}) \subset (\mathrm{Pic}^S \mathcal{X})^* \otimes \mathbb{R}$ generated by the S-extended degrees of S-decorated orbifold stable maps. One can easily see that

$$\mathrm{NE}^S(\mathcal{X}) \cong \mathrm{NE}(\mathcal{X}) \times \mathbb{R}^{|S|}_{\geqslant 0}$$

under the standard decomposition

$$(\mathrm{Pic}^S \mathcal{X})^* \otimes \mathbb{R} \cong ((\mathrm{Pic}\,\mathcal{X})^* \otimes \mathbb{R}) \oplus \mathbb{R}^{|S|} \tag{15}$$

induced from (14), where $\mathrm{NE}(\mathcal{X})$ denotes the usual Mori cone.

Remark 19. We can think of elements of S as 'states' to be inserted at markings of a stable map. If an S-decorated orbifold stable map has a degree $d \in H_2(X, \mathbb{Z})$ and each 'state' $b \in S$ is inserted n_b times into it, the S-extended degree with respect to (L, φ) is given by

$$\int_d c_1(f^*L) + \sum_{b\in S} n_b \varphi(b).$$

The value $\varphi(b)$ can be viewed as the degree of the variable dual to $b \in S$.

Remark 20. When $\mathcal{X}$ is Gorenstein and the subset S consists of those box elements of age 1, the S-extended degree of a stable map is essentially the same thing as the orbifold Neron–Severi degree defined by Bryan and Graber [BG09, § 2].

4.1 Extended degrees for toric stacks

Suppose now that $\mathcal{X} = \mathcal{X}(\boldsymbol{\Sigma})$ is the toric Deligne–Mumford stack associated to a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \rho)$, and that S is a finite set equipped with a map $S \to N_\Sigma = \{c \in N : \bar{c} \in |\Sigma|\}$. By composing it with a natural projection $N_\Sigma \to \operatorname{Box}(\boldsymbol{\Sigma})$ we obtain a map $S \to \operatorname{Box}(\boldsymbol{\Sigma})$. We now identify $\mathbb{L}^{S\vee}$ with $\operatorname{Pic}^S \mathcal{X}(\boldsymbol{\Sigma})$.

Let $m = |S|$ and let $s_1, \ldots, s_m \in N_\Sigma$ be the images of elements of S in N_Σ. The fan sequence and the S-extended fan sequence fit into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{L} \arrow[r, "\iota"] \arrow[d] & \mathbb{L}^S \arrow[r] \arrow[d] & \mathbb{Z}^m \arrow[d, equal] & \\
0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d, "\rho"'] & \mathbb{Z}^{n+m} \arrow[r] \arrow[d, "\rho^S"'] & \mathbb{Z}^m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, equal] & N \arrow[r] & 0 &
\end{tikzcd} \tag{16}$$

with exact rows and columns. We give a splitting of the first row over the rational numbers. Define

$$\mu : \mathbb{Q}^m \to \mathbb{L}^S \otimes \mathbb{Q}$$

by sending the jth standard basis vector to

$$e_{j+n} - \sum_{i \in \sigma(j)} s_{ji} e_i \in \mathbb{L}^S \otimes \mathbb{Q} \subset \mathbb{Q}^{n+m}$$

where $\sigma(j)$ is the minimal cone containing $\bar{s}_j$ and the positive numbers s_{ji} are determined by $\sum_{i \in \sigma(j)} s_{ji} \bar{\rho}_i = \bar{s}_j$. The map μ defines a splitting of the first row of (16) over $\mathbb{Q}$:

$$\mathbb{L}^S \otimes \mathbb{Q} \cong (\mathbb{L} \otimes \mathbb{Q}) \oplus \mathbb{Q}^m. \tag{17}$$

Let r_j be the order of the image of $s_j \in N_\Sigma$ in $N / \sum_{i \in \sigma(j)} \mathbb{Z}\rho_i$. Then we have $r_j s_{ji} \in \mathbb{Z}$. Therefore the dual of μ gives

$$\mu^* : \mathbb{L}^{S\vee} \longrightarrow (\mathbb{L}^S)^* \longrightarrow \bigoplus_{j=1}^m r_j^{-1}\mathbb{Z}.$$

One can check that the map μ^* together with the canonical map $\iota^* : \mathbb{L}^\vee \to \mathbb{L}^{S\vee}$ fits into the exact sequence

$$0 \longrightarrow \mathbb{L}^{S\vee} \xrightarrow{(\iota^*, \mu^*)} \mathbb{L}^\vee \oplus \bigoplus_{j=1}^m r_j^{-1}\mathbb{Z} \xrightarrow{(\mathrm{res}, \mathrm{can})} \bigoplus_{j=1}^m r_j^{-1}\mathbb{Z}/\mathbb{Z} \longrightarrow 0$$

where res maps an element of $\mathbb{L}^\vee \cong \operatorname{Pic}(\mathcal{X})$ to the ages of the corresponding line bundle at the box elements given by $s_1, \ldots, s_m$ and can is the canonical projection. Thus we obtain the following proposition.

Proposition 21. *We have* $\operatorname{Pic}^S \mathcal{X}(\boldsymbol{\Sigma}) \cong \mathbb{L}^{S\vee}$.

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

We have $(\operatorname{Pic}\mathcal{X}(\mathbf{\Sigma}))^*\otimes\mathbb{R} \cong \mathbb{L}\otimes\mathbb{R}$ and $(\operatorname{Pic}^S\mathcal{X}(\mathbf{\Sigma}))^*\otimes\mathbb{R} \cong \mathbb{L}^S\otimes\mathbb{R}$. The standard decomposition (15) matches with the splitting (17). The Mori cone and the S-extended Mori cone are described, as subsets of $\mathbb{L}\otimes\mathbb{R}$ and $\mathbb{L}^S\otimes\mathbb{R}$, as follows:

$$\begin{aligned}\mathrm{NE}(\mathcal{X}(\mathbf{\Sigma})) &= \sum_{\sigma\in\Sigma} C_\sigma^\vee,\\ \mathrm{NE}^S(\mathcal{X}(\mathbf{\Sigma})) &= \iota(\mathrm{NE}(\mathcal{X}(\mathbf{\Sigma}))) + \mu((\mathbb{R}_{\geqslant 0})^m)\\ &\cong \mathrm{NE}(\mathcal{X}(\mathbf{\Sigma}))\times(\mathbb{R}_{\geqslant 0})^m \quad \text{under (17)}.\end{aligned}$$

Here $C_\sigma^\vee \subset \mathbb{L}\otimes\mathbb{R}$ is the dual cone of

$$C_\sigma = \sum_{\substack{i:1\leqslant i\leqslant n,\\ i\notin\sigma}} \mathbb{R}_{\geqslant 0} D_i \subset \mathbb{L}^\vee\otimes\mathbb{R}.$$

Our semi-projectivity assumption implies that the Mori cone $\mathrm{NE}(\mathcal{X}(\mathbf{\Sigma}))$ is strictly convex.

Definition 22. Recall that $\mathbb{L}^S\subset\mathbb{Z}^{n+m}$, where $m=|S|$. For a cone $\sigma\in\Sigma$, denote by $\Lambda_\sigma^S\subset\mathbb{L}^S\otimes\mathbb{Q}$ the subset consisting of elements

$$\lambda = \sum_{i=1}^{n+m} \lambda_i e_i$$

such that $\lambda_{n+j}\in\mathbb{Z}, 1\leqslant j\leqslant m$, and $\lambda_i\in\mathbb{Z}$ if $i\notin\sigma$ and $i\leqslant n$. Set $\Lambda^S := \bigcup_{\sigma\in\Sigma}\Lambda_\sigma^S$.

Definition 23. The *reduction function* is

$$\begin{aligned} v^S : \Lambda^S &\longrightarrow \mathrm{Box}(\mathbf{\Sigma})\\ \lambda &\longmapsto \sum_{i=1}^n \lceil\lambda_i\rceil\rho_i + \sum_{j=1}^m\lceil\lambda_{n+j}\rceil s_j.\end{aligned}$$

This sends an element of Λ_σ^S to $\mathrm{Box}(\sigma)$ as we have $\overline{v^S(\lambda)} = \sum_{i=1}^n\langle-\lambda_i\rangle\bar{\rho}_i\in\sigma$ for $\lambda\in\Lambda_\sigma^S$. Note that $v^S(\lambda)_i = \langle-\lambda_i\rangle$. For a box element $b\in\mathrm{Box}(\mathbf{\Sigma})$, we set

$$\Lambda_b^S := \{\lambda\in\Lambda^S : v^S(\lambda) = b\}$$

and define

$$\begin{aligned}\Lambda E^S &:= \Lambda^S\cap\mathrm{NE}^S(\mathcal{X}(\mathbf{\Sigma})),\\ \Lambda E_b^S &:= \Lambda_b^S\cap\mathrm{NE}^S(\mathcal{X}(\mathbf{\Sigma})).\end{aligned}$$

We have $\Lambda_b^S\subset\Lambda_\sigma^S$ if $b\in\mathrm{Box}(\sigma)$.

Remark 24. Elements of ΛE_b^S can be interpreted as the S-extended degrees of certain orbifold stable maps, as follows. Let $f:(\mathcal{C},x_1,\ldots,x_k,x_\infty)\to\mathcal{X}$ be an orbifold stable map such that $f|_{x_\infty}$ gives the box element $\hat{b}\in\mathrm{Box}(\mathbf{\Sigma})$ and the rest of the markings $x_1,\ldots,x_k$ are S-decorated, i.e. each x_i is assigned an element of S that maps to the box element $f|_{x_i}$. Then f is naturally an $(S\sqcup\{\hat{b}\})$-decorated stable map and has the $(S\sqcup\{\hat{b}\})$-extended degree of the form

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

$(\kappa, 1) \in \mathbb{L}^{S\sqcup\{\hat{b}\}} \subset \mathbb{Z}^{n+m} \times \mathbb{Z}$ for some $\kappa \in \mathbb{Z}^{n+m}$. On the other hand, we have a 'wrong-way map' $\epsilon : \mathbb{L}^{S\sqcup\{\hat{b}\}} \otimes \mathbb{Q} \to \mathbb{L}^S \otimes \mathbb{Q}$ defined by the commutative diagram

$$\begin{tikzcd} \mathbb{L}^{S\sqcup\{\hat{b}\}} \otimes \mathbb{Q} \arrow[r, "\cong"] \arrow[d, "\epsilon"'] & (\mathbb{L}\otimes\mathbb{Q}) \oplus \mathbb{Q}^{m+1} \arrow[d, "\text{projection}"] \\ \mathbb{L}^S \otimes \mathbb{Q} \arrow[r, "\cong"] & (\mathbb{L}\otimes\mathbb{Q}) \oplus \mathbb{Q}^{m} \end{tikzcd}$$

where the horizontal arrows are the splitting given in (17). The map ϵ induces a bijection

$$\epsilon : \{(\kappa,1) \in \mathbb{L}^{S\sqcup\{\hat{b}\}} : \kappa \in \mathbb{Z}^{n+m}\} \cong \Lambda_b^S, \quad \text{where } \epsilon(\kappa,1)_i = \begin{cases} \kappa_i + \hat{b}_i & 1 \leqslant i \leqslant n, \\ \kappa_i & n+1 \leqslant i \leqslant n+m, \end{cases}$$

and sends the $(S \sqcup \{\hat{b}\})$-extended degree

$$\deg^{S\sqcup\{\hat{b}\}}(f) = (\kappa, 1) \in \mathbb{L}^{S\sqcup\{\hat{b}\}} \cap \mathrm{NE}^{S\sqcup\{\hat{b}\}}(\mathcal{X}(\mathbf{\Sigma}))$$

to the S-extended degree $\deg^S(f) \in \Lambda E_b^S$. Here we regard f as being S-decorated by forgetting the last marking x_∞; we need to generalize Definition 18 by allowing orbifold stable maps with domain curves having unmarked stacky points (and in this case $\deg^S(f)(L,\varphi)$ is not necessarily an integer).

