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SLICE MAPS AND MULTIPLIERS OF INVARIANT SUBSPACES

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ABSTRACT. Let $\overline{D^2}$ be the closed bidisc and T^2 be its distinguished boundary. For $(\alpha,\beta) \in \overline{D^2}$, let $\Phi_{\alpha\beta}$ be a slice map, that is, $(\Phi_{\alpha\beta}f)(\lambda) = f(\alpha\lambda, \beta\lambda)$ for $\lambda \in D$ and $f \in H^2(D^2)$. Then $\ker\Phi_{\alpha\beta}$ is an invariant subspace, and it is not difficult to describe $\ker\Phi_{\alpha\beta}$ and $\mathcal{M}(\ker\Phi_{\alpha\beta}) = \{\phi \in L^\infty(T^2) : \phi\ker\Phi_{\alpha\beta} \subset H^2(D^2)\}$. In this paper, we study the set $\mathcal{M}(M)$ of all multipliers for an invariant subspace M such that the common zero set of M contains that of $\ker\Phi_{\alpha\beta}$.

1. **Introduction.** Let D^2 be the open unit disc in $\mathbf{C}^2$ and T^2 be its distinguished boundary. Normalized Lebesgue measure on T^2 is denoted by dm. For $1 \leq p \leq \infty$, $H^p(D^2)$ is the Hardy space and $L^p(T^2)$ is the Lebesgue space on T^2. Let $N(D^2)$ denote the Nevanlinna class. Each f in $N(D^2)$ has radial limits f^* defined on T^2 a.e. Moreover, there is a singular measure $d\sigma_f$ on T^2 determined by f such that the least harmonic majorant $u(\log|f|)$ of $\log|f|$ is given by $u(\log|f|)(\zeta) = P_\zeta(\log|f^*|+d\sigma_f)$ where P_ζ denotes Poisson integration and $\zeta = (z,w) \in D^2$. Put $N_*(D^2) = \{f \in N(D^2)\ ;\ d\sigma_f \leq 0\}$; then $H^p(D^2) \subset N_*(D^2) \subset N(D^2)$ and $H^p(D^2) = N_*(D^2) \cap L^p(T^2) \subset N(D^2) \cap L^p(T^2)$. These facts are shown in [6, Theorem 3.3.5].

A closed subspace M of $H^2(D^2)$ is said to be *invariant* if $zM \subset M$ and $wM \subset M$. For an invariant subspace M of $H^2(D^2)$, set

$$\mathcal{M}(M) = \{\phi \in L^\infty(T^2)\ ;\ \phi M \subseteq H^2(D^2)\}.$$

If $M = qH^2(D^2)$ for some inner function q, it is trivial to see $\mathcal{M}(M) = \bar{q}H^\infty(D^2)$. In the case of one variable, an arbitrary invariant subspace M has the form $qH^2(D)$ for some inner function q by the famous Beurling theorem [1]. Hence $\mathcal{M}(M) = \bar{q}H^\infty(D)$. Hence the map : $M \longrightarrow \mathcal{M}(M)$ is one-to-one. However this result for invariant subspaces of $H^2(D^2)$ is not true. The author [4] studied the relation between M and $\mathcal{M}(M)$. To study $\mathcal{M}(M)$, R. G. Douglas and K. Yan [2] introduced the common zero set $Z(M)$ and the singular measure $Z_\partial(M)$, that is,

$$Z(M) = \{\zeta \in D^2\ ; f(\zeta) = 0 \text{ for } f \in M\}$$

and

$$Z_\partial(M) = \inf\{-d\sigma_f\ ; f \in M, f \neq 0\}.$$

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They showed that if the real 2-dimensional Hausdorff measure of $Z(M)$ is zero and $Z_{\partial}(M) = 0$, then $\mathcal{M}(M) = H^\infty(D^2)$. In this paper, we are interested in invariant subspaces M of $H^2(D^2)$ such that the real 2-dimensional Hausdorff measure of $Z(M)$ is positive and $Z_{\partial}(M) = 0$.

Fix $(\alpha,\beta) \in \overline{D^2}$. For f in $H^p(D^2)$,

$$(\Phi^p_{\alpha\beta}f)(\lambda) = f(\alpha\lambda, \beta\lambda) \qquad (\lambda \in D).$$