Lemma 25. *Let σ, σ' be top-dimensional cones of Σ such that $\sigma|\sigma'$. Let $b \in \mathrm{Box}(\sigma)$ and let $b' \in \mathrm{Box}(\sigma')$. Recall the set $\Lambda E_{\sigma,b}^{\sigma,b'} \subset \mathbb{L} \otimes \mathbb{Q}$ from Definition 14. Addition in $\mathbb{L}^S \otimes \mathbb{Q}$ induces a map $\Lambda E_{\sigma,b}^{\sigma',b'} \times \Lambda_{b'}^S \to \Lambda_b^S$. Moreover, for fixed $d \in \Lambda E_{\sigma,b}^{\sigma',b'}$, the map $\lambda' \mapsto \lambda' + d$ induces a bijection $\Lambda_{b'}^S \cong \Lambda_b^S$.*

Proof. Take $\lambda' \in \Lambda_{b'}^S$ and $l(c,\sigma,j) \in \Lambda E_{\sigma,b}^{\sigma',b'}$. Here j is the index defined in Notation 8 and c is a positive number such that $\langle c \rangle = \hat{b}_j$ and that (13) holds for some $q' \in \mathbb{Z}_{\geqslant 0}$. We need to show that $\lambda := \lambda' + l(c,\sigma,j) \in \Lambda_b^S$. It suffices to show that $v^S(\lambda) = b$. First, we show that $\lambda \in \Lambda_\sigma^S$. As described in Definition 12, $l(c,\sigma,j)$ is given by the relation of the form

$$c\bar{\rho}_j + c'\bar{\rho}_{j'} + \sum_{i \in \sigma\cap\sigma'} c_i \bar{\rho}_i = 0. \tag{18}$$

Thus the ith component of $\lambda \in \mathbb{L}^S \otimes \mathbb{Q} \subset \mathbb{Q}^{n+m}$ is given by

$$\lambda_i = \begin{cases} \lambda'_j + c & i = j, \\ \lambda'_{j'} + c' & i = j', \\ \lambda'_i + c_i & i \in \sigma \cap \sigma', \\ \lambda'_i & \text{otherwise.} \end{cases}$$

To show that $\lambda \in \Lambda_\sigma^S$, it suffices to see that $\lambda'_{j'} + c' \in \mathbb{Z}$. Since $v^S(\lambda') = b'$, we have $\langle -\lambda'_{j'} \rangle = b'_{j'}$. On the other hand, (13) together with the relation (18) shows that

$$\lfloor c' \rfloor = q' \quad \text{and} \quad \langle c' \rangle = b'_{j'}.$$

This proves that $\lambda'_{j'} + c' \in \mathbb{Z}$ and hence that $\lambda \in \Lambda^S_\sigma$. Now we show that $v^S(\lambda) = b$. We already know that $v^S(\lambda)$ lies in $\mathrm{Box}(\sigma)$. On the other hand,

$$\begin{aligned} v^S(\lambda) &= \sum_{i\notin\sigma\cup\sigma'} \lceil\lambda'_i\rceil\rho_i + \sum_{i=1}^{m}\lceil\lambda'_{n+i}\rceil s_{n+i} + \sum_{i\in\sigma\cap\sigma'}\lceil\lambda'_i + c_i\rceil\rho_i + \lceil\lambda'_j + c\rceil\rho_j + \lceil\lambda'_{j'} + c'\rceil\rho_{j'} \\ &= b' + (\lceil\lambda'_j + c\rceil - \lceil\lambda'_j\rceil)\rho_j + (\lceil\lambda'_{j'} + c'\rceil - \lceil\lambda'_{j'}\rceil)\rho_{j'} + \sum_{i\in\sigma\cap\sigma'}(\lceil\lambda'_i + c_i\rceil - \lceil\lambda'_i\rceil)\rho_i \\ &\equiv b' + \lceil c\rceil\rho_j + q'\rho_{j'} \mod \sum_{i\in\sigma\cap\sigma'}\mathbb{Z}\rho_i \end{aligned}$$

where we used $\lambda'_j \in \mathbb{Z}$ and $\lceil\lambda'_{j'} + c'\rceil - \lceil\lambda'_{j'}\rceil = \lambda'_{j'} + c' - \lceil\lambda'_{j'}\rceil = c' - \langle-\lambda'_{j'}\rangle = c' - b'_{j'} = q'$. The last expression is congruent to b modulo $\sum_{i\in\sigma}\mathbb{Z}\rho_i$ by (13). Therefore, $v^S(\lambda) = b$ as claimed. For the converse, if $\lambda \in \Lambda^S_b$, one can argue similarly to show that $\lambda - l(c,\sigma,j)$ lies in $\Lambda^S_{b'}$. □

5. Toric mirror theorem

In this section we state the main result of this paper, Theorem 31.

Notation 26. Let $\sigma \in \Sigma$ be a top-dimensional cone. We write $u_k(\sigma)$ for the character of $\mathbb{T}$ given by the restriction of the line bundle u_k to the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_\sigma$.

Notation 27. Let S be a finite set equipped with a map $S \to N_\Sigma$ and set $m = |S|$. For $\lambda \in \mathbb{L}^S \otimes \mathbb{Q}$, we write

$$\lambda = (d,k), \quad d \in \mathbb{L}\otimes\mathbb{Q}, k \in \mathbb{Q}^m$$

under the splitting $\mathbb{L}^S \otimes \mathbb{Q} \cong (\mathbb{L}\otimes\mathbb{Q}) \oplus \mathbb{Q}^m$ in (17). If $\lambda \in \Lambda E^S \subset \mathbb{L}^S \otimes \mathbb{Q}$, we have $k \in (\mathbb{Z}_{\geqslant 0})^m$ and $d \in \mathrm{NE}(\mathcal{X}(\mathbf{\Sigma})) \cap H_2(X(\mathbf{\Sigma}),\mathbb{Z})$. In this case we write

$$\tilde{Q}^\lambda = Q^d x^k = Q^d x_1^{k_1}\cdots x_m^{k_m} \in \Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x]\!]$$

where $\Lambda^{\mathbb{T}}_{\mathrm{nov}}$ is the $\mathbb{T}$-equivariant Novikov ring (2) and $x = (x_1,\ldots,x_m)$ are variables. We call $\tilde{Q} = (Q,x)$ the *S-extended Novikov variables.*

DEFINITION 28. Let $\mathbf{\Sigma} = (N,\Sigma,\rho)$ be a stacky fan, and let S be a finite set equipped with a map $S \to N_\Sigma$. Set $m = |S|$ and regard $\mathbb{L}^S \otimes \mathbb{Q}$ as a subspace of $\mathbb{Q}^{n+m}$. The *S-extended $\mathbb{T}$-equivariant I-function* of $\mathcal{X}(\mathbf{\Sigma})$ is

$$I^S_{\mathcal{X}(\mathbf{\Sigma})}(\tilde{Q},z) := ze^{\sum_{i=1}^n u_it_i/z}\sum_{b\in\mathrm{Box}(\mathbf{\Sigma})}\sum_{\lambda\in\Lambda E^S_b}\tilde{Q}^\lambda e^{\lambda t}\left(\prod_{i=1}^{n+m}\frac{\prod_{\langle a\rangle=\langle\lambda_i\rangle, a\leqslant 0}(u_i+az)}{\prod_{\langle a\rangle=\langle\lambda_i\rangle, a\leqslant\lambda_i}(u_i+az)}\right)y^b. \tag{19}$$

Some explanations are in order.

(i) The summation range $\Lambda E^S_b \subset \mathbb{L}^S \otimes \mathbb{Q}$ was introduced in Definition 23.

(ii) For each $\lambda \in \Lambda E^S_b$, we write λ_i for the ith component of λ as an element of $\mathbb{Q}^{n+m}$. We have $\langle\lambda_i\rangle = \hat{b}_i$ for $1 \leqslant i \leqslant n$ and $\langle\lambda_i\rangle = 0$ for $n+1 \leqslant i \leqslant n+m$.

(iii) $u_i := 0$ if $n+1 \leqslant i \leqslant n+m$. For $i = 1,\ldots,n$, u_i is the $\mathbb{T}$-equivariant first Chern class of the line bundle discussed in § 3.3.

(iv) y^b is the identity class supported on the twisted sector $I\mathcal{X}(\mathbf{\Sigma})_b$ associated to $b \in \mathrm{Box}(\mathbf{\Sigma})$; see § 3.4.

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

(v) $t = (t_1, \dots, t_n)$ are variables, and $e^{\lambda t} := \prod_{i=1}^{n} e^{(D_i \cdot d) t_i}$.

$I^S_{\mathcal{X}(\boldsymbol{\Sigma})}(\tilde{Q}, z)$ is a formal power series in Q, x, t with coefficients in the localized equivariant Chen–Ruan cohomology $H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}\times\mathbb{C}^\times}$, i.e.

$$I^S_{\mathcal{X}(\boldsymbol{\Sigma})}(\tilde{Q}, z) \in H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}\times\mathbb{C}^\times}[\![\mathrm{NE}(X) \cap H_2(X, \mathbb{Z})]\!][\![x, t]\!]$$

where $S_{\mathbb{T}\times\mathbb{C}^\times} = \mathrm{Frac}(H^{\bullet}_{\mathbb{T}\times\mathbb{C}^\times}(\mathrm{pt}))$ and z is identified with the $\mathbb{C}^\times$-equivariant parameter; see Remark 1.

Definition 29. If $\boldsymbol{\Sigma} = (N, \Sigma, \rho)$ and S are as above and the coarse moduli space of $\mathcal{X}(\boldsymbol{\Sigma})$ is projective, then we define the *S-extended (non-equivariant) I-function* of $\mathcal{X}(\boldsymbol{\Sigma})$ by the same equation (19), but with u_i, $1 \leqslant i \leqslant n$, replaced by the non-equivariant first Chern class D_i.

Remark 30. One can replace the summation range ΛE^S_b in the formula (19) with Λ^S_b without changing the I-function. This is because the summand for $\lambda \in \Lambda^S_b$ contains a factor $(\prod_{i:\lambda_i \in \mathbb{Z}_{<0}} u_i) y^b$ which vanishes unless $\{\bar{\rho}_i : \lambda_i \in \mathbb{Z}_{<0} \text{ or } \lambda_i \notin \mathbb{Z}\}$ spans a cone; in particular, the summand for λ automatically vanishes unless λ lies in $\mathrm{NE}^S(\mathcal{X}(\boldsymbol{\Sigma}))$.

We now state the main result of this paper.

Theorem 31 (Toric mirror theorem). *Let $\boldsymbol{\Sigma} = (N, \Sigma, \rho)$ be a stacky fan giving rise to a smooth toric Deligne–Mumford stack $\mathcal{X}(\boldsymbol{\Sigma})$ with semi-projective coarse moduli space, and let S be a finite set equipped with a map $S \to N_\Sigma$. The S-extended $\mathbb{T}$-equivariant I-function $I^S_{\mathcal{X}(\boldsymbol{\Sigma})}(\tilde{Q}, -z)$ is a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x, t]\!]$-valued point of the Lagrangian cone $\mathcal{L}_{\mathcal{X}(\boldsymbol{\Sigma})}$ for the $\mathbb{T}$-equivariant Gromov–Witten theory of $\mathcal{X}(\boldsymbol{\Sigma})$.*

Corollary 32. *Suppose that $\boldsymbol{\Sigma} = (N, \Sigma, \rho)$ and S are as in Theorem 31, and that the coarse moduli space of $\mathcal{X}(\boldsymbol{\Sigma})$ is projective. Then the S-extended non-equivariant I-function of $\mathcal{X}(\boldsymbol{\Sigma})$ is a $\Lambda_{\mathrm{nov}}[\![x, t]\!]$-valued point of the Lagrangian cone $\mathcal{L}_{\mathcal{X}(\boldsymbol{\Sigma})}$ for the non-equivariant Gromov–Witten theory of $\mathcal{X}(\boldsymbol{\Sigma})$.*

Proof. Since the coarse moduli space of $\mathcal{X}(\boldsymbol{\Sigma})$ is projective, the non-equivariant Chen–Ruan cohomology, S-extended non-equivariant I-function of $\mathcal{X}(\boldsymbol{\Sigma})$, and non-equivariant Gromov–Witten theory of $\mathcal{X}(\boldsymbol{\Sigma})$ are well defined. Pass to the non-equivariant limit in Theorem 31. □

Remark 33. Theorem 31 and Corollary 32 take a particularly simple form when the pair $(\mathcal{X}(\boldsymbol{\Sigma}), S)$ is *weak Fano*. Roughly speaking, in this case the S-extended I-function $I^S_{\mathcal{X}(\boldsymbol{\Sigma})}$ coincides with (a suitable restriction of) the J-function of $\mathcal{X}(\boldsymbol{\Sigma})$. See [Iri09, § 4.1] for more details.

Remark 34. The non-extended I-function (i.e. the S-extended I-function with $S = \emptyset$) typically only determines the restriction of the J-function to the 'very small parameter space' $H^2(\mathcal{X}; \mathbb{C}) \subset H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{C})$. Taking S to be non-trivial in Theorem 31 and Corollary 32, however, in practice often allows one to determine the J-function along twisted sectors too. But it is convenient to take S not to be too large (not equal to the whole of $\mathrm{Box}(\mathcal{X})$, for example) as otherwise we may lose control over the asymptotics of the I-function. We will elaborate on these points elsewhere.

Remark 35. The S-extended I-function arises from Givental's heuristic argument [Giv95] applied to the polynomial loop spaces (toric map spaces) associated to the S-extended quotient construction (10) of $\mathcal{X}(\boldsymbol{\Sigma})$. See [Giv98, Vla02, Iri06, CLCT09] for closely related discussions.

The remainder of this paper contains a proof of Theorem 31. We first give a criterion, in Theorem 41, that characterizes points on the Lagrangian cone $\mathcal{L}_{\mathcal{X}(\boldsymbol{\Sigma})}$. We then show, in § 7, that the S-extended I-function $I^S_{\mathcal{X}(\boldsymbol{\Sigma})}$ satisfies the criterion in Theorem 41.

6. Lagrangian cones in the toric case

Let $\mathcal{X} = \mathcal{X}(\mathbf{\Sigma})$ be the toric Deligne–Mumford stack associated to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, as in § 3.1. In this section we characterize those points of $\mathcal{H}$ which lie on Givental's Lagrangian cone $\mathcal{L}_{\mathcal{X}}$ associated to $\mathbb{T}$-equivariant Gromov–Witten theory of $\mathcal{X}(\mathbf{\Sigma})$ (see § 2.3). Recall that the $\mathbb{T}$-fixed points of $\mathcal{X}(\mathbf{\Sigma})$ are in bijection with top-dimensional cones of Σ: given a top-dimensional cone $\sigma \in \Sigma$, we have a fixed point

$$\mathcal{X}(\mathbf{\Sigma}/\sigma) \cong \mathcal{X}(\mathbf{\Sigma})_\sigma \subset \mathcal{X}(\mathbf{\Sigma}).$$

Note that $\mathcal{X}(\mathbf{\Sigma}/\sigma) \simeq BN(\sigma)$, where $N(\sigma) := N/N_\sigma$ and $N_\sigma \subset N$ is the subgroup generated by $\rho_i, i \in \sigma$.

Notation 36. For a top-dimensional cone $\sigma \in \Sigma$, we write $T_\sigma\mathcal{X}(\mathbf{\Sigma})$ for the tangent space at the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_\sigma$. This is a $\mathbb{T}$-equivariant vector bundle over $\mathcal{X}(\mathbf{\Sigma})_\sigma \cong BN(\sigma)$.

Notation 37. Let $\sigma \in \Sigma$ be a top-dimensional cone. We write $\mathcal{H}_\sigma$ for Givental's symplectic vector space associated to the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_\sigma$. We also write $\mathcal{H}_\sigma^{\mathrm{tw}}$ and $\mathcal{L}_\sigma^{\mathrm{tw}}$ for the symplectic vector space and Lagrangian cone corresponding to the Gromov–Witten theory of $\mathcal{X}(\mathbf{\Sigma})_\sigma$, twisted by the vector bundle $T_\sigma\mathcal{X}(\mathbf{\Sigma})$ and the $\mathbb{T}$-equivariant inverse Euler class $e_{\mathbb{T}}^{-1}$. More precisely:

$$\begin{aligned}\mathcal{H}_\sigma &:= H_{\mathrm{CR}}(\mathcal{X}(\mathbf{\Sigma})_\sigma) \otimes_{\mathbb{C}} S_{\mathbb{T}}(\!(z^{-1})\!)[\![\mathrm{NE}(X) \cap H_2(X, \mathbb{Z})]\!],\\ \mathcal{H}_\sigma^{\mathrm{tw}} &:= H_{\mathrm{CR}}(\mathcal{X}(\mathbf{\Sigma})_\sigma) \otimes_{\mathbb{C}} S_{\mathbb{T}}(\!(z)\!)[\![\mathrm{NE}(X) \cap H_2(X, \mathbb{Z})]\!].\end{aligned}$$

See §§ 2.3 and 2.4. Although there are no Novikov variables for the stacky point $\mathcal{X}(\mathbf{\Sigma})_\sigma$, we define $\mathcal{H}_\sigma$, $\mathcal{H}_\sigma^{\mathrm{tw}}$ over the Novikov ring of $\mathcal{X}(\mathbf{\Sigma})$ by extending scalars.

Notation 38. By the Atiyah–Bott localization theorem, we have the isomorphism

$$H^\bullet_{\mathrm{CR},\mathbb{T}}(\mathcal{X}(\mathbf{\Sigma})) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}} \simeq \bigoplus_{\sigma \in \Sigma:\ \text{top-dimensional}} H^\bullet_{\mathrm{CR}}(\mathcal{X}(\mathbf{\Sigma})_\sigma) \otimes_{\mathbb{C}} S_{\mathbb{T}} \tag{20}$$

given by restricting to $\mathbb{T}$-fixed points, and thus an isomorphism of vector spaces,

$$\mathcal{H} \simeq \bigoplus_{\sigma \in \Sigma:\ \text{top-dimensional}} \mathcal{H}_\sigma.$$

Under this isomorphism, the symplectic form on $\mathcal{H}$ corresponds to the direct sum of $e_{\mathbb{T}}^{-1}$-twisted symplectic forms on $\bigoplus_\sigma \mathcal{H}_\sigma$. For $\mathbf{f} \in \mathcal{H}$ and $\sigma \in \Sigma$ a top-dimensional cone, we write $\mathbf{f}_\sigma \in \mathcal{H}_\sigma$ for the component of $\mathbf{f}$ along $\mathcal{H}_\sigma \subset \mathcal{H}$. Thus $\mathbf{f}_\sigma$ is the restriction of $\mathbf{f}$ to the inertia stack $I\mathcal{X}(\mathbf{\Sigma})_\sigma$ of the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_\sigma$. We write $\mathbf{f}_{(\sigma,b)}$ for the restriction of $\mathbf{f}_\sigma$ to the component $I\mathcal{X}(\mathbf{\Sigma})_{\sigma,b}$ of $I\mathcal{X}(\mathbf{\Sigma})_\sigma$ corresponding to $b \in \mathrm{Box}(\sigma)$. The component $I\mathcal{X}(\mathbf{\Sigma})_{\sigma,b}$ is contained in both $I\mathcal{X}(\mathbf{\Sigma})_\sigma$ and $I\mathcal{X}(\mathbf{\Sigma})_b$.

Definition 39 (Recursion coefficient). Let $\mathbf{\Sigma} = (N, \Sigma, \rho)$ be a stacky fan, and let σ, $\sigma' \in \Sigma$ satisfy $\sigma|\sigma'$. Let j be as in Notation 8. Fix $b \in \mathrm{Box}(\sigma)$, and let c be a positive rational number such that $\langle c \rangle = \hat{b}_j$ with $\hat{b} = \mathrm{inv}(b)$. The *recursion coefficient* associated to (σ, σ', b, c) is the element of $S_{\mathbb{T}} = \mathrm{Frac}(H^\bullet_{\mathbb{T}}(\mathrm{pt})) \cong \mathbb{C}(\chi_1, \ldots, \chi_d)$ given by

$$\mathbf{RC}(c)^{(\sigma',b')}_{(\sigma,b)} := \frac{1}{c}\Bigg(\prod_{i\in\sigma:b_i=0} u_i(\sigma)\Bigg)\frac{(c/u_j(\sigma))^{\lfloor c\rfloor}}{\lfloor c\rfloor!}\frac{(-c/u_j(\sigma))^{\lfloor c'\rfloor}}{\lfloor c'\rfloor!} \times \prod_{i\in\sigma\cap\sigma'}\frac{\prod_{\langle a\rangle=\hat{b}_i,a<0}(u_i(\sigma)+u_j(\sigma)a/(-c))}{\prod_{\langle a\rangle=\hat{b}_i,a\leqslant c_i}(u_i(\sigma)+u_j(\sigma)a/(-c))}$$

where c', c_i are as in Definition 12, i.e. $c' = D_{j'}\cdot l(c,\sigma,j)$ and $c_i = D_i\cdot l(c,\sigma,j)$ for $i\in\sigma\cap\sigma'$.

Remark 40. The recursion coefficient $\mathbf{RC}(c)^{(\sigma',b')}_{(\sigma,b)}$ depends only on σ, σ', b, and c. The box element $b'\in\mathrm{Box}(\sigma')$ is determined by these data, via Remark 11.

Theorem 41. *Let $\mathcal{X}=\mathcal{X}(\mathbf{\Sigma})$ be a smooth toric Deligne–Mumford stack associated to a stacky fan $\mathbf{\Sigma}=(N,\Sigma,\rho)$. Let $x=(x_1,\ldots,x_m)$ be formal variables. Let $\mathbf{f}$ be an element of $\mathcal{H}[\![x]\!]$ such that $\mathbf{f}|_{Q=x=0}=-1z$. Then $\mathbf{f}$ is a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x]\!]$-valued point of $\mathcal{L}_{\mathcal{X}}$ if and only if the following three conditions hold.*

(C1) *For each top-dimensional cone $\sigma\in\Sigma$ and each $b\in\mathrm{Box}(\sigma)$, the restriction $\mathbf{f}_{(\sigma,b)}$ is a power series in Q and x such that each coefficient of this power series is an element of $S_{\mathbb{T}\times\mathbb{C}^\times}\cong\mathbb{C}(\chi_1,\ldots,\chi_d,z)$ and, as a function in z, it is regular except possibly for a pole at $z=0$, a pole at $z=\infty$, and simple poles at*

$$\left\{\frac{u_j(\sigma)}{c} : \exists\sigma'\in\Sigma \text{ such that } \sigma|\sigma' \text{ and } j\in\sigma\backslash\sigma',\ c>0 \text{ is such that } \langle c\rangle=\hat{b}_j\right\}.$$

Here we use Notation 8.

(C2) *The residues of $\mathbf{f}_{\sigma,b}$ at the simple poles satisfy the following recursion relations: given any σ, $\sigma'\in\Sigma$ such that $\sigma|\sigma'$, $b\in\mathrm{Box}(\sigma)$ and $c>0$ with $\langle c\rangle=\hat{b}_j$, we have*

$$\mathrm{Res}_{z=u_j(\sigma)/c}\,\mathbf{f}_{(\sigma,b)}(z)\,dz = -Q^{l(c,\sigma,j)}\,\mathbf{RC}(c)^{(\sigma',b')}_{(\sigma,b)}\,\mathbf{f}_{(\sigma',b')}(z)|_{z=u_j(\sigma)/c}.$$

Here we use Notation 8, Definition 12 and Definition 39.

(C3) *The Laurent expansion of $\mathbf{f}_\sigma$ at $z=0$ is a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x]\!]$-valued point of $\mathcal{L}^{\mathrm{tw}}_\sigma$.*

Remark 42. Condition (C1) ensures that the right-hand side of the recursion relation in (C2) is well defined. Note that the $\mathbb{T}$-weights $\{u_i(\sigma') : i\in\sigma'\}$ of the tangent space $T_{\sigma'}\mathcal{X}(\mathbf{\Sigma})$ form a basis of $\mathrm{Lie}(\mathbb{T})^*$ and all simple poles of $\mathbf{f}_{\sigma',b'}(z)$ are contained in the cone $\sum_{i\in\sigma'}\mathbb{R}_{\geqslant 0}u_i(\sigma')$ by (C1). On the other hand, if we take the representable morphism $f:\mathbb{P}_{r_1,r_2}\to\mathcal{X}(\mathbf{\Sigma})$ associated to σ,c,j,b in Proposition 10, then $u_j(\sigma)/c$ and $u_{j'}(\sigma')/c'$ are the induced $\mathbb{T}$-weights of the tangent spaces at 0 and ∞ of the *coarse* domain curve $|\mathbb{P}_{r_1,r_2}|\cong\mathbb{P}^1$ (see Definition 12 for c'). Therefore we have $u_j(\sigma)/c=-u_{j'}(\sigma')/c'$, and it follows that $\mathbf{f}_{\sigma',b'}(z)$ is regular at $z=u_j(\sigma)/c=-u_{j'}(\sigma')/c'$.

Remark 43. Note that (C3) involves analytic continuation: (C1) implies that each coefficient of Q^dx^k in $\mathbf{f}_\sigma(z)$ is a rational function in $z,\chi_1,\ldots,\chi_d$, and so it makes sense to take the Laurent expansion at $z=0$.

Proof of Theorem 41. In outline, Brown's proof for toric bundles [Bro14, Theorem 2] works for toric Deligne–Mumford stacks too. In detail, we argue as follows.

Suppose first that $\mathbf{f}$ is a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x]\!]$-valued point on $\mathcal{L}_{\mathcal{X}}$. Then

$$\mathbf{f} = -1z+\mathbf{t}(z)+\sum_{n=0}^{\infty}\sum_{d\in\mathrm{NE}(\mathcal{X})}\sum_{\alpha}\frac{Q^d}{n!}\left\langle\mathbf{t}(\bar{\psi}),\ldots,\mathbf{t}(\bar{\psi}),\frac{\phi_\alpha}{-z-\bar{\psi}}\right\rangle^{\mathbb{T}}_{0,n+1,d}\phi^\alpha \tag{21}$$

for some $\mathbf{t}(z) \in \mathcal{H}_+[\![x]\!]$ with $\mathbf{t}|_{Q=x=0} = 0$; here once again we expand the expression $\phi_\alpha/(-z-\bar{\psi})$ as a power series in z^{-1}. Under the isomorphism (20), the identity class $1 \in H^\bullet_{\mathrm{CR},\mathbb{T}}(\mathcal{X}(\mathbf{\Sigma}))$ and $\mathbf{t}(z) \in \mathcal{H}_+[\![x]\!]$ correspond to

$$\bigoplus_{\sigma\in\Sigma\text{: top-dimensional}} 1_\sigma \quad \text{and} \quad \bigoplus_{\sigma\in\Sigma\text{: top-dimensional}} \mathbf{t}_\sigma(z)$$

where 1_σ is the identity element in $H^\bullet_{\mathrm{CR},\mathbb{T}}(\mathcal{X}(\mathbf{\Sigma})_\sigma)$ and $\mathbf{t}_\sigma(z) \in \mathcal{H}_{\sigma,+}$, and we have

$$\mathbf{f}_\sigma = -1_\sigma z + \mathbf{t}_\sigma(z) + \iota_\sigma^\star\left[\sum_{n=0}^{\infty}\sum_{d\in\mathrm{NE}(\mathcal{X})}\sum_\alpha \frac{Q^d}{n!}\left\langle \mathbf{t}(\bar{\psi}),\ldots,\mathbf{t}(\bar{\psi}), \frac{\phi_\alpha}{-z-\bar{\psi}}\right\rangle^{\mathbb{T}}_{0,n+1,d}\phi^\alpha\right]$$

where $\iota_\sigma : \mathcal{X}(\mathbf{\Sigma})_\sigma \to \mathcal{X}(\mathbf{\Sigma})$ is the inclusion of the $\mathbb{T}$-fixed point. Furthermore,

$$\mathbf{f}_{(\sigma,b)} = -\delta_{b,0}z + \mathbf{t}_{(\sigma,b)}(z) + \sum_{n=0}^{\infty}\sum_{d\in\mathrm{NE}(\mathcal{X})}\frac{Q^d}{n!}\left\langle \frac{1^{\sigma,b}}{-z-\bar{\psi}}, \mathbf{t},\ldots,\mathbf{t}\right\rangle^{\mathbb{T}}_{0,n+1,d} \tag{22}$$

where $1^{\sigma,b} := |N(\sigma)| e_{\mathbb{T}}(N_{\sigma,b}) 1_{\sigma,\hat{b}}$, $N_{\sigma,b}$ is the normal bundle to $I\mathcal{X}(\mathbf{\Sigma})_{\sigma,b}$ in $I\mathcal{X}(\mathbf{\Sigma})_b$, and $1_{\sigma,\hat{b}}$ is the fundamental class of $I\mathcal{X}(\mathbf{\Sigma})_{\sigma,\hat{b}}$ with $\hat{b} = \mathrm{inv}(b)$.