$\Phi^p_{\alpha\beta}$ is called a *slice map*. $\Phi^2_{\alpha\beta}$ maps $H^2(D^2)$ into $L^2_a(D)$, where $L^2_a(D)$ is the Bergman space (*cf.* [6, p. 53]). In this paper, we study the kernel $\ker\Phi^p_{\alpha\beta}$ and the range $\operatorname{ran}\Phi^p_{\alpha\beta}$ for $p = 2, \infty$. $\ker\Phi^2_{\alpha\beta}$ is an invariant subspace of $H^2(D^2)$ and the closure of $\operatorname{ran}\Phi^2_{\alpha\beta}$ is an invariant subspace of $L^2_a(D)$. Put

$$D_{\alpha\beta} = \{(\alpha\lambda, \beta\lambda) \in D^2 \ ; \ \lambda \in \mathbf{C}\} \ ;$$

then $Z(\ker\Phi^2_{\alpha\beta}) = \mathcal{D}_{\alpha\beta}$ if $(\alpha,\beta) \in T^2 \cup T\times D \cup D\times T$. The 2-dimensional Hausdorff measure of $Z(\ker\Phi^2_{\alpha\beta})$ is positive and $Z_{\partial}(\ker\Phi^2_{\alpha\beta}) = 0$. In this paper, we show $\mathcal{M}(M) = H^\infty(D^2)$ when $Z(M) = \mathcal{D}_{\alpha\beta}$ for some $(\alpha,\beta) \in T^2$ and $Z_{\partial}(M) = 0$ and M satisfies some additional natural condition. The main result in this paper is Theorem 4 in Section 3. Theorem 1 of [2] has a lot of corollaries on the rigidity of invariant subspaces. Similarly Theorem 3 in this paper has such corollaries. Hence our results can be seen as the generalizations of results of R. G. Douglas and K. Yan.

For f in $N(D^2)$, $f(\zeta) = \sum_{j=0}^\infty F_j(\zeta)$ is a homogeneous expansion of f and F_j is a polynomial which is homogeneous of degree j. The smallest $j = j(f)$ such that F_j is not the zero-polynomial is called the *order of the zero* which f has at $\zeta = (0,0)$. For $p \in D^2$, the order of the zero of f at p is simply the order of the zero of $f(p+\zeta)$ at $\zeta = (0,0)$. We will write $f_p(\zeta) = f(p+\zeta)$.

2. Slice maps.

In this section, we study the slice map $\Phi_{\alpha\beta} = \Phi^p_{\alpha\beta}$ for $(\alpha,\beta) \in \overline{D^2}$.

PROPOSITION 1. *Let $(\alpha,\beta) \in \overline{D^2}$.*
(1) $\Phi^2_{\alpha\beta}$ is a contractive map from $H^2(D^2)$ to $L^2_a(D)$.
(2) If $(\alpha,\beta) \in D^2$, then ran *$\Phi^2_{\alpha\beta}$ is a subset of analytic functions on $\bar{D}$.*
(3) If $(\alpha,\beta) \in T^2$, then $\Phi^2_{\alpha\beta}$ is an onto map from $H^2(D^2)$ to $L^2_a(D)$ with $\|\Phi^2_{\alpha\beta}\| = 1$.
(4) If $(\alpha,\beta) \in T\times D \cup D\times T$, then $\Phi^2_{\alpha\beta}$ is an onto map from $H^2(D^2)$ to $H^2(D)$ with $\|\Phi^2_{\alpha\beta}\| \le (1-|\beta|^2)^{-1}$.

PROOF. (1) For $f \in H^2(D^2)$, let $f(z,w) = \sum_{j=0}^\infty F_j(z,w)$ be a homogeneous expansion of f. Then $F_j(z,w) = \sum_{\ell=0}^j a_\ell z^{j-\ell}w^\ell$ and $\int |F_j|^2\,dm = \sum_{\ell=0}^j |a_\ell|^2$. Moreover

$$\int |f|^2\,dm = \sum_{j=0}^\infty \int |F_j|^2\,dm = \sum_{j=0}^\infty \sum_{\ell=0}^j |a_\ell|^2 < \infty.$$

$(\Phi_{\alpha\beta}f)(\lambda) = \sum_{j=0}^\infty F_j(\alpha,\beta)\lambda^j$ and

$$|F_j(\alpha,\beta)|^2 \le \Big(\sum_{\ell=0}^j |a_\ell|^2\Big)\Big(\sum_{\ell=0}^j |\beta|^{2\ell}\Big) \le (j+1)\Big(\sum_{\ell=0}^j |a_\ell|^2\Big).$$

Hence

$$\int_0^1 \int_0^{2\pi} |\Phi_{\alpha\beta} f|^2 (re^{i\theta}) r \, d\theta \, dr/\pi = \int_0^1 \sum_{j=0}^{\infty} |F_j(\alpha,\beta)|^2 r^{2j+1} 2 \, dr$$
$$= \sum_{j=0}^{\infty} |F_j(\alpha,\beta)|^2 \frac{1}{j+1} \le \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} |a_\ell|^2$$
$$= \int |f|^2 \, dm.$$

Thus $\Phi_{\alpha\beta} f \in L_a^2(D)$ and $\|\Phi_{\alpha\beta}\| \le 1$.

(2) is clear. (3): For $g \in L_a^2(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$, put $f(z,w) = \sum_{j=0}^{\infty} \frac{b_j}{j+1} (\bar{\beta} w)^{j-\ell} (\bar{\alpha} z)^{\ell}$. Then $f \in H^2(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This and (1) imply (3). (4): We may assume $(\alpha,\beta) \in T \times D$. Then

$$|F_j(\alpha,\beta)|^2 \le (1-|\beta|^2)^{-1} \sum_{\ell=0}^{j} |a_\ell|^2$$

and hence

$$\int_0^{2\pi} |\Phi_{\alpha\beta} f|^2 (re^{i\theta}) \, d\theta/2\pi \le \sum_{j=0}^{\infty} \frac{1}{1-|\beta|^2} \sum_{\ell=0}^{j} |a_\ell|^2 \le \frac{1}{1-|\beta|^2} \int |f|^2 \, dm.$$

For $g \in H^2(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$, put $f(z,w) = \sum_{j=0}^{\infty} b_j (\bar{\alpha} z)^j$. Then $f \in H^2(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This implies (4).

(3) of Proposition 1 is essentially known (see [6, p. 53]). Now we study the slice map $\Phi_{\alpha\beta}^{\infty}$ on $H^{\infty}(D^2)$. Let L be the norm closed linear span of $\overline{H^{\infty}(D^2)} H^{\infty}(D^2)$ in $L^{\infty}(T^2)$. Then $L \ne L^{\infty}(T^2)$ (see [5]).

PROPOSITION 2. *Let* $(\alpha,\beta) \in \overline{D^2}$.

(1) $\Phi_{\alpha\beta}^{\infty}$ *is a contractive homomorphism from* $H^{\infty}(D^2)$ *to* $H^{\infty}(D)$.

(2) If $(\alpha,\beta) \in T^2 \cup T \times D \cup D \times T$, *then* $\Phi_{\alpha\beta}^{\infty}$ *is a contractive homomorphism from* $H^{\infty}(D^2)$ *onto* $H^{\infty}(D)$.

(3) If $(\alpha,\beta) \in T^2$, *there exists a contractive* $*$-*homomorphism* $\tilde{\Phi}_{\alpha\beta}^{\infty}$ *from L onto* $L^{\infty}(T)$ *such that* $\tilde{\Phi}_{\alpha\beta}^{\infty} \mid H^{\infty}(T^2) = \Phi_{\alpha\beta}^{\infty} \mid H^{\infty}(T^2)$.

PROOF. (1) is clear. (2): If $g \in H^{\infty}(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$ and $|\alpha| = 1$, then $f(z,w) = \sum_{j=0}^{\infty} b_j (\bar{\alpha} z)^j \in H^{\infty}(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This and (1) imply (2).

(3): For $f_j, g_j \in H^{\infty}(D^2)$ and $j = 1, \ldots, n$, put

$$\left\{ \tilde{\Phi}_{\alpha\beta} \left(\sum_{j=1}^{n} f_j \bar{g}_j \right) \right\} (\lambda) = \sum_{j=1}^{n} f_j(\alpha\lambda, \beta\lambda) \overline{g_j(\alpha\lambda, \beta\lambda)}$$

for $\lambda \in D$, then $\tilde{\Phi}_{\alpha\beta}(\sum_{j=1}^{n} f_j \bar{g}_j)$ can be seen as an element in $L^{\infty}(T)$ by its radial limits.

Hence for a.e. $\lambda \in T$

$$\begin{aligned}\Big|\Phi_{\alpha\beta}\Big(\sum_{j=1}^n f_j\bar g_j\Big)(\lambda)\Big| &\le \operatorname*{ess\,sup}_{\lambda\in T}\Big|\sum_{j=1}^n f_j(\alpha\lambda,\beta\lambda)\overline{g_j(\alpha\lambda,\beta\lambda)}\Big| \\ &\le \operatorname*{ess\,sup}_{(z,w)\in T^2}\Big|\sum_{j=1}^n f_j(\alpha z,\beta w)\overline{g_j(\alpha z,\beta w)}\Big| \\ &= \Big\|\sum_{j=1}^n f_j\bar g_j\Big\|_\infty\end{aligned}$$

because $(\alpha,\beta)\in T^2$. Then $\tilde\Phi_{\alpha\beta}$ is the extension of $\Phi_{\alpha\beta}$ from $H^\infty(D^2)$ to L, and then $\tilde\Phi_{\alpha\beta}$ is a contractive $*$-homomorphism from L to $L^\infty(T)$. If $U(\lambda)=\sum_{j=1}^n F_j(\lambda)\bar G_j(\lambda)$ a.e. on T where $F_j,G_j\in H^\infty(D)$, then $u(z,w)=\sum_{j=1}^n F_j(\bar\alpha z)\bar G_j(\beta w)$ belongs to L and $(\tilde\Phi_{\alpha\beta}u)(\lambda)=U(\lambda)$ a.e. on T. Since arbitrary function U in $L^\infty(T)$ can be approximated by such functions, $\tilde\Phi_{\alpha\beta}$ is onto.