We compute the sum in (22) using localization in $\mathbb{T}$-equivariant cohomology. Chiu-Chu Melissa Liu has produced a detailed and beautifully written introduction to localization in $\mathbb{T}$-equivariant Gromov–Witten theory of toric stacks [Liu13]; we follow her notation closely. $\mathbb{T}$-fixed strata in the moduli space $\overline{\mathcal{M}}_{0,n+1}(\mathcal{X}, d)$ are indexed by decorated trees Γ, where:

- each vertex v of Γ is labelled by a top-dimensional cone $\sigma_v \in \Sigma$;
- each edge e of Γ is labelled by a codimension-one cone $\tau_e \in \Sigma$ and a positive integer d_e;
- each flag[2] (e, v) of Γ is labelled with an element $k_{(e,v)}$ of the isotropy group G_v of the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_{\sigma_v}$;
- there are markings $\{1, 2, \ldots, n+1\}$ and a map $s : \{1, 2, \ldots, n+1\} \to V(\Gamma)$ that assigns markings to vertices of Γ;
- the marking $j \in \{1, 2, \ldots, n+1\}$ is labelled with an element $k_j \in G_v$, where $v = s(j)$;
- a number of compatibility conditions hold.

The compatibility conditions that we require are spelled out in detail in [Liu13, Definition 9.6]; they include, for example, the requirement that if (e, v) is a flag of Γ then the $\mathbb{T}$-fixed point determined by σ_v is contained in the closure of the one-dimensional $\mathbb{T}$-orbit determined by τ_e. We denote the set of all decorated trees satisfying the compatibility conditions by $G_{0,n+1}(\mathcal{X}, d)$, so that $\mathbb{T}$-fixed strata in $\overline{\mathcal{M}}_{0,n+1}(\mathcal{X}, d)$ are indexed by decorated trees $\Gamma \in G_{0,n+1}(\mathcal{X}, d)$.

We rewrite equation (22) as

$$\mathbf{f}_{(\sigma,b)} = -\delta_{b,0}z + \mathbf{t}_{(\sigma,b)}(z) + \sum_{n=0}^{\infty}\sum_{d\in\mathrm{NE}(\mathcal{X})}\frac{Q^d}{n!}\sum_{\Gamma\in G_{0,n+1}(\mathcal{X},d)}\mathrm{Contr}_{\sigma,b}(\Gamma) \tag{23}$$

where $\mathrm{Contr}_{\sigma,b}(\Gamma)$ denotes the contribution to the $\mathbb{T}$-equivariant Gromov–Witten invariant

$$\left\langle \frac{1^{\sigma,b}}{-z-\bar{\psi}}, \mathbf{t},\ldots,\mathbf{t}\right\rangle^{\mathbb{T}}_{0,n+1,d}$$

[2] A flag (e, v) of Γ is an edge–vertex pair such that e is incident to v.

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

from the $\mathbb{T}$-fixed stratum $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{0,n+1}(\mathcal{X}, d)$ corresponding to Γ. We will need some notation for graphs. For a decorated graph $\Gamma \in G_{0,n+1}(\mathcal{X}, d)$, we write:

- $V(\Gamma)$ for the set of vertices of Γ;
- $E(\Gamma)$ for the set of edges of Γ;
- $F(\Gamma)$ for the set of flags of Γ;
- S_v for the set of markings assigned to the vertex $v \in V(\Gamma)$,
$$S_v = \{j \in \{1, 2, \ldots, n+1\} : s(j) = v\};$$
- E_v for the set of edges incident to the vertex $v \in V(\Gamma)$,
$$E_v = \{e \in E(\Gamma) : (e, v) \in F(\Gamma)\};$$
- $\operatorname{val}(v) = |E_v| + |S_v|$ for the valence of the vertex $v \in V(\Gamma)$.

Liu [Liu13, Theorem 9.32] shows that the contribution from $\mathcal{M}_\Gamma$ to the Gromov–Witten invariant
$$\langle \gamma_1 \bar{\psi}_1^{a_1}, \ldots, \gamma_{n+1} \bar{\psi}_{n+1}^{a_{n+1}} \rangle_{0,n+1,d}$$
is
$$c_\Gamma \prod_{e \in E(\Gamma)} \frac{e_{\mathbb{T}}(H^1(\mathcal{C}_e, f_e^\star T\mathcal{X})^{\mathrm{mov}})}{e_{\mathbb{T}}(H^0(\mathcal{C}_e, f_e^\star T\mathcal{X})^{\mathrm{mov}})} \prod_{(e,v) \in F(\Gamma)} e_{\mathbb{T}}((T_{\sigma_v}\mathcal{X})^{k_{(e,v)}}) \prod_{v \in V(\Gamma)} \Bigg(\prod_{j : s(j) = v} \iota_{\sigma_v}^\star \gamma_j \Bigg)$$
$$\prod_{v \in V(\Gamma)} \int_{[\overline{\mathcal{M}}^{\overrightarrow{b(v)}}_{0,\operatorname{val}(v)}(BG_v)]^w} \frac{\prod_{j \in S_v} \bar{\psi}_j^{a_j}}{\prod_{e \in E_v} (w_{(e,v)} - \bar{\psi}_{(e,v)}/r_{(e,v)})} \cup e_{\mathbb{T}}^{-1}((T_{\sigma_v}\mathcal{X})_{0,\operatorname{val}(v),0}), \tag{24}$$
where
$$c_\Gamma = \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{e \in E(\Gamma)} \frac{1}{(d_e |G_e|)} \prod_{(e,v) \in F(\Gamma)} \frac{|G_v|}{r_{(e,v)}};$$

G_e is the generic stabilizer of the one-dimensional toric substack $\mathcal{X}(\boldsymbol{\Sigma}/\tau_e)$;
$f_e : \mathcal{C}_e \to \mathcal{X}$ is the toric map to the one-dimensional toric substack $\mathcal{X}(\boldsymbol{\Sigma}/\tau_e)$ determined by the edge e and the decorations $\tau_e, d_e, \{k_{(e,v)} : v$ is a vertex incident to $e\}$;
$H^i(\mathcal{C}_e, f_e^* T\mathcal{X})^{\mathrm{mov}}$ denotes the moving part with respect to the $\mathbb{T}$-action;
$w_{(e,v)} = e_{\mathbb{T}}(T_{\mathfrak{y}(e,v)}\mathcal{C}_e)$, where $\mathfrak{y}(e,v)$ is the marked point on $\mathcal{C}_e$ determined by (e,v);
$r_{(e,v)}$ is the order of $k_{(e,v)} \in G_v$;
$\overrightarrow{b(v)}$ is determined by the decorations k_j, $j \in S_v$, and $k_{(e,v)}$, $e \in E_v$;
$(T_{\sigma_v}\mathcal{X})_{0,\operatorname{val}(v),0}$ is the twisting bundle associated to the vector bundle $T_{\sigma_v}\mathcal{X}$ over the $\mathbb{T}$-fixed point $\mathcal{X}(\boldsymbol{\Sigma}/\sigma_v)$ (see § 2.4);
$\overline{\mathcal{M}}^{\overrightarrow{b(v)}}_{0,\operatorname{val}(v)}(BG_v)$ is taken to be a point if $\operatorname{val}(v) = 1$ or $\operatorname{val}(v) = 2$.

The integrals over $\overline{\mathcal{M}}^{\overrightarrow{b(v)}}_{0,\operatorname{val}(v)}(BG_v)$ here in the unstable cases $\operatorname{val}(v) = 1$ and $\operatorname{val}(v) = 2$ are defined as in [Liu13, (9.12)–(9.14)]. The twisting bundles $(T_{\sigma_v}\mathcal{X})_{0,\operatorname{val}(v),0}$ in the unstable cases $\operatorname{val}(v) = 1$ and $\operatorname{val}(v) = 2$ are defined to be $(T_{\sigma_v}\mathcal{X})^{k_{(e,v)}}$; see the end of [Liu13, § 9.3.4].

Consider now the graph sum in (23). Each graph Γ in the sum contains a distinguished vertex v that carries the first marked point. We may assume both that $\sigma_v = \sigma$ and that the label k_1 of the first marking is equal to $\hat{b}$, as otherwise the contribution of Γ is zero. There are two possibilities:

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

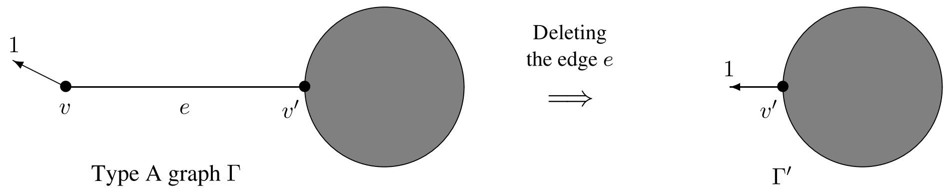

FIGURE 1. A graph of type A.

(A) v is 2-valent;

(B) v has valence at least 3.

In the first case we say that Γ has type A, and in the second case we say that Γ has type B; see Figures 1 and 2. As we will see below, the contributions from type A graphs have simple poles at points of the form $u_j(\sigma)/c$ as described in the statement of the theorem, and the contributions from type B graphs are polynomials in z^{-1}. Condition (C1) then follows.

Consider a graph Γ of type A. Let $e \in E(\Gamma)$ be the edge incident to v. Then Γ is obtained from another decorated graph Γ' by adding the decorated vertex v and the decorated edge e. See Figure 1. Let v' be the other vertex incident to e. The graph Γ' is assigned the first marking at v' instead of the edge e. The map $f_e : \mathcal{C}_e \to \mathcal{X}$ determined by the edge e has $\mathcal{C}_e \simeq \mathbb{P}_{r_{(e,v)},r_{(e,v')}}$, $f_e(0) = \mathcal{X}_{\sigma_v}$, and $f_e(\infty) = \mathcal{X}_{\sigma_{v'}}$; the restriction $f_e|_0 : B\mu_{r_{(e,v)}} \to \mathcal{X}_{\sigma_v}$ gives $\hat{b} \in \mathrm{Box}(\sigma_v)$. Let $c \in \mathbb{Q}$ and $b' \in \mathrm{Box}(\sigma_{v'})$ be the rational number and box element determined by applying Proposition 10 and Remark 11 to f_e, and write $\sigma' = \sigma_{v'}$. Since $\bar{\psi}_1 = -r_{(e,v)} w_{(e,v)}$, we obtain

$$\begin{aligned}\mathrm{Contr}_{\sigma,b}(\Gamma) &= \frac{c_\Gamma}{c_{\Gamma'}} \frac{e_{\mathbb{T}}(H^1(\mathcal{C}_e, f_e^\star T\mathcal{X})^{\mathrm{mov}})}{e_{\mathbb{T}}(H^0(\mathcal{C}_e, f_e^\star T\mathcal{X})^{\mathrm{mov}})} e_{\mathbb{T}}((T_\sigma \mathcal{X})^{k_{(e,v)}}) e_{\mathbb{T}}((T_{\sigma'}\mathcal{X})^{k_{(e,v')}}) \\ &\quad \times \int_{[\overline{\mathcal{M}}_{0,2}^{(\hat{b},b)}(BG_v)]^w} \frac{|N(\sigma_v)| e_{\mathbb{T}}(N_{\sigma_v,b})}{-z + r_{(e,v)} w_{(e,v)}} \frac{1}{w_{(e,v)} - \bar{\psi}_2 / r_{(e,v)}} \cup e_{\mathbb{T}}^{-1}((T_{\sigma_v}\mathcal{X})^{k_{(e,v)}}) \\ &\quad \times \frac{r_{(e,v')}}{|N(\sigma')| e_{\mathbb{T}}(N_{\sigma',b'})} \mathrm{Contr}_{\sigma',b'}(\Gamma')|_{z=-r_{(e,v')}w_{(e,v')}}.\end{aligned}$$

Calculating the ratio $c_\Gamma / c_{\Gamma'}$ and evaluating the integral over $\overline{\mathcal{M}}_{0,2}^{(\hat{b},b)}(BG_v)$ using [Liu13, (9.14)] yields

$$\mathrm{Contr}_{\sigma,b}(\Gamma) = \frac{|G_v|}{d_e |G_e|} \frac{e_{\mathbb{T}}(H^1(\mathcal{C}_e, f_e^\star T\mathcal{X})^{\mathrm{mov}})}{e_{\mathbb{T}}(H^0(\mathcal{C}_e, f_e^\star T\mathcal{X})^{\mathrm{mov}})} \frac{e_{\mathbb{T}}(N_{\sigma,b})}{-z + r_{(e,v)} w_{(e,v)}} [\mathrm{Contr}_{\sigma',b'}(\Gamma')]_{z=-r_{(e,v')}w_{(e,v')}}.$$

Liu has computed the ratio of Euler classes here [Liu13, Lemma 9.25], and in our notation this gives

$$\mathrm{Contr}_{\sigma,b}(\Gamma) = \frac{\mathbf{RC}(c)_{(\sigma,b)}^{(\sigma',b')}}{-z + u_j(\sigma)/c} [\mathrm{Contr}_{\sigma',b'}(\Gamma')]_{z=u_j(\sigma)/c}$$

where we used $r_{(e,v)} w_{(e,v)} = u_j(\sigma)/c = -r_{(e,v')} w_{(e,v')}$. (See Remark 44 below for a detailed comparison between Liu's notation and ours.) Note that the degree of the map $f_e : \mathcal{C}_e \to \mathcal{X}$ is $l(c, \sigma, j)$; see Definition 12. Note also that if we hold the decorated vertex v and the decorated edge e constant (or in other words, if we hold the map $f_e : \mathcal{C}_e \to \mathcal{X}$ constant) then the sum of $\mathrm{Contr}(\Gamma')_{\sigma',b'}$ over all compatible trees Γ' is exactly[3] the graph sum that defines $\mathbf{f}_{(\sigma',b')}$. Thus the

[3] We elide a subtle detail here: the unstable terms $[-\delta_{b',0} z + \mathbf{t}_{(\sigma',b')}(z)]_{z=u_j(\sigma)/c}$ in $[\mathbf{f}_{(\sigma',b')}]_{z=u_j(\sigma)/c}$ arise from the graphs Γ in the sum such that Γ' is unstable, with only one vertex and one or two markings attached to it.