The following lemma will be used in the proofs in the following proposition and the main theorem. We can prove it by an approximation method as in [4] but we prove it using Proposition 2.

LEMMA. *If $\phi\in L^\infty(T^2)$, $(\alpha,\beta)\in T^2$ and $\phi(z,w)(\beta z-\alpha w)\in H^\infty(D^2)$, then $\phi\in H^\infty(D^2)$.*

PROOF. Note that $\beta z-\alpha w\in\ker\Phi_{\alpha\beta}$. If $\phi(\beta z-\alpha w)=g$ for some $g\in H^\infty(D^2)$, then g belongs to $\ker\Phi_{\alpha\beta}$. In fact, $\hat\phi(\beta z-\alpha w)^\wedge=\hat g$ on $\operatorname{Spec}L^\infty(T^2)$ which is the maximal ideal space of $L^\infty(T^2)$ and $(\beta z-\alpha w)^\wedge=0$ on hull$(\ker\tilde\Phi_{\alpha\beta})$. Hence $\hat g=0$ on hull$(\ker\tilde\Phi_{\alpha\beta})\cap\operatorname{Spec}L^\infty(T^2)$. Since L is a commutative C^*-algebra, every element of $\operatorname{Spec}L$ extends to an element of $\operatorname{Spec}L^\infty(T^2)$. Therefore $\hat g=0$ on hull$(\ker\tilde\Phi_{\alpha\beta})$. Thus $g\in(\ker\tilde\Phi_{\alpha\beta})\cap H^\infty(D^2)=\ker\Phi_{\alpha\beta}$. Hence if $g=\sum_{j=0}^\infty G_j$ and $G_j(z,w)=\sum_{\ell=0}^j b_\ell z^{j-\ell}w^\ell$, then

$$G_j(z,w)=z^j\sum_{\ell=0}^j b_\ell(\bar z w)^\ell=k\prod_{\ell=1}^j(w-k_\ell z)$$

where $k\in\mathbf{C}$ and $k_\ell\in\mathbf{C}$ for $1\le\ell\le j$ and $G_j(\alpha\lambda,\beta\lambda)\equiv 0$ for $\lambda\in D$ because $g\in\ker\Phi^2_{\alpha\beta}$. Thus $G_j(z,w)=m(\beta z-\alpha w)\prod_{\ell=2}^j(w-m_\ell z)$ where $m\in\mathbf{C}$ and $m_\ell\in\mathbf{C}$ for $2\le\ell\le j$ and hence $g/(\beta z-\alpha w)$ is analytic on D^2. Since $d\sigma_{\beta z-\alpha w}=0$, $g/(\beta z-\alpha w)\in N_*(D^2)\cap L^\infty(T^2)=H^\infty(D^2)$ and hence ϕ belongs to $H^\infty(D^2)$.

PROPOSITION 3. *Let $(\alpha,\beta)\in\overline{D^2}$.*

(1) For any $r\in(0,1]$, $\ker\Phi^2_{\alpha\beta}=\ker\Phi^2_{r\alpha,r\beta}$.

(2) $\ker\Phi^2_{\alpha\beta}$ is an invariant subspace of $H^2(D^2)$,

$$Z(\ker\Phi^2_{\alpha\beta})=\mathcal{D}_{\alpha\beta}\ \text{and}\ Z_\partial(\ker\Phi^2_{\alpha\beta})=0.$$

For any $p\in\mathcal{D}_{\alpha\beta}$, $\beta z-\alpha w\in\ker\Phi^2_{\alpha\beta}$ has a zero of order 1 at p.

(3) If $(\alpha,\beta)\in T^2$, then $(\beta z-\alpha w)H^2(D^2)$ is dense in $\ker\Phi^2_{\alpha\beta}$ but $\ker\Phi^2_{\alpha\beta}\ne(\beta z-\alpha w)H^2(D^2)$. If $(\alpha,\beta)\in T\times D\cup D\times T$, then $\ker\Phi^2_{\alpha\beta}=(\beta z-\alpha w)H^2(D^2)$.

(4) If $(\alpha,\beta)\in T^2$, *then* $\mathcal{M}(\ker\Phi^2_{\alpha\beta}) = H^\infty(D^2)$ *and if* $(\alpha,\beta)\in T\times D\cup D\times T$, *then* $\mathcal{M}(\ker\Phi^2_{\alpha\beta}) = (\beta z-\alpha w)^{-1}H^\infty(D^2)$.

(5) If $(\alpha,0)\in\bar{D}\times D$ *and* $\alpha\neq 0$, *then* $\ker\Phi^2_{\alpha 0} = wH^2(D^2)$ *and hence* $\mathcal{M}(\ker\Phi^2_{\alpha 0}) = w^{-1}H^\infty(D^2)$.