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

contribution to

$$\mathbf{f}_{(\sigma,b)} = -\delta_{b,0}z + \mathbf{t}_{(\sigma,b)}(z) + \sum_{n=0}^{\infty} \sum_{d\in \mathrm{NE}(\mathcal{X})} \sum_{\Gamma \in G_{0,n+1}(\mathcal{X},d)} \frac{Q^d}{n!} \operatorname{Contr}_{\sigma,b}(\Gamma)$$

from all graphs Γ of type A is

$$\sum_{\sigma':\sigma|\sigma'} \sum_{\substack{c\in\mathbb{Q}:c>0,\\ \langle c\rangle=\hat{b}_j}} Q^{l(c,\sigma,j)} \frac{\mathbf{RC}(c)^{(\sigma',b')}_{(\sigma,b)}}{-z+u_j(\sigma)/c} [\mathbf{f}_{(\sigma',b')}]_{z=u_j(\sigma)/c}. \tag{25}$$

This proves (C2).

Write

$$\boldsymbol{\tau}_{(\sigma,b)}(z) := \mathbf{t}_{(\sigma,b)}(z) + (\text{the quantity in (25)})$$

and

$$\boldsymbol{\tau}_\sigma(z) := \sum_{b\in \mathrm{Box}(\sigma)} \boldsymbol{\tau}_{(\sigma,b)}(z)1_b.$$

We have that

$$\begin{aligned}\mathbf{f}_\sigma &= \sum_{b\in\mathrm{Box}(\sigma)} \mathbf{f}_{(\sigma,b)}1_b \\ &= -1_\sigma z + \boldsymbol{\tau}_\sigma(z) + \sum_{n=0}^{\infty} \sum_{d\in\mathrm{NE}(\mathcal{X})} \sum_{b\in\mathrm{Box}(\sigma)} \sum_{\substack{\Gamma\in G_{0,n+1}(\mathcal{X},d):\\ \Gamma \text{ is of type B}}} \frac{Q^d}{n!} \operatorname{Contr}_{\sigma,b}(\Gamma)1_{\sigma,b}. \end{aligned}\tag{26}$$

Consider the contribution to (26) given by the sum over decorated graphs Γ of type B such that the distinguished vertex v has valence l and that the label k_1 of the distinguished vertex is equal to $\hat{b} \in \mathrm{Box}(\sigma)$. Each such graph Γ gives contributions of the form (24). We evaluate these contributions by integrating over all the factors $\overline{\mathcal{M}}^{\vec{b'}}_{0,\mathrm{val}(v')}(BG_{v'})$ except that associated with the distinguished vertex v, obtaining an expression of the form[4]

$$\frac{1}{|\operatorname{Aut}_{\Gamma_2,\dots,\Gamma_l}|} \int_{\overline{\mathcal{M}}^{\hat{b},b^2,\dots,b^l}_{0,l}(BG_v)} \frac{1^{\sigma,b}}{-z-\bar{\psi}_1} \cup h_2(\mathbf{t},\bar{\psi}_2) \cup \cdots \cup h_l(\mathbf{t},\bar{\psi}_l) \cup e_{\mathbb{T}}^{-1}((T_{\sigma_v}\mathcal{X})_{0,l,0})\, 1_{\sigma,b}$$

for some elements $b^2,\dots,b^l \in \mathrm{Box}(\sigma)$ and some polynomials $h_i(\mathbf{t},\bar{\psi}_i)$ in t_0, $t_1,\dots,Q$, and $\bar{\psi}_i$. Suppose that Γ is obtained from type A subgraphs $\Gamma_2,\dots,\Gamma_l$ by joining them at the distinguished vertex v, as in Figure 2. If Γ_i consists of one vertex with two markings (such as Γ_4 in Figure 2) then $h_i(\mathbf{t},\bar{\psi}_i) = \mathbf{t}_{(\sigma,b^i)}(\bar{\psi}_i)$. Otherwise $h_i(\mathbf{t},\bar{\psi}_i)$ records a more complicated contribution determined by the subgraph Γ_i; we have

$$h_i(\mathbf{t},\bar{\psi}_i) = Q^{d_i} \operatorname{Contr}_{\sigma,b^i}(\Gamma_i)|_{z=\bar{\psi}_i}$$

[4] Here $\operatorname{Aut}_{\Gamma_2,\dots,\Gamma_l}$ is the subgroup of the symmetric group $\mathfrak{S}_{l-1}$ which leaves the $(l-1)$-tuple $(\Gamma_2,\dots,\Gamma_l)$ of decorated graphs invariant.

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

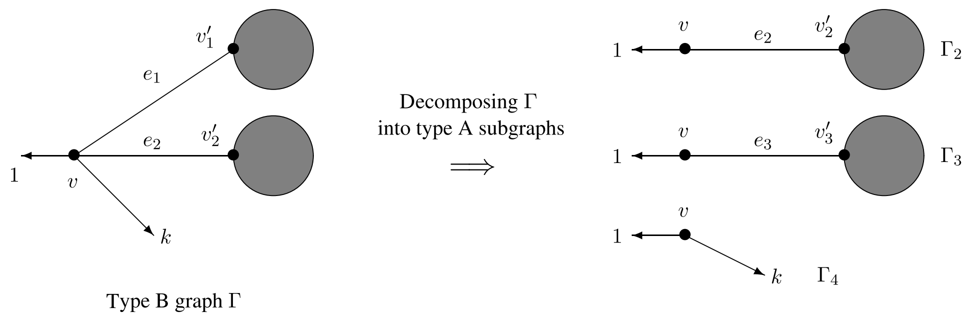

FIGURE 2. A graph of type B.

where d_i is the total degree of the contribution from Γ_i. Now fix v and all other parts of Γ except the subtree Γ_i, and sum over all possible subtrees Γ_i: the total contribution of the $h_i(\mathbf{t}, \bar{\psi}_i)$ is (25) with $b = b^i$ and $z = \bar{\psi}_i$. Thus the contribution to (26) given by the sum over decorated graphs Γ of type B such that the distinguished vertex has valence l and that the label k_1 of the distinguished vertex is equal to $\hat{b} \in \mathrm{Box}(\sigma)$ is

$$\frac{1}{(l-1)!}\int_{\overline{\mathcal{M}}_{0,l}(BG_v)} \frac{1^{\sigma,b}}{-z-\bar{\psi}_1} \cup \boldsymbol{\tau}_\sigma(\bar{\psi}_2) \cup \cdots \cup \boldsymbol{\tau}_\sigma(\bar{\psi}_l) \cup e_{\mathbb{T}}^{-1}((T_\sigma\mathcal{X})_{0,l,0}).$$

These are twisted Gromov–Witten invariants of the $\mathbb{T}$-fixed point $\mathcal{X}(\boldsymbol{\Sigma})_\sigma$. Summarizing, we see that (26) becomes

$$\mathbf{f}_\sigma = -1_\sigma z + \boldsymbol{\tau}_\sigma(z) + \sum_{l=3}^{\infty} \sum_{b\in\mathrm{Box}(\sigma)} \frac{1}{(l-1)!}\left\langle \frac{1^{\sigma,b}}{-z-\bar{\psi}}, \boldsymbol{\tau}_\sigma(\psi), \ldots, \boldsymbol{\tau}_\sigma(\psi)\right\rangle_{0,l,0}^{\mathrm{tw}} 1_{\sigma,b}.$$

The superscript 'tw' indicates that these are Gromov–Witten invariants of $\mathcal{X}(\boldsymbol{\Sigma})_\sigma$ twisted by the vector bundle $T_\sigma\mathcal{X}(\boldsymbol{\Sigma})$ and the $\mathbb{T}$-equivariant inverse Euler class $e_{\mathbb{T}}^{-1}$. Using (7), we see that the Laurent expansion at $z = 0$ of $\mathbf{f}_\sigma$ lies in $\mathcal{L}_\sigma^{\mathrm{tw}}$. Thus, we have proved (C3).

Conversely, suppose that $\mathbf{f} \in \mathcal{H}[\![x]\!]$ satisfies $\mathbf{f}|_{Q=x=0} = -1z$ and conditions (C1)–(C3) in the statement of the theorem. Conditions (C1) and (C2) together imply that

$$\mathbf{f}_\sigma = -1_\sigma z + \mathbf{t}_\sigma + \sum_{b\in\mathrm{Box}(\sigma)} 1_b \sum_{\sigma':\sigma|\sigma'} \sum_{\substack{c\in\mathbb{Q}:c>0,\\ \langle c\rangle=\hat{b}_j}} Q^{l(c,\sigma,j)} \frac{\mathbf{RC}(c)_{(\sigma,b)}^{(\sigma',b')}}{-z+u_j(\sigma)/c} \mathbf{f}_{(\sigma',b')}|_{z=u_j(\sigma)/c} + O(z^{-1}) \quad (27)$$

for some $\mathbf{t}_\sigma \in \mathcal{H}_{\sigma,+}[\![x]\!]$ with $\mathbf{t}_\sigma|_{Q=t=0} = 0$. The remainder $O(z^{-1})$ is a formal power series in Q and x with coefficients in $z^{-1}S_{\mathbb{T}}[z^{-1}]$. Let $\mathbf{t}_{\mathrm{GW}} \in \mathcal{H}_+$ be the unique element such that its restriction to $I\mathcal{X}(\boldsymbol{\Sigma})_\sigma$ is $\mathbf{t}_\sigma$, and let $\mathbf{f}_{\mathrm{GW}}$ be the element of $\mathcal{L}_{\mathcal{X}}$ defined by (21) with $\mathbf{t} = \mathbf{t}_{\mathrm{GW}}$. Then, in view of the first part of the proof, we have that $\mathbf{f}_{\mathrm{GW}}$ and $\mathbf{f}$ both satisfy conditions (C1)–(C3), and both give rise to the same values $\mathbf{t}_\sigma$ in (27). It therefore suffices to show that $\mathbf{f}$ can be reconstructed uniquely from the collection

$$(\mathbf{t}_\sigma : \sigma \in \Sigma \text{ is a top-dimensional cone}) \quad (28)$$

using condition (C3).

We argue by induction on the degree with respect to Q and x. Pick a Kähler class ω of $\mathcal{X}(\boldsymbol{\Sigma})$ and assign the degree $\int_d \omega + \sum_{i=1}^m k_i$ to the monomial $Q^d x_1^{k_1} \cdots x_m^{k_m}$. Let $\kappa_0 = \int_{d_0} \omega > 0$ be the minimal possible degree of a non-constant stable map. Suppose that $\mathbf{f}$ is uniquely determined from the collection (28) up to order κ. We shall show that $\mathbf{f}$ is determined up to order $\kappa+\kappa_0$. We know by (27) that $\mathbf{f}_\sigma$ is determined up to order $\kappa+\kappa_0$ except for the term $O(z^{-1})$. On the other hand, under the Laurent expansion at $z=0$, all the quantities in the first line of (27) lie in $\mathcal{H}^{\mathrm{tw}}_{\sigma,+}$. Therefore, in view of (7), the term $O(z^{-1})$ is uniquely determined up to order $\kappa+\kappa_0$ from the quantities in the first line by condition (C3), i.e. that the Laurent expansion at $z=0$ of $\mathbf{f}_\sigma$ lies in $\mathcal{L}^{\mathrm{tw}}_\sigma$. This completes the induction and the proof of Theorem 41. □

Remark 44. For the convenience of the reader, we compare Liu's notation [Liu13, Lemma 9.25] with ours. Consider a decorated graph Γ occurring in the proof above, and an edge $e\in E(\Gamma)$ with incident vertices v, $v'\in V(\Gamma)$. The edge e corresponds to a toric representable morphism $f=f_e:\mathbb{P}_{r_1,r_2}\to\mathcal{X}(\boldsymbol{\Sigma})$ given by σ,σ',c,b in Proposition 10, where $\sigma=\sigma_v$ and $\sigma'=\sigma_{v'}$. Let j and j' be the indices in Notation 8. Recall (from Definition 12) that the degree $l(c,\sigma,j)\in\mathbb{L}\otimes\mathbb{Q}$ of the map f is given by the relation

$$c\bar\rho_j+c'\bar\rho_{j'}+\sum_{i\in\sigma\cap\sigma'}c_i\bar\rho_i=0.$$

Set $\tau=\tau_e=\sigma\cap\sigma'$. Then Liu's quantities[5] $\mathbf{w}(\tau,\sigma)$, $\mathbf{w}(\tau,\sigma')$, $\mathbf{w}(\tau_i,\sigma)$, $\mathbf{w}(\tau_i',\sigma')$, $r(\tau,\sigma)$, $\mathbf{u}=r(\tau,\sigma)\mathbf{w}(\tau,\sigma)$, $d=d_e$, a_i, ϵ_i, $r_{(e,v)}$, $r_{(e,v')}$, $w_{(e,v)}$, $w_{(e,v')}$ are given in our notation as:

$$\begin{aligned}
\mathbf{w}(\tau,\sigma)&=u_j(\sigma),\quad \mathbf{w}(\tau,\sigma')=u_{j'}(\sigma');\\
\mathbf{w}(\tau_i,\sigma)&=u_i(\sigma),\quad \mathbf{w}(\tau_i',\sigma')=u_i(\sigma'),\quad \text{for } i\in\sigma\cap\sigma';\\
r(\tau,\sigma)&:=\text{the order of the stabilizer at } \mathcal{X}(\boldsymbol{\Sigma})_\sigma \text{ of the rigidification } \mathcal{X}(\boldsymbol{\Sigma})^{\mathrm{rig}}_\tau \text{ of } \mathcal{X}(\boldsymbol{\Sigma})_\tau\\
&=\frac{|G_v|}{|G_e|}=\frac{|N(\sigma)|}{|N(\tau)_{\mathrm{tor}}|}=\text{the norm of the image of } \rho_j \text{ in } \overline{N(\tau)}\cong\mathbb{Z};\\
\mathbf{u}&=r(\tau,\sigma)u_j(\sigma);\\
d&:=\text{the degree of the map } (f:|\mathbb{P}_{r_1,r_2}|\to|\mathcal{X}(\boldsymbol{\Sigma})_\tau|) \text{ between the coarse curves } (\cong\mathbb{P}^1)\\
&=r(\tau,\sigma)c=r(\tau,\sigma')c';\\
a_i&:=\int_{\mathcal{X}(\boldsymbol{\Sigma})^{\mathrm{rig}}_\tau}u_i=\frac{c_i}{d}=\frac{c_i}{c}r(\tau,\sigma)^{-1};\\
\epsilon_i&=\hat b_i,\quad \text{for } i\in\sigma\cap\sigma';\\
r_{(e,v)}&=r_1,\quad r_{(e,v')}=r_2,\\
r_{(e,v)}w_{(e,v)}&=u_j(\sigma)/c,\quad r_{(e,v')}w_{(e,v')}=-r_{(e,v)}w_{(e,v)}=u_{j'}(\sigma')/c'.
\end{aligned}$$

Here we set $N(\tau)=N/\sum_{i\in\tau}\mathbb{Z}\rho_i$, $\overline{N(\tau)}=N(\tau)/N(\tau)_{\mathrm{tor}}$, and $N(\tau)_{\mathrm{tor}}$ is the torsion part of $N(\tau)$. Our recursion coefficient $\mathbf{RC}(c)^{(\sigma',b')}_{(\sigma,b)}$ coincides with $(1/c)e_{\mathbb{T}}(N_{\sigma,b})\mathbf{h}(e)$ where $\mathbf{h}(e)$ is in [Liu13, (9.26)].

7. Proof of Theorem 31

In this section we complete the proof of Theorem 31, by showing that the S-extended I-function

$$I^S_{\mathcal{X}(\boldsymbol{\Sigma})}(\tilde Q,-z)$$

[5] The definition of a_i in [Liu13, § 8.6] contains a typo; it should be the integral over the *rigidification* of $\mathcal{X}(\boldsymbol{\Sigma})_\tau$.

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

satisfies the conditions in Theorem 41. This amounts to proving Propositions 45–47 below. Note that the sign of z should be flipped when we consider the I-function.

For a top-dimensional cone $\sigma \in \Sigma$ and $b \in \mathrm{Box}(\sigma)$, we write $I^S_\sigma(\tilde{Q}, z)$ and $I^S_{(\sigma,b)}(\tilde{Q}, z)$ for the restrictions of $I^S_{\mathcal{X}(\mathbf{\Sigma})}(\tilde{Q}, z)$ to the inertia stack $I\mathcal{X}(\mathbf{\Sigma})_\sigma$ of the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_\sigma$ and the component $I\mathcal{X}(\mathbf{\Sigma})_{\sigma,b}$ of $I\mathcal{X}(\mathbf{\Sigma})_\sigma$, respectively.

Proposition 45. *The extended I-function satisfies condition (C1) in Theorem 41. In other words, for each top-dimensional cone $\sigma \in \Sigma$ and $b \in \mathrm{Box}(\sigma)$, $I^S_{(\sigma,b)}(\tilde{Q}, z)$ is a power series in the extended Novikov variables $\tilde{Q}$ and t such that each coefficient of this power series lies in $S_{\mathbb{T}\times\mathbb{C}^\times} = \mathbb{C}(\chi_1, \ldots, \chi_d, z)$ and, as a function of z, it is regular except possibly for a pole at $z = 0$, a pole at $z = \infty$, and simple poles at*

$$\left\{ \frac{-u_j(\sigma)}{c} : \exists \sigma' \in \Sigma \text{ such that } \sigma|\sigma' \text{ and } j \in \sigma\backslash\sigma',\ c > 0 \text{ is such that } \langle c\rangle = \hat{b}_j \right\}.$$

Here we use Notation 8.

Proposition 46. *The extended I-function satisfies condition (C2) in Theorem 41. In other words, for any σ, $\sigma' \in \Sigma$ such that $\sigma|\sigma'$, we have*

$$\mathrm{Res}_{z=-(u_j(\sigma)/c)}\, I^S_{(\sigma,b)}(\tilde{Q}, z)\, dz = Q^{l(c,\sigma,j)}\, \mathbf{RC}(c)^{(\sigma',b')}_{(\sigma,b)}\, I^S_{(\sigma',b')}(\tilde{Q}, z)|_{z=-(u_j(\sigma)/c)}.$$

Proposition 47. *The extended I-function satisfies condition (C3) in Theorem 41. In other words, if $\sigma \in \Sigma$ is a top-dimensional cone, then the Laurent expansion at $z = 0$ of $I^S_\sigma(\tilde{Q}, -z)$ is a $\Lambda^{\mathbb{T}}_{\mathrm{nov}}[\![x, t]\!]$-valued point of $\mathcal{L}^{\mathrm{tw}}_\sigma$.*

7.1 Poles of the extended *I*-function

In this subsection we prove Proposition 45. Let σ be a top-dimensional cone and take $b \in \mathrm{Box}(\sigma)$. The restriction $I^S_{(\sigma,b)}$ of the I-function to the fixed point $I\mathcal{X}(\mathbf{\Sigma})_{\sigma,b}$ takes the form

$$I^S_{(\sigma,b)}(\tilde{Q}, z) = z e^{\sum_{i=1}^n u_i(\sigma)t_i/z} \sum_{\lambda \in \Lambda E^S_b} \tilde{Q}^\lambda e^{\lambda t} \left(\prod_{i \notin \sigma} \frac{\prod_{a \leqslant 0, \langle a\rangle = 0} az}{\prod_{a \leqslant \lambda_i, \langle a\rangle = 0} az} \right) \left(\prod_{i \in \sigma} \frac{\prod_{a \leqslant 0, \langle a\rangle = \langle\lambda_i\rangle} (u_i(\sigma) + az)}{\prod_{a \leqslant \lambda_i, \langle a\rangle = \langle\lambda_i\rangle} (u_i(\sigma) + az)} \right) \tag{29}$$

where the index i ranges over $\{1, \ldots, n + m\}$ and we regard $\sigma \subset \{1, \ldots, n\}$ as a subset of $\{1, \ldots, n + m\}$. We also used $u_i(\sigma) = 0$ for $i \notin \sigma$. For $\lambda \in \Lambda^S_b$, we have that $\lambda_i \in \mathbb{Z}$ for all $i \notin \sigma$ because $\langle\lambda_i\rangle = \hat{b}_i$ and $b \in \mathrm{Box}(\sigma)$. Note also that one may assume that $\lambda_i \in \mathbb{Z}_{\geqslant 0}$ for $i \notin \sigma$ in the above sum, as otherwise the contribution is zero. We see that $I^S_{(\sigma,b)}$ has poles possibly at $z = 0$ and $z = \infty$ and simple poles at

$$-u_i(\sigma)/a \quad \text{with } 0 < a \leqslant \lambda_i,\ \langle a\rangle = \langle\lambda_i\rangle = \hat{b}_i,\ i \in \sigma,$$

for $\lambda \in \Lambda E^S_b$ contributing to the sum. It suffices to see that if $\lambda_{i_0} > 0$ for some $i_0 \in \sigma$, then there exists a top-dimensional cone σ' such that $\sigma|\sigma'$ and $i_0 \in \sigma\backslash\sigma'$, i.e. $i_0 = j$ in Notation 8. We have

$$\sum_{i\in\sigma} (-\lambda_i)\bar{\rho}_i = \sum_{\substack{i:1\leqslant i\leqslant n\\ i\notin\sigma}} \lambda_i \bar{\rho}_i + \sum_{i=1}^m \lambda_{n+i} \bar{s}_i \tag{30}$$

where $s_1, \ldots, s_m$ are the images of elements of S in N_Σ. As we remarked above, we may assume that $\lambda_i \in \mathbb{Z}_{\geqslant 0}$ for $i \notin \sigma$ and hence the right-hand side belongs to the support $|\Sigma|$ of the fan. Therefore $\sum_{i\in\sigma}(-\lambda_i)\bar{\rho}_i \in |\Sigma|$. Because $|\Sigma|$ is convex, the positivity of λ_{i_0} implies that there exists a top-dimensional cone $\sigma' \in \Sigma$ such that $\sigma|\sigma'$ and $i_0 \in \sigma\backslash\sigma'$. Proposition 45 is proved.

7.2 Recursion for the extended I-function

In this subsection we prove Proposition 46. Let $\sigma \in \Sigma$ be a top-dimensional cone and let $b \in \mathrm{Box}(\sigma)$. Fix another top-dimensional cone σ' with $\sigma|\sigma'$ and a positive rational number c such that $\langle c\rangle = \hat{b}_j$, where j is the index in Notation 8. We examine the residue of $I^S_{(\sigma,b)}$ at $z = -u_j(\sigma)/c$. Write

$$\triangleright_{\lambda,i,\sigma}(z) = \frac{\prod_{\langle a\rangle=\langle\lambda_i\rangle, a\leqslant 0}(u_i(\sigma)+az)}{\prod_{\langle a\rangle=\langle\lambda_i\rangle, a\leqslant\lambda_i}(u_i(\sigma)+az)}$$

for $\lambda \in \Lambda^S$ and $1 \leqslant i \leqslant n+m$. The residue of (29) at $z = -u_j(\sigma)/c$ is given by

$$\left(-\frac{u_j(\sigma)}{c}\right)e^{(\sum_{i=1}^n u_i(\sigma)t_i/-u_j(\sigma)/c)}\frac{1}{c}\sum_{\substack{\lambda\in\Lambda_b^S\\ \lambda_j\geqslant c}}\tilde{Q}^\lambda e^{\lambda t}\frac{\prod_{i:i\neq j}\triangleright_{\lambda,i,\sigma}(-u_j(\sigma)/c)}{\prod_{\substack{0<a\leqslant\lambda_j,\langle a\rangle=\langle\lambda_j\rangle\\ a\neq c}}(u_j(\sigma)-a(u_j(\sigma)/c))}. \tag{31}$$

Recall from Remark 30 that the summation range can be taken to be Λ_b^S instead of ΛE_b^S. Let $l(c,\sigma,j) \in \Lambda E_{\sigma,b}^{\sigma',b'} \subset \mathbb{L}\otimes\mathbb{Q}$ be the degree from Definition 12. We now consider the change of variables

$$\lambda = \lambda' + l(c,\sigma,j)$$

and replace the sum over $\lambda \in \Lambda_b^S$ with the sum over $\lambda' \in \Lambda_{b'}^S$ using Lemma 25. We write c_i for the components of $l(c,\sigma,j) \in \mathbb{L}\otimes\mathbb{Q} \subset \mathbb{L}^S\otimes\mathbb{Q}$ as an element of $\mathbb{Q}^{n+m}$. Using the notation in Definition 12, we have $c_i = D_i\cdot l(c,\sigma,j)$ for $1 \leqslant i \leqslant n$, $c_j = c$, $c_{j'} = c'$ and $c_i = 0$ for $n+1 \leqslant i \leqslant n+m$.