(6) Let M *be an invariant subspace of* $H^2(D^2)$ *with* $\ker\Phi^2_{\alpha\beta}\subsetneq M$, $\mathcal{M}(M) = H^\infty(D^2)$. *If* $(\alpha,\beta)\in T^2$, *then* $Z(M) = \{(\alpha a_j,\beta a_j)\in D^2\ ;\ \sum_{j=0}^\infty(1-|a_j|)\times[-\log(1-|a_j|)]^{1-\varepsilon}<\infty$ *for all* $\varepsilon>0\}$. *If* $(\alpha,\beta)\in T\times D\cup D\times T$, *then* $Z(M) = \{(\alpha a_j,\beta a_j)\in D^2\ ;\ \sum_{j=1}^\infty(1-|a_j|)<\infty\}$. *If* $(\alpha,0)\in\bar{D}\times D$ *and* $\alpha\neq 0$, *then* $M = qH^2(D)\oplus wH^2(D^2)$ *where* q *is a one variable inner function with* $q=q(z)$ *and hence* $Z(M) = \{(s,0)\in D^2\ ;\ q(s)=0$ *and* $s\in D\}$.

PROOF. (1) and (2) are clear. (3): Let $(\alpha,\beta)\in T^2$. If $f\in\ker\Phi_{\alpha\beta}$, $f=\sum_{j=0}^\infty F_j$ and $F_j(z,w)=\sum_{\ell=0}^j a_\ell z^{j-\ell}w^\ell$, then $F_j(z,w)=c(\beta z-\alpha w)\prod_{\ell=2}^j(w-c_\ell z)$ and hence f can be approximated by the functions in $(\beta z-\alpha w)H^2(D^2)$. This implies that $(\beta z-\alpha w)H^2(D^2)$ is dense in $\ker\Phi^2_{\alpha\beta}$. Suppose $\ker\Phi_{\alpha\beta}=(\beta z-\alpha w)H^2(D^2)$; then the multiplication operator by $\beta z-\alpha w$ is a left invertible operator from $H^2(D^2)$ to $\ker\Phi_{\alpha\beta}$. Hence there exists a positive constant ε such that

$$\int_{T^2}|g|^2|\beta z-\alpha w|^2\,dm \geq \varepsilon\int_{T^2}|g|^2\,dm$$

for all $g\in H^\infty(D^2)$ and so

$$\int_{T^2}u|\beta z-\alpha w|^2\,dm \geq \varepsilon\int_{T^2}u\,dm$$

for all nonnegative continuous functions u on T^2. Thus $|\beta z-\alpha w|^2\geq\varepsilon>0$ a.e. on T^2 and this contradiction implies that $\ker\Phi_{\alpha\beta}\neq(\beta z-\alpha w)H^2(D^2)$. Let $(\alpha,\beta)\in T\times D\cup D\times T$. Since $\beta z-\alpha w$ is invertible in L^∞, $\ker\Phi_{\alpha\beta}=(\beta z-\alpha w)H^2(D^2)$ because $(\beta z-\alpha w)H^2(D^2)$ is dense in $\ker\Phi_{\alpha\beta}$.

(4): Let $(\alpha,\beta)\in T^2$. If $\phi\in\mathcal{M}(\ker\Phi_{\alpha\beta})$, then $\phi(\beta z-\alpha w)=g$ for some $g\in H^\infty(D^2)$. By the Lemma, ϕ belongs to $H^\infty(D^2)$ and hence $\mathcal{M}(\ker\Phi_{\alpha\beta})=H^\infty(D^2)$. Let $(\alpha,\beta)\in T\times D\cup D\times T$ and $\phi\in\mathcal{M}(\ker\Phi_{\alpha\beta})$. By (3), $\ker\Phi_{\alpha\beta}=(\beta z-\alpha w)H^2(D^2)$ and hence

$$\phi(\beta z-\alpha w)H^2(D^2)\subset H^2(D^2).$$

This implies that $\phi(\beta z-\alpha w)\in H^\infty(D^2)$ and so $\mathcal{M}(\ker\Phi_{\alpha\beta})=(\beta z-\alpha w)^{-1}H^\infty(D^2)$. (5) is easy to see. (6): If $(\alpha,\beta)\in T^2$ and $\ker\Phi_{\alpha\beta}\subsetneq M$, then by (3) of Proposition 1 and [3, Corollary 3.6], $Z([\Phi_{\alpha\beta}M]_2)=\{a_j\in D\ ;\ \sum_{j=1}^\infty(1-|a_j|)[-\log(1-|a_j|)]^{1-\varepsilon}<\infty$ for all $\varepsilon>0\}$. This and Theorem 1 in [2] imply the first part. For $(\alpha,\beta)\in T\times D\cup D\times T$, we can show similarly by (4) of Proposition 1.