Lemma 48. *Let λ, λ' be as above. We have*

$$u_i(\sigma) = u_i(\sigma') + \frac{c_i}{c}u_j(\sigma), \tag{32}$$

$$\frac{\sum_{i=1}^n u_i(\sigma)t_i}{-u_j(\sigma)/c} + \lambda t = \frac{\sum_{i=1}^n u_i(\sigma')t_i}{-u_j(\sigma)/c} + \lambda' t, \tag{33}$$

$$\triangleright_{\lambda,i,\sigma}\left(-\frac{u_j(\sigma)}{c}\right) = \triangleright_{\lambda',i,\sigma'}\left(-\frac{u_j(\sigma)}{c}\right)\frac{\prod_{a\leqslant 0,\langle a\rangle=\langle\lambda_i\rangle}(u_i(\sigma)-(a/c)u_j(\sigma))}{\prod_{a\leqslant c_i,\langle a\rangle=\langle\lambda_i\rangle}(u_i(\sigma)-(a/c)u_j(\sigma))}, \quad \text{for } i\neq j, \tag{34}$$

$$\prod_{\substack{0<a\leqslant\lambda_j,\langle a\rangle=\langle\lambda_j\rangle\\ a\neq c}}\left(u_j(\sigma)-a\frac{u_j(\sigma)}{c}\right) = \prod_{\substack{-c<a\leqslant\lambda_j',a\in\mathbb{Z}\\ a\neq 0}}\left(-a\frac{u_j(\sigma)}{c}\right). \tag{35}$$

Proof. Formulas (33) and (34) follow easily from (32); formula (35) is obvious if we notice that $\langle c\rangle = \hat{b}_j = \langle\lambda_j\rangle$ and $\lambda_j = \lambda_j' + c$. It suffices to show (32). Equality (32) is obvious for $n+1 \leqslant i \leqslant m$, so we restrict to the case where $1 \leqslant i \leqslant n$. Consider the representable morphism $f : \mathbb{P}_{r_1,r_2} \to \mathcal{X}(\mathbf{\Sigma})$ given by (σ,σ',b,c) via Proposition 10. By the localization formula, we obtain

$$c_i = D_i\cdot l(c,\sigma,j) = \int_{\mathbb{P}_{r_1,r_2}} f^*D_i = \int_{\mathbb{P}_{r_1,r_2}}^{\mathbb{T}} f^*u_i = \frac{u_i(\sigma)}{u_j(\sigma)/c} + \frac{u_i(\sigma')}{-u_j(\sigma)/c}$$

where we use the fact that $u_j(\sigma)/c$ and $-u_j(\sigma)/c$ are the induced $\mathbb{T}$-weights at 0 and ∞ of the coarse domain curve $|\mathbb{P}_{r_1,r_2}|$: see Remark 42. The lemma follows. □

https://doi.org/10.1112/S0010437X15007356 Published online by Cambridge University Press

Applying Lemma 48, we see that (31) equals

$$\left(-\frac{u_j(\sigma)}{c}\right)e^{(\sum_{i=1}^n u_i(\sigma')t_i)/(-u_j(\sigma)/c)}\frac{1}{c}\sum_{\substack{\lambda'\in\Lambda_{b'}^S\\ \lambda'_j\geqslant 0}}\tilde{Q}^{\lambda'}e^{\lambda' t}\frac{\prod_{i:i\neq j}\rhd_{\lambda',i,\sigma'}(-u_j(\sigma)/c)}{\prod_{\substack{-c<a\leqslant\lambda'_j,a\in\mathbb{Z}\\ a\neq 0}}(-a(u_j(\sigma)/c))}$$
$$\times\prod_{i:i\neq j}\frac{\prod_{a\leqslant 0,\langle a\rangle=\langle\lambda_i\rangle}(u_i(\sigma)-(a/c)u_j(\sigma))}{\prod_{a\leqslant c_i,\langle a\rangle=\langle\lambda_i\rangle}(u_i(\sigma)-(a/c)u_j(\sigma))}$$
$$=\frac{1}{c}\frac{1}{\prod_{0<a<c,a\in\mathbb{Z}}(a(u_j(\sigma)/c))}\prod_{i\in\sigma'}\frac{\prod_{a\leqslant 0,\langle a\rangle=\langle\lambda_i\rangle}(u_i(\sigma)-(a/c)u_j(\sigma))}{\prod_{a\leqslant c_i,\langle a\rangle=\langle\lambda_i\rangle}(u_i(\sigma)-(a/c)u_j(\sigma))}I^S_{(\sigma',b')}(\tilde{Q},z)|_{z=-u_j(\sigma)/c}$$

multiplied by $Q^{l(c,\sigma,j)}$. It is now straightforward to check that the last expression coincides with

$$\mathbf{RC}(c)_{(\sigma,b)}^{(\sigma',b')}I^S_{(\sigma',b')}(\tilde{Q},z)|_{z=-u_j(\sigma)/c}.$$

(Here we used $u_{j'}(\sigma)=0$ and $\lambda_{j'}\in\mathbb{Z}$.) Proposition 46 is proved.

7.3 Restriction of the extended I-function to fixed points

In this subsection we prove Proposition 47. Let $\sigma\in\Sigma$ be a top-dimensional cone. By (29) and the discussion in § 7.1, the restriction $I^S_\sigma(\tilde{Q},-z)$ of the S-extended I-function to the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_\sigma$ is

$$-ze^{-\sum_{i=1}^n u_i(\sigma)t_i/z}\sum_{\substack{\lambda\in\Lambda_\sigma^S\\ \lambda_i\in\mathbb{Z}_{\geqslant 0}\text{ if }i\notin\sigma}}\frac{\tilde{Q}^\lambda e^{\lambda t}}{\prod_{i\notin\sigma}\lambda_i!(-z)^{\lambda_i}}\left(\prod_{j\in\sigma}\frac{\prod_{\langle a\rangle=\langle\lambda_j\rangle,a\leqslant 0}(u_j(\sigma)-az)}{\prod_{\langle a\rangle=\langle\lambda_j\rangle,a\leqslant\lambda_j}(u_j(\sigma)-az)}\right)1_{v^S(\lambda)}$$

where $1_{v^S(\lambda)}\in H^\bullet_{\mathrm{CR}}(\mathcal{X}(\mathbf{\Sigma})_\sigma)$ is the identity class supported on the twisted sector corresponding to $v^S(\lambda)\in\mathrm{Box}(\sigma)$. We want to show that this lies on the Lagrangian cone $\mathcal{L}_\sigma^{\mathrm{tw}}$. We claim that it suffices to show that $I^S_\sigma(\tilde{Q},-z)|_{t=0}$ lies on $\mathcal{L}_\sigma^{\mathrm{tw}}$. By the String Equation, $\mathcal{L}_\sigma^{\mathrm{tw}}$ is invariant under multiplication by $e^{-\sum_{i=1}^n u_i(\sigma)t_i/z}$ and thus we can remove the factor $e^{-\sum_{i=1}^n u_i(\sigma)t_i/z}$. Since the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_\sigma$ has no Novikov variables, we can regard $\tilde{Q}$ in $I^S_\sigma(\tilde{Q},-z)$ as *variables* rather than elements of the ground ring. (In other words, $\mathcal{L}_\sigma^{\mathrm{tw}}$ is defined over $S_{\mathbb{T}}$.) Therefore, we can absorb the factor $e^{\lambda t}$ into $\tilde{Q}$ by rescaling $\tilde{Q}$. The claim follows.

Define rational numbers a_{ij} for $i\notin\sigma, j\in\sigma$ by $\bar{\rho}_i=\sum_{j\in\sigma}a_{ij}\bar{\rho}_j$ for $1\leqslant i\leqslant n$ and $\bar{s}_{i-n}=\sum_{j\in\sigma}a_{ij}\bar{\rho}_j$ for $n+1\leqslant i\leqslant n+m$. Then (30) shows that

$$\lambda_j=-\sum_{i\notin\sigma}\lambda_i a_{ij} \tag{36}$$

for $\lambda\in\Lambda_\sigma^S$ and $j\in\sigma$. Henceforth we regard λ_j for $j\in\sigma$ as a linear function of $(\lambda_i:i\notin\sigma)$ via this relation. We introduce variables $(q_i:i\notin\sigma)$ dual to $(\lambda_i:i\notin\sigma)$ and consider the change of variables

$$\tilde{Q}^\lambda=\prod_{i\notin\sigma}q_i^{\lambda_i}.$$

We also have

$$v^S(\lambda)=\sum_{j\in\sigma}\lceil\lambda_j\rceil\rho_j+\sum_{i\notin\sigma,i\leqslant n}\lambda_i\rho_i+\sum_{i=1}^m\lambda_{n+i}s_i\equiv\sum_{i\notin\sigma}\lambda_i b^i\mod N_\sigma$$

where

$$b^i = \begin{cases} \text{the image of } \rho_i \text{ in } N(\sigma) \cong \operatorname{Box}(\sigma) & 1 \leqslant i \leqslant n, \\ \text{the image of } s_{i-n} \text{ in } N(\sigma) \cong \operatorname{Box}(\sigma) & n+1 \leqslant i \leqslant n+m. \end{cases}$$

Now it suffices to show that

$$-z \sum_{(\lambda_i : i \notin \sigma) \in (\mathbb{Z}_{\geqslant 0})^\ell} \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i}}{\lambda_i!(-z)^{\lambda_i}} \right) \left(\prod_{j \in \sigma} \frac{\prod_{\langle a \rangle = \langle \lambda_j \rangle, a \leqslant 0} (u_j(\sigma) - az)}{\prod_{\langle a \rangle = \langle \lambda_j \rangle, a \leqslant \lambda_j} (u_j(\sigma) - az)} \right) 1_{\sum_{i \notin \sigma} \lambda_i b^i} \tag{37}$$

is a $S_{\mathbb{T}}[\![q]\!]$-valued point on $\mathcal{L}_\sigma^{\mathrm{tw}}$, where $\ell = n + m - \dim \mathcal{X}(\mathbf{\Sigma})$.

Jarvis and Kimura [JK02] calculated the Gromov–Witten theory of BG with G a finite group, and it follows from their result that the J-function of $BN(\sigma) \cong \mathcal{X}(\mathbf{\Sigma})_\sigma$ is

$$J_{BN(\sigma)}\left(\sum_{i \notin \sigma} q_i 1_{b^i}, -z \right) = -z \sum_{(\lambda_i : i \notin \sigma) \in (\mathbb{Z}_{\geqslant 0})^\ell} \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i}}{\lambda_i!(-z)^{\lambda_i}} \right) 1_{\sum_{i \notin \sigma} \lambda_i b^i}. \tag{38}$$

(See [CCIT09, Proposition 6.1].) Comparing this with (37), we find that expression (37) is the *hypergeometric modification* of $J_{BN(\sigma)}$, in the sense of [CG07, CCIT09]. The J-function (38) lies on the Lagrangian cone of the Gromov–Witten theory of $BN(\sigma)$ (see Remark 3), and we now use the argument of [CCIT09] to show that the hypergeometric modification of the J-function (37) lies on the cone $\mathcal{L}_\sigma^{\mathrm{tw}}$ of the twisted theory.

We briefly recall the setting from [CCIT09]. Let F be the direct sum $\bigoplus_{j=1}^d F^{(j)}$ of d vector bundles and consider a universal multiplicative characteristic class

$$\mathbf{c}(F) = \prod_{j=1}^d \exp\left(\sum_{k=0}^\infty s_k^{(j)} \operatorname{ch}_k(F^{(j)}) \right)$$

where $s_0^{(j)}, s_1^{(j)}, s_2^{(j)}, \ldots$ are formal indeterminates. As in § 2.4, one can define $(F, \mathbf{c})$-twisted Gromov–Witten invariants and a Lagrangian cone for the $(F, \mathbf{c})$-twisted theory. The Lagrangian cone here is defined over a certain formal power series ring $\Lambda_{\mathrm{nov}}[\![\mathbf{s}]\!]$ in infinitely many variables $s_k^{(j)}$, $0 \leqslant k < \infty$, $1 \leqslant j \leqslant d$. We apply this setting to the case where $F = T_\sigma \mathcal{X}(\mathbf{\Sigma})$, which is the direct sum of line bundles $u_j|_\sigma$, $j \in \sigma$, over $\mathcal{X}(\mathbf{\Sigma})_\sigma$. Denote by $\mathcal{L}^{\mathbf{s}}$ the Lagrangian cone of the $(T_\sigma \mathcal{X}(\mathbf{\Sigma}), \mathbf{c})$-twisted theory of the $\mathbb{T}$-fixed point $\mathcal{X}(\mathbf{\Sigma})_\sigma$. By specializing the parameters $s_k^{(j)}$, $j \in \sigma$, as

$$s_k^{(j)} = \begin{cases} -\log u_j(\sigma) & k = 0, \\ (-1)^k (k-1)! u_j(\sigma)^{-k} & k \geqslant 1, \end{cases}$$

we recover the $(T_\sigma \mathcal{X}(\mathbf{\Sigma}), e_{\mathbb{T}}^{-1})$-twisted theory of $\mathcal{X}(\mathbf{\Sigma})_\sigma$. This specialization ensures that

$$\mathbf{s}^{(j)}(x) := \exp\left(\sum_{k=0}^\infty s_k^{(j)} \frac{x^k}{k!} \right) \quad \text{coincides with } (u_j(\sigma) + x)^{-1}.$$

It now suffices to establish the following lemma.

Lemma 49. *Let:*

$$I_{\mathbf{s}}(q) = \sum_{(\lambda_i : i \notin \sigma) \in (\mathbb{Z}_{\geqslant 0})^\ell} \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i}}{\lambda_i!(-z)^{\lambda_i}} \right) \left(\prod_{j \in \sigma} \frac{\prod_{\langle a \rangle = \langle \lambda_j \rangle, a \leqslant 0} \exp(-\mathbf{s}^{(j)}(-az))}{\prod_{\langle a \rangle = \langle \lambda_j \rangle, a \leqslant \lambda_j} \exp(-\mathbf{s}^{(j)}(-az))} \right) 1_{\sum_{i \notin \sigma} \lambda_i b^i}$$

where λ_j with $j \in \sigma$ is a linear function of $(\lambda_i : i \notin \sigma)$ via (36). Then $I_{\mathbf{s}}(q)$ defines a $\mathbb{C}[\![\mathbf{s}]\!][\![q]\!]$-valued point on $\mathcal{L}^{\mathbf{s}}$.