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3. **Multipliers.** By (4) of Proposition 3, we know the set of all multipliers $\mathcal{M}(M)$ of an invariant subspace such that $\ker \Phi^2_{\alpha\beta} \subseteq M \subseteq H^2(D^2)$ when $(\alpha,\beta) \in T^2 \cup T \times D \cup D \times T$ or $(\alpha, 0) \in D \times D \setminus (0,0)$. When $(\alpha,\beta) \in D \times D$ and $|\alpha| = |\beta| > 0$, there exists $(\alpha_0,\beta_0) \in T \times T$ such that $\alpha = r\alpha_0$ and $\beta = r\beta_0$ for some $r \in (0,1)$. When $(\alpha,\beta) \in D \times D$ and $0 \leq |\alpha| < |\beta|$, there exists $(\alpha_0,\beta_0) \in D \times T$ such that $\alpha = r\alpha_0$ and $\beta = r\beta_0$ for some $r \in (0,1)$. (1) of Proposition 3 implies $\ker \Phi^2_{\alpha\beta} = \ker \Phi^2_{\alpha_0\beta_0}$. Hence for arbitrary $(\alpha,\beta) \in \bar{D} \times \bar{D} \setminus (0,0)$, we can describe $\mathcal{M}(M)$ by Proposition 3. In this section, we study $\mathcal{M}(M)$ without such a condition. In this section, for example, we study $\mathcal{M}(M)$ of an invariant subspace such that $M \subseteq \ker \Phi^2_{\alpha\beta}$. In fact, we study such a problem more generally, that is, when the 2-dimensional Hausdorff measure of $Z(M) \cap \mathcal{D}^c_{\alpha\beta}$ is zero. For $\Lambda \subset T^2 \cup T \times D \cup D \times T$, put

$$\mathcal{D}_\Lambda = \{\cup \mathcal{D}_{\alpha\beta}\ ;\ (\alpha,\beta) \in \Lambda\} \setminus \{(0,0)\}.$$

Note that if $Z(M) \supseteq \mathcal{D}_\Lambda$ and Λ is an infinite set such that $\mathcal{D}_{\alpha\beta} \cap \mathcal{D}_{\gamma\delta} = \{(0,0)\}$ when $(\alpha,\beta) \neq (\gamma,\delta)$, then $M = \{0\}$.

THEOREM 4. *Let Λ be a finite set of T^2. If M is an invariant subspace of $H^2(D^2)$ which satisfies the following (1)–(3), then $\mathcal{M}(M) = H^\infty(D^2)$.*

(1) *For any $p \in Z(M) \cap \mathcal{D}_\Lambda$, there exists a function f in M such that f has a zero of order 1 at p.*
(2) *The 2-dimensional Hausdorff measure of $Z(M) \cap \mathcal{D}^c_\Lambda$ is zero.*
(3) *$Z_{\partial}(M) = 0$.*

PROOF. Suppose $\phi \in \mathcal{M}(M)$. Fix $p \in Z(M) \cap \mathcal{D}_\Lambda$. By (1), let f be a function in M such that f has a zero of order 1 at p. Let $(\alpha,\beta) \in \Lambda$ with $p \in \mathcal{D}_{\alpha\beta}$. By definition of $\mathcal{M}(M)$, $\phi f = g$ for some $g \in H^2(D^2)$. Put $k(z,w) = \beta z - \alpha w$; then $k_p(\zeta) = k(\zeta + p) = k(\zeta)$ and $k_p(\zeta)\phi_p(\zeta)f_p(\zeta) = k(\zeta)g_p(\zeta)$. Suppose $f_p(\zeta) = \sum_{j=0}^\infty F_j(\zeta)$ is a homogeneous expansion of f_p. Since $1 = s(f_p)$, $F_1(0,w) = cw$ for $c \neq 0$. By the Weierstrass preparation theorem (*cf.* [6, Theorem 1.2.1]), there exists a polydisc Δ in $\mathbf{C}^2$, centered at $(0,0)$, such that

$$f_p(z,w) = W(z,w)h(z,w)$$

for $(z,w) \in \Delta$ where h is analytic in Δ, h has no zero in Δ, $W(z,w) = w + b_0(z)$ and b_0 is analytic in Δ with $b_0(0) = 0$. Since $f_p(\alpha\lambda, \beta\lambda) \equiv 0$ on D, $\beta\lambda + b_0(\alpha\lambda) = 0$ if $(\alpha\lambda,\beta\lambda) \in \Delta$ and hence $b_0(\alpha\lambda) = -\beta\lambda$. Thus $b_0(z) = -\frac{\beta}{\alpha} z$ and $W(z,w) = -\frac{1}{\alpha}(\beta z - \alpha w)$. Therefore $k_p\phi_p = kg_p/f_p$ is analytic in Δ and so $k\phi$ is analytic in $\Delta + p$, in a sense of R. G. Douglas and K. Yan [2]. Therefore

$$\prod_{(\alpha,\beta)\in\Lambda} (\beta z - \alpha w)\phi(z,w)$$

is analytic in a neighborhood of $Z(M) \cap \mathcal{D}_\Lambda$.