Proof. Introduce the function

$$G_y^{(j)}(x,z) := \sum_{l,m\geqslant 0} s_{l+m-1}^{(j)} \frac{B_m(y)}{m!}\frac{x^l}{l!} z^{m-1} \in \mathbb{C}[y,x,z,z^{-1}][\![s_0^{(j)}, s_1^{(j)}, s_2^{(j)}, \ldots]\!]$$

as in [CCIT09]. We have

$$\begin{aligned} G_y^{(j)}(x,z) &= G_0^{(j)}(x+yz,z),\\ G_0^{(j)}(x+z,z) &= G_0^{(j)}(x,z) + \mathbf{s}^{(j)}(x). \end{aligned} \tag{39}$$

We apply the differential operator $\exp(-\sum_{j\in\sigma} G_0^{(j)}(z\theta_j,z))$ with $\theta_j = \sum_{i\notin\sigma} a_{ij} q_i(\partial/\partial q_i)$ to the J-function (38) of $BN(\sigma)$ and obtain

$$\begin{aligned} \mathbf{f} &:= e^{-\sum_{j\in\sigma} G_0^{(j)}(z\theta_j,z)} J_{BN(\sigma)}\bigg(\sum_{i\notin\sigma} q_i 1_{b^i}, -z\bigg)\\ &= -z \sum_{(\lambda_i : i\notin\sigma)\in(\mathbb{Z}_{\geqslant 0})^\ell} \bigg(\prod_{i\notin\sigma} \frac{q_i^{\lambda_i}}{\lambda_i!(-z)^{\lambda_i}}\bigg) \exp\bigg(-\sum_{j\in\sigma} G_0^{(j)}(-z\lambda_j,z)\bigg) 1_{\sum_{i\notin\sigma}\lambda_i b^i} \end{aligned}$$

where we used (36). The argument in the paragraph after [CCIT09, (14)] shows that $\mathbf{f}$ lies on the Lagrangian cone $\mathcal{L}^{\mathrm{un}}$ of the untwisted theory of $BN(\sigma)$. (This is where we use Theorem 2.) On the other hand, Tseng's quantum Riemann–Roch operator for $\bigoplus_{j\in\sigma} u_j|_\sigma$ is

$$\Delta_{\mathbf{s}} = \bigoplus_{b\in\mathrm{Box}(\sigma)} \exp\bigg(\sum_{j\in\sigma} G_{b_j}^{(j)}(0,z)\bigg).$$

This operator maps the untwisted cone $\mathcal{L}^{\mathrm{un}}$ to the twisted cone $\mathcal{L}^{\mathbf{s}}$ [Tse10]. Therefore

$$\Delta_{\mathbf{s}}\mathbf{f} = -z \sum_{(\lambda_i : i\notin\sigma)\in(\mathbb{Z}_{\geqslant 0})^\ell} \bigg(\prod_{i\notin\sigma} \frac{q_i^{\lambda_i}}{\lambda_i!(-z)^{\lambda_i}}\bigg) \exp\bigg(\sum_{j\in\sigma}\Big(G_{\langle-\lambda_j\rangle}^{(j)}(0,z) - G_0^{(j)}(-z\lambda_j,z)\Big)\bigg) 1_{\sum_{i\notin\sigma}\lambda_i b^i}$$

lies on $\mathcal{L}^{\mathbf{s}}$. Here we used the fact that $b_j = \langle -\lambda_j\rangle$ for the box element $b = \sum_{i\notin\sigma}\lambda_i b^i$. After a straightforward calculation using (39), the lemma follows. □

This completes the proof of Proposition 47, and thus completes the proof of our mirror theorem.

Acknowledgements

We benefited greatly from discussions with Alexander Givental, Melissa Liu, and Yongbin Ruan. T.C. was supported in part by a Royal Society University Research Fellowship, ERC Starting Investigator Grant number 240123, and the Leverhulme Trust. A.C. was supported in part by EPSRC grants EP/E022162/1 and EP/I008128/1. H.I. was supported in part by EPSRC grant EP/E022162/1 and JSPS Grant-in-Aid for Scientific Research (C) 25400069. H.-H.T. was supported in part by a Simons Foundation Collaboration Grant.

References

AGV02 D. Abramovich, T. Graber and A. Vistoli, *Algebraic orbifold quantum products*, in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, Contemporary Mathematics, vol. 310 (American Mathematical Society, Providence, RI, 2002), 1–24; MR 1950940 (2004c:14104).

AGV08 D. Abramovich, T. Graber and A. Vistoli, *Gromov–Witten theory of Deligne–Mumford stacks*, Amer. J. Math. **130** (2008), 1337–1398; MR 2450211 (2009k:14108).

BC10 A. Bayer and C. Cadman, *Quantum cohomology of* $[\mathbb{C}^N/\mu_r]$, Compositio Math. **146** (2010), 1291–1322; MR 2684301 (2012d:14095).

BCS05 L. A. Borisov, L. Chen and G. G. Smith, *The orbifold Chow ring of toric Deligne–Mumford stacks*, J. Amer. Math. Soc. **18** (2005), 193–215 (electronic); MR 2114820 (2006a:14091).

BC11 A. Brini and R. Cavalieri, *Open orbifold Gromov–Witten invariants of* $[\mathbb{C}^3/\mathbb{Z}_n]$*: localization and mirror symmetry*, Selecta Math. (N.S.) **17** (2011), 879–933; MR 2861610 (2012i:14068).

Bro14 J. Brown, *Gromov–Witten invariants of toric fibrations*, Int. Math. Res. Not. IMRN **2014**(19) (2014), 5437–5482; MR 3267376.

BG09 J. Bryan and T. Graber, *The crepant resolution conjecture*, in *Algebraic geometry—Seattle 2005. Part 1*, Proceedings of Symposia in Pure Mathematics, vol. 80 (American Mathematical Society, Providence, RI, 2009), 23–42; MR 2483931 (2009m:14083).

CC09 C. Cadman and R. Cavalieri, *Gerby localization,* Z_3*-Hodge integrals and the GW theory of* $[\mathbb{C}^3/Z_3]$, Amer. J. Math. **131** (2009), 1009–1046; MR 2543921 (2010e:14051).

CCLT13 K. Chan, C.-H. Cho, S.-C. Lau and H.-H. Tseng, *Gross fibrations, SYZ mirror symmetry, and open Gromov–Witten invariants for toric Calabi–Yau orbifolds*, Preprint (2013), arXiv:1306.0437 [math.SG].

CCLT14 K. Chan, C.-H. Cho, S.-C. Lau and H.-H. Tseng, *Lagrangian Floer superpotentials and crepant resolutions for toric orbifolds*, Comm. Math. Phys. **328** (2014), 83–130; MR 3196981.

CR02 W. Chen and Y. Ruan, *Orbifold Gromov–Witten theory*, in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, Contemporary Mathematics, vol. 310 (American Mathematical Society, Providence, RI, 2002), 25–85; MR 1950941 (2004k:53145).

CR04 W. Chen and Y. Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248** (2004), 1–31; MR 2104605 (2005j:57036).

CCFK14 D. Cheong, I. Ciocan-Fontanine and B. Kim, *Orbifold quasimap theory*, Preprint (2014), arXiv:1405.7160 [math.AG].

CK14 I. Ciocan-Fontanine and B. Kim, *Wall-crossing in genus zero quasimap theory and mirror maps*, Algebr. Geom. **1** (2014), 400–448; MR 3272909.

CCIT09 T. Coates, A. Corti, H. Iritani and H.-H. Tseng, *Computing genus-zero twisted Gromov–Witten invariants*, Duke Math. J. **147** (2009), 377–438; MR 2510741 (2010a:14090).

CCIT14 T. Coates, A. Corti, H. Iritani and H.-H. Tseng, *Some applications of the mirror theorem for toric stacks*, Preprint (2014), arXiv:1401.2611 [math.AG].

CG07 T. Coates and A. Givental, *Quantum Riemann–Roch, Lefschetz and Serre*, Ann. of Math. (2) **165** (2007), 15–53; MR 2276766 (2007k:14113).

CIT09 T. Coates, H. Iritani and H.-H. Tseng, *Wall-crossings in toric Gromov–Witten theory. I. Crepant examples*, Geom. Topol. **13** (2009), 2675–2744; MR 2529944 (2010i:53173).

CLCT09 T. Coates, Y.-P. Lee, A. Corti and H.-H. Tseng, *The quantum orbifold cohomology of weighted projective spaces*, Acta Math. **202** (2009), 139–193; MR 2506749 (2010f:53155).

Cor07 A. Corti, Quantum orbifold cohomology of weak Fano toric DM stacks, Talk at Institute Henri Poincaré, February 15, 2007, as part of the Workshop on Quantum Cohomology of Stacks and String Theory, February 12–16, 2007.

CG11 A. Corti and V. Golyshev, *Hypergeometric equations and weighted projective spaces*, Sci. China Math. **54** (2011), 1577–1590; MR 2824960.

CLS11 D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124 (American Mathematical Society, Providence, RI, 2011); MR 2810322 (2012g:14094).

FLT12 B. Fang, C.-C. M. Liu and H.-H. Tseng, *Open-closed Gromov–Witten invariants of 3-dimensional Calabi–Yau smooth toric DM stacks*, Preprint (2012), arXiv:1212.6073 [math.AG].

FMN10 B. Fantechi, E. Mann and F. Nironi, *Smooth toric Deligne–Mumford stacks*, J. Reine Angew. Math. **648** (2010), 201–244; MR 2774310 (2012b:14097).

Giv95 A. B. Givental, *Homological geometry. I. Projective hypersurfaces*, Selecta Math. (N.S.) **1** (1995), 325–345; MR 1354600 (97c:14052).

Giv98 A. B. Givental, *A mirror theorem for toric complete intersections*, in *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, Progress in Mathematics, vol. 160 (Birkhäuser, Boston, 1998), 141–175; MR 1653024 (2000a:14063).

Giv01 A. B. Givental, *Gromov–Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), 551–568; 645, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary; MR 1901075 (2003j:53138).

Giv04 A. B. Givental, *Symplectic geometry of Frobenius structures*, in *Frobenius manifolds*, Aspects Mathematics, E36 (Friedr. Vieweg, Wiesbaden, 2004), 91–112; MR 2115767 (2005m:53172).

GW12a E. Gonzalez and C. T. Woodward, *Quantum cohomology and toric minimal model programs*, Preprint (2012), arXiv:1207.3253 [math.AG].

GW12b E. Gonzalez and C. T. Woodward, *A wall-crossing formula for Gromov–Witten invariants under variation of GIT quotient*, Preprint (2012), arXiv:1208.1727 [math.AG].

GS14 M. Guest and H. Sakai, *Orbifold quantum D-modules associated to weighted projective spaces*, Comment. Math. Helv. **89** (2014), 273–297; MR 3225449.

Iri06 H. Iritani, *Quantum D-modules and equivariant Floer theory for free loop spaces*, Math. Z. **252** (2006), 577–622; MR 2207760 (2007e:53118).

Iri09 H. Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. **222** (2009), 1016–1079; MR 2553377 (2010j:53182).

Iri11 H. Iritani, *Quantum cohomology and periods*, Ann. Inst. Fourier (Grenoble) **61** (2011), 2909–2958; MR 3112512.

Iwa09a I. Iwanari, *The category of toric stacks*, Compositio Math. **145** (2009), 718–746; MR 2507746 (2011b:14113).

Iwa09b I. Iwanari, *Logarithmic geometry, minimal free resolutions and toric algebraic stacks*, Publ. Res. Inst. Math. Sci. **45** (2009), 1095–1140; MR 2597130 (2011f:14084).

JK02 T. J. Jarvis and T. Kimura, *Orbifold quantum cohomology of the classifying space of a finite group*, in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, Contemporary Mathematics, vol. 310 (American Mathematical Society, Providence, RI, 2002), 123–134; MR 1950944 (2004a:14056).

Jia08 Y. Jiang, *The orbifold cohomology ring of simplicial toric stack bundles*, Illinois J. Math. **52** (2008), 493–514; MR 2524648 (2011c:14059).

JT08 Y. Jiang and H.-H. Tseng, *Note on orbifold Chow ring of semi-projective toric Deligne–Mumford stacks*, Comm. Anal. Geom. **16** (2008), 231–250; MR 2411474 (2009h:14092).

Joh14 P. Johnson, *Equivariant GW theory of stacky curves*, Comm. Math. Phys. **327** (2014), 333–386; MR 3183403.

JPT11 P. Johnson, R. Pandharipande and H.-H. Tseng, *Abelian Hurwitz–Hodge integrals*, Michigan Math. J. **60** (2011), 171–198; MR 2785870 (2012c:14025).

LS14 Y.-P. Lee and M. Shoemaker, *A mirror theorem for the mirror quintic*, Geom. Topol. **18** (2014), 1437–1483; MR 3228456.

Liu13 C.-C. M. Liu, *Localization in Gromov–Witten theory and orbifold Gromov–Witten theory*, in *Handbook of moduli. Vol. II*, Advanced Lectures in Mathematics (ALM), vol. 25 (International Press, Somerville, MA, 2013), 353–425; MR 3184181.

Man08 E. Mann, *Orbifold quantum cohomology of weighted projective spaces*, J. Algebraic Geom. **17** (2008), 137–166; MR 2357682 (2008k:14106).

MT08 T. E. Milanov and H.-H. Tseng, *The spaces of Laurent polynomials, Gromov–Witten theory of* $\mathbb{P}^1$*-orbifolds, and integrable hierarchies*, J. Reine Angew. Math. **622** (2008), 189–235; MR 2433616 (2010e:14053).

Tse10 H.-H. Tseng, *Orbifold quantum Riemann–Roch, Lefschetz and Serre*, Geom. Topol. **14** (2010), 1–81; MR 2578300 (2011c:14147).

Vla02 Y. Vlassopoulos, *Quantum cohomology and Morse theory on the loop space of toric varieties*, Preprint (2002), arXiv:math/0203083.

Woo12 C. T. Woodward, *Quantum Kirwan morphism and Gromov–Witten invariants of quotients I*, Preprint (2012), arXiv:1204.1765 [math.AG].

Woo14a C. T. Woodward, *Quantum Kirwan morphism and Gromov–Witten invariants of quotients II*, Preprint (2014), arXiv:1408.5864 [math.AG].

Woo14b C. T. Woodward, *Quantum Kirwan morphism and Gromov–Witten invariants of quotients III*, Preprint (2014), arXiv:1408.5869 [math.AG].

Tom Coates t.coates@imperial.ac.uk
Department of Mathematics, Imperial College London,
180 Queen's Gate, London SW7 2AZ, UK

Alessio Corti a.corti@imperial.ac.uk
Department of Mathematics, Imperial College London,
180 Queen's Gate, London SW7 2AZ, UK

Hiroshi Iritani iritani@math.kyoto-u.ac.jp
Department of Mathematics, Graduate School of Science,
Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

Hsian-Hua Tseng hhtseng@math.ohio-state.edu
Department of Mathematics, Ohio State University,
100 Math Tower, 231 West 18th Ave., Columbus OH 43210, USA