If $p \notin Z(M)$, then there exists a function k in M such that k has no zeros in some polydisc Δ_p, centered at p. As in the proof above, $\phi(z,w)$ is analytic in Δ_p and hence ϕ is analytic in $D^2 \setminus Z(M)$. Thus $\prod(\beta z - \alpha w)\phi(z,w)$ is analytic in $D^2 \setminus Z(M) \cap \mathcal{D}^c_\Lambda$. By

(2), $Z(M) \cap \mathcal{D}_\Lambda^c$ is a removable singularity for analytic functions, and hence $\psi(z,w) = \prod(\beta z - \alpha w)\phi(z,w)$ is analytic in D^2. By the proof of [2, Theorem 1], $\psi \in N(D^2) \cap L^\infty(T^2)$ and $d\sigma_\psi \le -Z_\partial(M)$ because $d\sigma_\phi = d\sigma_\psi$. By (3), $d\sigma_\psi = 0$ and hence $\psi \in H^\infty(D^2)$. By the Lemma, ϕ belongs to $H^\infty(D^2)$ and hence $\mathcal{M}(M) = H^\infty(D^2)$.

THEOREM 5. *Let Λ be a finite set of $T \times D \cup D \times T$. If M is an invariant subspace of $H^2(D^2)$ which $Z(M) \supseteq \mathcal{D}_\Lambda$ and satisfies the following (1)–(3), then*

$$\mathcal{M}(M) = \prod_{(\alpha,\beta)\in\Lambda} (\beta z - \alpha w)^{-1} H^\infty(D^2).$$

(1) For any $p \in Z(M)$, there exists a function f in M such that f has a zero of order 1 at p.

(2) The 2-dimensional Hausdorff measure of $Z(M) \cap \mathcal{D}_\Lambda^c$ is zero.

(3) $Z_\partial(M) = 0$.

PROOF. By the proof of Theorem 4, if $\phi \in \mathcal{M}(M)$, then $\prod(\beta z - \alpha w)\phi(z,w) \in H^\infty(D^2)$ where (α,β) ranges over Λ. Hence $\phi \in \prod(\beta z - \alpha w)^{-1}H^\infty(D^2)$. Conversely if $\phi \in \prod(\beta z - \alpha w)^{-1}H^\infty(D^2)$ and $f \in M$, then $f = 0$ on $\mathcal{D}_\Lambda$; hence by the Weierstrass preparation theorem, $\prod(\beta z - \alpha w)^{-1}f(z,w)$ is analytic in D^2 and ϕ belongs to $\mathcal{M}(M)$.

4. **Two general cases and remarks.** Let a and b be two functions in $H^\infty(D)$ with $\|a\|_\infty \le 1$ and $\|b\|_\infty \le 1$. For f in $H^p(D^2)$,

$$(\Phi^p_{ab} f)(\lambda) = f\big(a(\lambda), b(\lambda)\big) \qquad (\lambda \in D).$$

If $a(\lambda) = \alpha(\lambda)$ and $b(\lambda) = \beta\lambda$, then Φ^p_{ab} was called a slice map $\Phi^p_{\alpha\beta}$ in the previous sections. For an arbitrary pair a and b, we know only very trivial results. It is easy to see that Φ^∞_{ab} maps $H^\infty(D^2)$ into $H^\infty(D)$. If $\|a\|_\infty < 1$ and $\|b\|_\infty < 1$, then Φ^p_{ab} maps $H^p(D^2)$ into $H^\infty(D)$. In general, $\ker \Phi^2_{ab}$ is still an invariant subspace of $H^2(D^2)$, and

$$Z(\ker \Phi^2_{ab}) \supseteq \mathcal{D}_{ab} = \Big\{\big(a(\lambda), b(\lambda)\big) \in D^2 \,;\, \lambda \in D\Big\}.$$

The function $b(z) - a(w)$ may not belong to $\ker \Phi^2_{ab}$. If $a(\lambda) = \alpha\lambda$ and $b(\lambda) = \beta\lambda$, then $(b \circ a)(\lambda) = (a \circ b)(\lambda)$ for $\lambda \in D$, and hence $b(z) - a(w)$ belongs to $\ker \Phi^2_{ab}$. If $a(\lambda) = \lambda$ and $b(\lambda)$ is an inner function, then $(b \circ a)(\lambda) = (a \circ b)(\lambda)$ for $\lambda \in D$, and hence $b(z) - a(w) = b(z) - w$ belongs to $\ker \Phi^2_{ab}$. In this case, $Z_\partial(\ker \Phi^2_{ab}) = 0$. For any $p \in \mathcal{D}_{ab}$, $b(z) - w$ has a zero of order 1 at $p \in \mathcal{D}_{ab}$. If $\phi \in L^\infty(T^2)$ and $\big(b(z) - w\big)\phi(z,w) \in H^\infty(D^2)$, then $\phi \in H^\infty(D^2)$. This can be shown as in [4, Proposition 3 and Theorem 7]. This implies (4) of Proposition 3. The proof of the following theorem is almost parallel to that of Theorem 4.

THEOREM 6. *Let $a(\lambda) = \lambda$ and $b(\lambda)$ be an inner function. If M is an invariant subspace of $H^2(D^2)$ which satisfies the following (1)–(3), then $\mathcal{M}(M) = H^\infty(D^2)$.*

(1) For any $p \in Z(M) \cap \mathcal{D}_{ab}$, there exists a function f in M such that f has a zero of order 1 at $p \in Z(M) \cap \mathcal{D}_{ab}$.

(2) The 2-dimensional Hausdorff measure of $Z(M) \cap \mathcal{D}_{ab}^c$ is zero.
(3) $Z_{\partial}(M) = 0$.

If $a(\lambda) = \lambda$ and $b(\lambda) = cq(\lambda)$ where c is a constant with $|c| < 1$ and q is an inner function, we can show a version of Theorem 5 as Theorem 6 which is that of Theorem 4.

Let D^n be the open unit polydisc in $\mathbf{C}^n$ and T^n be its distinguished boundary. Fix $\alpha = (\alpha_1, \ldots, \alpha_n) \in \overline{D^n}$. For f in $H^p(D^n)$

$$(\Phi_\alpha^p f)(\lambda) = f(\alpha_1\lambda, \ldots, \alpha_n\lambda) \qquad (\lambda \in D).$$

(1), (2) and (3) of Proposition 1 can be proved for arbitrary n. If $\alpha_j \in T$ for some j with $1 \leq j \leq n$ and $\alpha_i \in D$ for all i with $1 \leq i \leq n$ and $i \neq j$, we can show that Φ_α^2 is an onto map from $H^2(D^n)$ to $H^2(D)$ with $\|\Phi_\alpha^2\| \leq \prod_{i \neq j}(1 - |\alpha_j|^2)^{-1}$. This is a generalization of (4) of Proposition 1. Similarly we can generalize Proposition 2. If $\phi \in L^\infty(T^n)$ and $(\alpha_i z_j - \alpha_j z_i)\phi(z_1, \ldots, z_n) \in H^\infty(D^n)$ where $1 \leq i \neq j \leq n$ and $\alpha = (\alpha_i, \ldots, \alpha_n) \in T^n$, then $\phi \in H^\infty(D^n)$. This also can be shown as in [4, Proposition 3 and Theorem 7]. $\ker \Phi_\alpha^2$ is an invariant subspace and a generalization of (1) and (2) of Proposition 3 is true. Suppose $n > 2$. If M is an invariant subspace of $H^2(D^n)$, $Z(M) = \mathcal{D}_\alpha = \{(\alpha_1\lambda, \ldots, \alpha_n\lambda)\ ;\ \lambda \in D\}$ for $\alpha \in T^n$ and $Z_\partial(M) = 0$, then $\mathcal{M}(M) = H^\infty(D^n)$. For it is a result of R. G. Douglas and K. Yan [2, Theorem 1] because the real $2n-2$ dimensional Hausdorff measure of $Z(M)$ is zero.

REMARK. (i): As in Theorem 1 of [2], Theorem 4 can be stated as the following: If M is an invariant subspace of $H^2(D^2)$ which satisfies (1) and (2), then $\phi \in \mathcal{M}(M)$ if and only if $\phi \in N(D^2) \cap L^\infty(T^2)$ and $d\sigma_\phi \leq Z_\partial(M)$. (ii): By Lemma 7 in [2] and Theorem 4, if M and N are quasi-similar invariant subspaces of $H^2(D^2)$ and M satisfies (1)–(3) in Theorem 4, then $M \subseteq N$. This is a generalization of Theorem 2 in [2]. Similarly we can generalize Corollaries 9 and 12. (iii): Let M, N be invariant subspaces of $H^2(D^2)$ satisfying (a) the 2-dimensional Hausdorff measures of $Z(M) \cap \mathcal{D}_\Lambda^c$ and $Z(N) \cap \mathcal{D}_\Lambda^c$ are zero. (b) $Z_\partial(M) = Z_\partial(N)$. (c) M and N satisfy the condition (1) in Theorem 4 about $Z(M) \cap \mathcal{D}_\Lambda$ and $Z(N) \cap \mathcal{D}_\Lambda$. If M and N are quasi-similar, then $M = N$.

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